Spontaneous Stochasticity Amplifies Even Thermal Noise to the Largest Scales of Turbulence in a Few Eddy Turnover Times

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How predictable are turbulent flows? Here, we use theoretical estimates and shell model simulations to argue that Eulerian spontaneous stochasticity, a manifestation of the nonuniqueness of the solutions to the Euler equation that is conjectured to occur in Navier-Stokes turbulence at high Reynolds numbers, leads to universal statistics at finite times, not just at infinite time as for standard chaos. These universal statistics are predictable, even though individual flow realizations are not. Any small-scale noise vanishing slowly enough with increasing Reynolds number can trigger spontaneous stochasticity, and here we show that thermal noise alone, in the absence of any larger disturbances, would suffice. If confirmed for Navier-Stokes turbulence, our findings would imply that intrinsic stochasticity of turbulent fluid motions at all scales can be triggered even by unavoidable molecular noise, with implications for modeling in engineering, climate, astrophysics, and cosmology.

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Spontaneous stochasticity is a recently discovered phenomenon [1–3] in turbulent flows [4,5], whereby solutions of model fluid equations remain unpredictable and stochastic due to diverging Lyapunov exponents in the high Reynolds number limit, even though random perturbations to the flow are negligible asymptotically. It is an open question as to whether or not it occurs in real fluids or in the Navier-Stokes equations. Here, we report that at large (but finite) Reynolds numbers, a stochastic wave propagating from small to large scales, as first postulated by Lorenz [6], rapidly randomizes the large-scale flow, *even if the only sources of noise to trigger the wave are molecular fluctuations at small scales*. Going beyond Lorenz, we show that flow fluctuations *at large scales* exhibit universal statistics due to spontaneous stochasticity and not directly due to whatever *small-scale* noise triggers the stochastic wave. Spontaneous stochasticity is not inevitable; for it to be triggered, the small-scale noise must become negligible in the large Reynolds number limit sufficiently slowly. The surprise is that even thermal noise satisfies this condition. These new indirect effects at large scales are distinct from a growing body of work showing that thermal noise directly alters the turbulent dissipation range below the Kolmogorov scale [7–16], and, in fact, other small-scale disturbances, typically much larger than thermal agitation, will produce indistinguishable large-scale stochasticity. To demonstrate our claim that the probability distributions of relevant flow quantities have a universal, non-delta-function form at large but finite Reynolds numbers, we are forced to employ a simplified but well-studied dynamical model [17] of turbulence, since the Navier-Stokes equations are computationally unfeasible at the large Reynolds numbers required for convincing convergence to universal statistics. These results suggest an essential indeterminism of turbulent flows at scales of practical interest, with potentially far-ranging implications for engineering, geophysics, and astrophysics.

Fluctuating hydrodynamics.—The fluctuating hydrodynamics of Landau-Lifshitz [18] describes the effect of thermal noise in fluid flows by including fluctuating stresses into the Navier-Stokes equation. It is expressed in a form nondimensionalized by large-scale velocity U and outer or integral length L as

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\triangle\mathbf{u} + \sqrt{\Theta}\nabla\cdot\boldsymbol{\xi} + F\mathbf{f}, \tag{1}$$

where the fluctuating stress is modeled as a Gaussian random field $\boldsymbol{\xi}$ with mean zero and covariance

$$\langle \xi_{ij}(\mathbf{x},t)\xi_{kl}(\mathbf{x}',t')\rangle = \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right) \times \delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{2}$$

Here, the Reynolds number is defined as $\mathrm{Re} \equiv UL/\nu$, where U is the large-scale velocity of the flow, L is the forcing scale at which energy is injected, and ν is kinematic viscosity. $\Theta \equiv 2\nu k_B T/\rho L^4 U^3$ appears due to the presence of noise and characterizes its strength according to the fluctuation-dissipation relation. k_B is Boltzmann's constant, T is absolute temperature, and ρ is mass density. To drive turbulence, we have added an external forcing $\mathbf{f}$ nondimensionalized by rms value f_{rms}, with prefactor $F = Lf_{\mathrm{rms}}/U^2$ setting the magnitude. Strictly speaking, Eq. (1) describes a coarse-grained velocity field on length scale Λ^{-1} larger than the mean free path, and some care is needed to interpret this mathematically; see Supplemental Material (Sec. I) [19] for details [41–44].

Using flow parameters characteristic of the atmospheric boundary layer (ABL) [45] $T = 300°\,\mathrm{K}$, $\nu = 1.5 \times 10^{-5}\ \mathrm{m}^2/\mathrm{sec}$, $\rho = 1.2\,\mathrm{kg}/\mathrm{m}^3$, $\varepsilon = 4 \times 10^{-2}\ \mathrm{m}^2/\mathrm{sec}^3$, $L = 10^3$ m, and $U = 3.42$ m/sec, where ε is the mean energy dissipation per mass, we find $\Theta \simeq 2.59 \times 10^{-39}$, naively justifying dropping the fluctuating stress term from the Landau-Lifshitz equations and supporting the conventional wisdom that physically relevant turbulent fluid flows are well modeled by the deterministic (cutoff) Navier-Stokes equations. This conclusion is consistent with numerical findings [12–14] that the turbulent steady-state statistics at scales larger than the Kolmogorov length $\eta = \nu^{3/4}\varepsilon^{-1/4}$ are unchanged by molecular fluctuations but not addressing the issue of the flow predictability.

Fully developed fluid turbulence in the ABL has $\mathrm{Re} \simeq 2.28 \times 10^8$, justifying dropping as well the viscous term proportional to Re^{-1} from the equations [46]. However, the limiting equations with no viscosity, no wave number cutoff, and no thermal noise are the continuum Euler equations, which do not have unique solutions and are formally ill posed [47–49]. Such nonuniqueness or "flexibility" of solutions suggests an intrinsic unpredictability of turbulent fluid motions at high Reynolds numbers known as *Eulerian spontaneous stochasticity* [3,50] and provides a possibility for tiny thermal noise to influence the predictability of all scales of the flow up to the largest.

Spontaneous stochasticity.—Spontaneous stochasticity can be given a precise meaning through a probability distribution on solutions of the governing stochastic differential equation. In the case of Eulerian spontaneous stochasticity triggered by thermal noise, the corresponding equation is (1). The solution of this equation may be expressed in terms of a transition probability density $P_{\mathrm{Re},\Theta}(\mathbf{u}_f, t_f|\mathbf{u}_i, t_i)$ as follows:

$$P_{\mathrm{Re},\Theta}(\mathbf{u}_f, t_f) = \int \mathcal{D}\mathbf{u}_i P_{\mathrm{Re},\Theta}(\mathbf{u}_f, t_f|\mathbf{u}_i, t_i) P_{\mathrm{Re},\Theta}(\mathbf{u}_i, t_i), \quad (3)$$

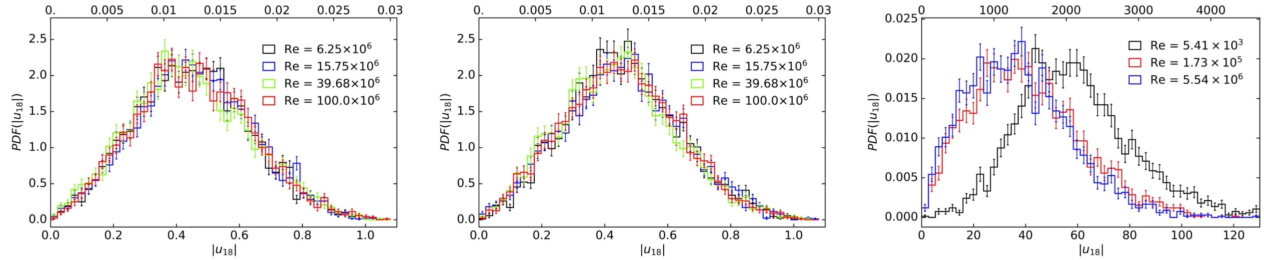
where $P_{\mathrm{Re},\Theta}(\mathbf{u}_{i/f}, t_{i/f})$ is a probability distribution of velocity fields at the initial or final time. The transition probability $P_{\mathrm{Re},\Theta}(\mathbf{u}_f, t_f|\mathbf{u}_i, t_i)$ satisfies the Fokker-Planck equation corresponding to Langevin equation (1), and it is parametrized by the nondimensional numbers Re and Θ. In the limit of zero noise $\Theta \to 0$ with Re fixed, the transition probability becomes deterministic, that is, expressed as a delta distribution on the unique solution of the limiting deterministic problem. The issue of uniqueness of solutions plays the central role in emergence of spontaneous stochasticity: See Supplemental Material Sec. II [19] for more details. Note that the limiting equations for the fluctuating hydrodynamics of Landau-Lifshitz in the limit $\Theta \to 0$ are Navier-Stokes equations with the finite cutoff Λ, so that uniqueness of solutions to the Cauchy initial-value problem is then elementary and well known. This physically necessary cutoff is crucial, since the uniqueness of Leray solutions of the continuum Navier-Stokes equations is a major open problem in pure mathematics [51]. However, if the zero-noise limit is taken together with $\mathrm{Re} \to \infty$ and $\Lambda \propto \mathrm{Re} \to \infty$, it leads to singular Euler dynamics with no unique solutions. Then the limiting transition probability may cease to become deterministic:

$$P_{\mathrm{Re},\Theta}(\mathbf{u}_f, t_f|\mathbf{u}_i, t_i) \xrightarrow[\mathrm{Re}\to\infty]{\Theta\to 0} P_\infty(\mathbf{u}_f, t_f|\mathbf{u}_i, t_i). \quad (4)$$

If such a nontrivial limiting transition probability exists, the limit is called spontaneously stochastic and corresponds to stochastic behavior of a formally deterministic Euler system, which each realization of the limiting distribution satisfies in a weak sense [1,3,50,52,53].

Because the spontaneously stochastic limit is a double limit Re^{-1}, $\Theta \to 0$, there is no unique way to arrive at it. Furthermore, if the noise strength is taken to zero sufficiently fast, the limit becomes deterministic. In many cases of practical importance, such as turbulent flows past a grid or a cylinder, experimental evidence [54,54,55] points at a nonvanishing energy dissipation with the limiting dissipation rate satisfying a relation first proposed by Taylor [56] $\varepsilon = CU^3/L$, where C is a dimensionless constant. In this scenario, we can control the macroscopic flow parameters U (or ε) and L, while ν, T, and ρ are fixed material parameters of the fluid. These considerations motivate as a "canonical limit" the one obtained by fixing the ratio U^3/L, along with ν, T, and ρ, while taking $\mathrm{Re} \to \infty$. This leads us to define the second nondimensional number as $\theta_\eta = 2k_B T/\rho\nu^{11/4}\varepsilon^{-1/4}$ (see [12]) and consider the limit $\Theta = \mathrm{Re}^{-15/4}\theta_\eta \to 0$ with θ_η held constant. This is the physically relevant continuum limit in which also $\Lambda \propto \mathrm{Re} \to \infty$, describing fully developed 3D hydrodynamic turbulence.

Numerical verification of Eulerian spontaneous stochasticity.—In order to check that the limit (4) is indeed spontaneously stochastic, we need to simulate Landau-Lifshitz equations at very high Reynolds numbers. State-of-the-art simulation can achieve only $\mathrm{Re} \approx 500$ [14,57] for incompressible flows, so we use here the Sabra model [17], a simplified dynamical model of turbulent cascade that preserves many key features of Navier-Stokes equations (1)



(a) Self-similar initial state with noise scaling $\alpha = 0$.

(b) Self-similar initial state with noise scaling $\alpha = 3$.

(c) "Burst" initial state with noise scaling $\alpha = 0$.

FIG. 1. Transition probability density function for the absolute values $|u_n|$ at a single fixed shell number $n = 18$ and time $t_f = 1$ (a),(b) and $t_f = 1.477 \times 10^{-3}$ (c) in inertial units for Reynolds numbers spanning almost two decades. The bottom axis represents the inertial range units, while the top axis represents the SI units for the ABL parameters. All the errors are estimated as standard errors using the bootstrap method. Reference [19] contains details on how the Reynolds number was varied, Sec. IV for the K41 initial condition and Sec. VI for the "burst" initial condition.

but discretizes length scales as $\ell_n = 2^{-n}L$ and keeps only one complex mode u_n to represent velocity increments $\delta_l u \propto |u_n|$ at scale ℓ_n. We also include a stochastic term to model thermal noise and a deterministic forcing f_n that acts only at large scales. In nondimensionalized form, this modified Sabra model is given by the stochastic ordinary differential equations:

$$\frac{du_n}{dt} = i\left(k_{n+1}u_{n+2}u_{n+1}^* - \frac{1}{2}k_n u_{n+1}u_{n-1}^* + \frac{1}{2}k_{n-1}u_{n-1}u_{n-2}\right) - \frac{1}{\mathrm{Re}}k_n^2 u_n + \sqrt{\Theta}k_n\xi_n(t) + Ff_n, \quad n = 1, \ldots, N, \tag{5}$$

where $k_n = 1/\ell_n$, covariance of the white noise $\langle \xi_n^*(t)\xi_m(t')\rangle = 2k_n^\alpha \delta_{nm}\delta(t - t)$, and the second nondimensional number group is $\Theta = \mathrm{Re}^{-\beta}\theta_\eta$, $\beta = 3(\alpha + 2)/4$. Here, we take the number of shells $N \propto \frac{3}{4}\log_2(\mathrm{Re})$, sufficient to resolve a few shells above the Kolmogorov wave number $k_\eta = 1/\eta$. The choice $\alpha = 3$ in the noise covariance is dimensionally identical to 3D Landau-Lifshitz, with $\beta = 15/4$, and it produces also an energy spectrum $\propto k^2$ in the dissipation range, the same as for 3D fluids, but violates the shell-model fluctuation-dissipation relation. On the other hand, the choice $\alpha = 0$ preserves this relation, although the equipartition energy spectrum in the dissipation range is $\propto k^{-1}$ rather than $\propto k^2$. Since it is impossible to match exactly all relevant properties of 3D Landau-Lifshitz equations with a single choice of α, we investigated both choices $\alpha = 0$ and $\alpha = 3$, and we find the overall results are insensitive to this choice. We emphasize that for either choice of α the noise does not serve as a large scale forcing, and, in fact, together with the viscous damping, it becomes vanishingly small in the limit $\mathrm{Re} \to \infty$. For more details on the numerical simulations, including the forcing f_n used and the choice of α, see Supplemental Material, Secs. III–VI [19].

We study the Cauchy problem for (5) with two different deterministic but "quasisingular" initial data that are not smooth uniformly in Reynolds number. It is convenient to study spontaneous stochasticity with such quasisingular initial data, since with large-scale initial data independent of Re one would otherwise have to wait for singularities to form by finite-time blowup [50]. The first is the Kolmogorov initial datum $u_n = -iA\varepsilon^{1/3}k_n^{-1/3}$, which is an exact stationary (but unstable) solution of the inviscid, deterministic Sabra model if suitable deterministic forces f_n are added to the two lowest shells $n = 1, 2$ [58]. The other initial datum is a "burst" state selected from the ensemble of turbulent steady states of the Sabra model at very high Re (see Supplemental Material, Sec. V [19]). This particular initial datum has approximately a power-law form $u_n \propto k_n^{-h}$ in the inertial range, with Hölder exponent $h \simeq 0.258$; by construction, this is not intended to be the scaling of the statistical steady state. Both of these initial data are quasisingular with exponent $h < 1$, regularized only at very high wave number either by the cutoff N or by viscosity ν. The numerical details of how Re was varied differs for the two initial data: See Supplemental Material, Sec. IV for the K41 case and Sec. VI for the burst case [19].

The key statistical quantities which we calculate are the probability density functions (PDFs) of local-in-scale variables, such as the absolute values of velocities at a fixed shell numbers n, fixed time t_f, at an increasing sequence of Reynolds numbers. These reduced PDFs are integrals over the transition probability densities in (3). Without external noise, these are delta distributions; see Supplemental Material, Sec. VII [19]. Presented in Fig. 1 are plots of the PDFs for shell $n = 18$ and time $t_f = 1$ (a),(b) and $t_f = 1.477 \times 10^{-3}$ (c), where Figs. 1(a) and 1(b) are for the K41 initial datum with noise exponents $\alpha = 0$ and $\alpha = 3$, respectively, and Fig. 1(c) is for the burst initial datum with $\alpha = 0$. As seen clearly, the PDFs converge with increasing Re to nondelta distributions and, therefore, do not become

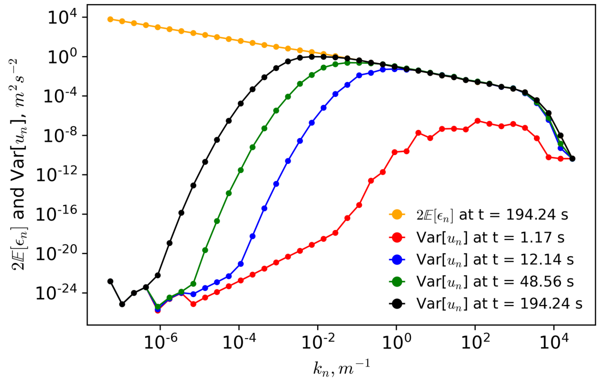


FIG. 2. Twice ensemble average energy $\mathbb{E}[\epsilon_n]$ (orange) for $\epsilon_n = \frac{1}{2}|u_n|^2$ and velocity variances (blue, green, and black) across the ensemble as a function of wave number in SI units for four increasing times. The smallest time in the variances plots the initial non-self-similar transient, and the subsequent three times show the self-similar propagation of the stochastic wave toward large scales. $2\mathbb{E}[\epsilon_n]$ is almost unchanged in time and forms the envelope of the propagating wave.

deterministic. The direct effects of thermal fluctuations at this scale can be estimated from θ_η, and the resulting rms velocity fluctuations are 4–5 orders of magnitude smaller than the ones shown in Fig. 1. Thus, the universal statistics reflect spontaneous stochasticity, not direct effects of thermal noise. We have obtained similar results for the PDFs of other scale-local variables, e.g., energy flux Π_n (see Supplemental Material, Sec. VIII [19]).

These observations constitute our crucial numerical evidence for Eulerian spontaneous stochasticity triggered by thermal noise in the Sabra model and, presumably, for the Landau-Lifshitz equations. The two cases in Figs. 1(a) and 1(b) correspond to the same initial datum and the same limiting equations when Re $\to \infty$ but a different scale-by-scale approach toward it. Nevertheless, the limiting probability distributions are the same and independent of the type of regularization and the type of noise that triggers random perturbations.

Inverse error cascade and stochastic wave.—What causes this unpredictability if the direct effects of thermal noise are too small? The mechanism was first suggested by Lorenz: an *inverse cascade of error* [6] that has since been extensively studied [59–62]. Perhaps the simplest way to illustrate this mechanism is to look at the time-dependent variances $\mathrm{Var}[u_n] = \mathbb{E}[|u_n - \mathbb{E}[u_n]|^2]$ calculated across an ensemble of noise realizations with fixed initial datum. These are shown in Fig. 2 for the K41 datum. Initially, variances at all scales exhibit diffusive linear growth in time, with higher rate at larger k_n. Next, modes become chaotic scale by scale, starting from high wave numbers, and eventually the variance for a particular shell saturates when it reaches twice the average energy at that scale. In the early stage of development of the stochastic wave, the total variance of the system $\mathrm{Var}(\mathbf{u}) = \sum_n \mathrm{Var}(u_n)$ grows exponentially (see Supplemental Material, Sec. IX [19]), and this regime is fully consistent with the work of Ruelle [10] on the effects of thermal noise in predictability of developed turbulence. However, when the stochastic front starts to propagate across the inertial range [50,63], the system enters the spontaneously stochastic regime. In the case of a self-similar initial state $u_{0,n} \propto Ak_n^{-h}$, the front is self-similar, located at length scale $\ell(t) = (At)^{1/(1-h)}$ with amplitude $u(t) = (At^h)^{1/(1-h)}$ at time t. Plotted as $\mathrm{Var}(u_n)/u^2(t)$ versus $k_n\ell(t)$, the curves collapse for the three late times $t = 12.14, 48.56, 194.24$ s. For more details on the stages of the stochastic wave formation and propagation, see Supplemental Material, Sec. IX [19]. Furthermore, after the stochastic front passes some scale, the statistics of Kolmogorov multipliers [64] at that scale converges to the steady state distribution. Such *super-universality* has been observed before in [50]; see Supplemental Material, Sec. X [19]. We draw attention to the striking resemblance of our Fig. 2 to Fig. 2 of Lorenz [6], which he obtained for two-dimensional Euler equations using a turbulence closure model. For the analogous plot with the burst initial state, see Supplemental Material, Sec. XI [19], where the same picture holds qualitatively, although there is no exact self-similarity. The large spontaneous fluctuations illustrated in Fig. 1(a) are, thus, due to effects of thermal noise in the dissipation range which are propagated up into the inertial range by nonlinear error cascade and not due to the direct local effects of thermal noise.

An important feature of this "inverse error cascade" is that in the inertial range the universal statistical distributions are achieved at each length scale ℓ in a time which is a constant multiple of the eddy-turnover time $\tau_\ell = \ell/u_\ell$, indifferent to the noise magnitude. This should be contrasted with a predictability horizon in conventional chaotic systems, which is dependent on the noise

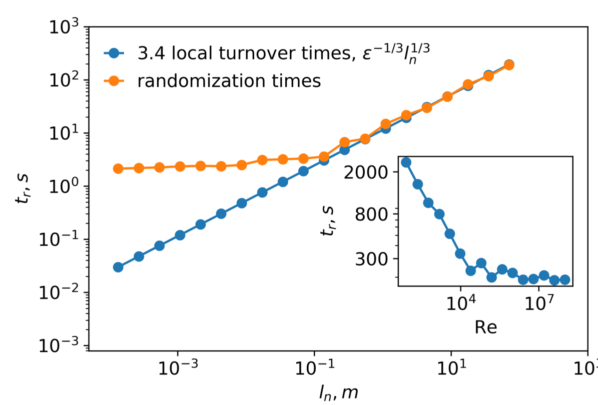


FIG. 3. Local randomization times $t_r(n)$ as a function of length scale $\ell_n = 2^{-n}L$ for the K41 initial datum. $t_r(n)$ is defined as the time in which the nth shell's variance reaches the ensemble average energy $\mathbb{E}[\epsilon_n]$. The inset plot depicts $t_r(18)$ as a function of the Reynolds number.

strength [3,6,65,66]. To illustrate this point, Fig. 3 shows the randomization time $t_r(n)$, defined as the time when the nth shell's variance reaches its ensemble average energy, plotted versus index n. As is clear from the figure, the randomization times above the length scale of 10 cm for the flow parameters of the ABL are given by $t_r(n) = 3.4\varepsilon^{-1/3}\ell_n^{2/3}$. Therefore, we conclude that the length scales of about the size of a coffee mug and above in 3D ABL turbulence behave in a spontaneously stochastic fashion. In Supplemental Material, Sec. XII [19], we provide a theoretical estimate on dimensional grounds of that length scale as a function of Re and Θ. Crucially, we observe that $t_r(n)$ approaches the asymptotic value $\propto \varepsilon^{-1/3}\ell_n^{2/3}$ for any shell n in the limit $\mathrm{Re} \to \infty$: See the inset in Fig. 3. Thus, *all* scales are spontaneously stochastic in that idealized limit.

Discussion.—It is important to emphasize our finding that the spontaneous large-scale statistics are universal with respect to the small-scale noise that triggers them, as long as the noise amplitude becomes negligible with respect to the deterministic equation more slowly than some Re-dependent threshold. On dimensional grounds, we estimate this threshold to be $\sim\exp(-\sqrt{\mathrm{Re}})$ as $\mathrm{Re} \to \infty$ (see Supplemental Material, Sec. XII [19]). Even the inevitably present molecular noise satisfies this criterion, and our simulations suggest that it is sufficient to trigger spontaneous stochasticity. In one turnover time of the largest 3D turbulent eddies, the unknown molecular motions will impact the evolution, rendering only statistical predictions possible.

Our work has implications for turbulence across multiple scales. For climate models, even if the projected goal of 1 km horizontal resolution in the next decade is achieved [66], such refined resolution will not obviate the need for stochastic models [65–68]. For the dynamics of galaxy formation, it has already been shown that microscale chaos and stochasticity lead to large variations in star-formation histories and distribution of stellar mass [69], and our results suggest that these effects may be even more severe than currently thought. At the large scale of hydrodynamic simulations of cosmological galaxy formation, the sensitivity of simulations to minute perturbations has also been examined with regard to chaotic dynamics [70] and would be expected to be amplified further by the results we have discussed [71]. Closer to home, there have been recent efforts to reconstruct best-fit individual solution trajectories of Navier-Stokes equations using variational data assimilation techniques [72,73]. It is already recognized that these reconstruction problems are highly ill conditioned due to chaotic dynamics. The inclusion of spontaneous stochasticity into this program poses even more severe limitations and implies that a well-posed problem is instead the reconstruction of statistical ensembles of solutions [74,75]. These examples show that there are many potential ramifications of spontaneous stochasticity in turbulence and related phenomena. It will be important to determine if our findings, based admittedly on a shell model, are valid beyond the necessary simplifications entailed in our work.

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