Instituto de Matemática Pura e Aplicada

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EDUARDO ALVES DA SILVA

Log Calabi-Yau geometry and Cremona maps

Rio de Janeiro
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PhD thesis presented to the Graduate Program of the Instituto Nacional de Matemática Pura e Aplicada - IMPA, as a partial requirement for obtaining the title of Doctor of Mathematics.

Advisor: Prof. Carolina Araujo

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Rio de Janeiro
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[2] Programa de Iniciação Científica Júnior.

[3] Programa de Iniciação Científica e Mestrado.

Dedication

I dedicate this work to my beloved and dear parents who have always done everything so that I could become the person I am today. I also dedicate it to my godmother Josefina (Zefa) (*in memoriam*) for the most worthy example of a person.

"The one who works hard can overcome a genius, but there is no point in working hard if you do not trust yourself..."

Rock Lee, a character in the manga Naruto

Abstract

This work explores interactions between log Calabi-Yau geometry and Cremona maps. In a 2-dimensional context, we investigate the decomposition group of a nonsingular plane cubic under the light of the log Calabi-Yau geometry. Using this approach we prove that an appropriate algorithm of the Sarkisov Program applied to an element of this group is automatically volume preserving. From this, we deduce some properties of the (volume preserving) Sarkisov factorization of its elements. We also negatively answer a question posed by Blanc, Pan and Vust asking whether the canonical complex of a nonsingular plane cubic is split. Within a similar context in dimension 3, we exhibit in detail an interesting counterexample for a possible generalization of a theorem by Pan, in which there exists a Sarkisov factorization produced by the usual algorithm that is not volume preserving. In a 3-dimensional context, we investigate the birational geometry of log Calabi-Yau pairs of coregularity 2, where the ambient variety is the 3-dimensional projective space and the boundary divisor is necessarily an irreducible normal quartic surface with canonical singularities. We completely classify which toric weighted blowups of a point are volume preserving and initiate a volume preserving Sarkisov link from this pair. Depending on the type of singularity, our results point out that some of these weights do not work generically for a general member of the corresponding coarse moduli space of quartics.

Key-words: Sarkisov Program, Calabi-Yau pairs, Cremona maps.

Resumo

Este trabalho explora as interações entre a geometria log Calabi-Yau e os mapas de Cremona. Em um contexto bidimensional, investigamos o grupo de decomposição de uma cúbica plana não singular sob a luz da geometria log Calabi-Yau. Usando esta abordagem provamos que um algoritmo apropriado do Programa de Sarkisov aplicado em um elemento deste grupo é automaticamente volume preserving. A partir disto, deduzimos algumas propriedades da fatoração de Sarkisov (volume preserving) de seus elementos. Também respondemos negativamente a uma questão colocada por Blanc, Pan e Vust perguntando se o complexo canônico de uma cúbica plana não singular cinde. Dentro de um contexto semelhante em dimensão 3, exibimos em detalhes um contraexemplo interessante para uma possível generalização de um teorema de Pan, no qual existe uma fatoração de Sarkisov produzida pelo algoritmo usual que não é volume preserving. Em um contexto tridimensional, investigamos a geometria birracional de pares log Calabi-Yau de corregularidade 2, onde a variedade ambiente é o espaço projetivo tridimensional e o divisor de fronteira é necessariamente uma superfície quártica normal irredutível com singularidades canônicas. Classificamos completamente quais os blowups com peso tóricos de um ponto são volume preserving e iniciam um link de Sarkisov volume preserving começando deste par. Dependendo do tipo de singularidade, nossos resultados apontam que alguns desses pesos não funcionam genericamente para um membro geral do correspondente moduli space grosseiro de quárticas.

Palavras chave: Programa de Sarkisov, pares de Calabi-Yau, mapas de Cremona.

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Chapter 1

Introduction

This work relies on the birational geometry of Calabi-Yau pairs and interactions with Cremona groups. The study of Calabi-Yau pairs has been an active research area in complex Algebraic Geometry. In part, this is because they can be seen as distinguished minimal models of the classical Minimal Model Program (MMP) or its log version. Moreover, maximal log Calabi-Yau pairs (see Definition 3.2.2) have notable properties predicted from mirror symmetry [HK1]. One important tool in the study of Calabi-Yau pairs is a relatively new version of the Sarkisov Program [Cor1, HM1] for volume preserving maps between Mori fibered Calabi-Yau pairs obtained by Corti & Kaloghiros [CK]. See Theorem 3.1.19.

The Sarkisov Program asserts that any birational map between Mori fibered spaces can be written as a composition of a finite sequence of elementary maps, called Sarkisov links. This is very useful to study Mori fibered spaces, which are outcomes of the MMP coming from uniruled varieties. The result by Corti & Kaloghiros [CK] can be interpreted as a generalization of this theorem with some additional structures, and aiming for an equilibrium between singularities of pairs and varieties.

A very nice application of this result is to the study of decomposition and inertia groups. In Algebraic Geometry, they are special subgroups of the Cremona group that preserve a certain subvariety of $\mathbb{P}^n$ as a set and pointwise, respectively. In [Cas, MM] and more recently in [Giz, Pan1, BPV1, BPV2, Bl1, Bl2, HZ, DHZ, Piñ], there exist numerous interesting results and descriptions of these groups in several cases.

In the particular case where this fixed subvariety is a hypersurface $D_{n+1} \subset \mathbb{P}^n$ of degree $n+1$, we have that $(\mathbb{P}^n, D_{n+1})$ is an example of a *Calabi-Yau pair*, that is, a pair (X, D) with mild singularities consisting of a normal projective variety X and a reduced Weil divisor on X such that $K_X + D \sim 0$. In other words, regarding $n = \dim(X)$, there exists a meromorphic volume form $\omega = \omega_{X,D} \in \Omega^n_X$ up to nonzero scaling, such that $D + \mathrm{div}(\omega) = 0$.

Under restrictions on the singularities of $(\mathbb{P}^n, D)$, see Proposition 3.1.20, the decomposition group of the hypersurface D, denoted by $\mathrm{Bir}(\mathbb{P}^n, D)$ or simply $\mathrm{Dec}(D)$, coincides with the group of birational self-maps of $\mathbb{P}^n$ that preserve the volume form ω, up to nonzero scaling. Such maps are naturally called *volume preserving*.

This notion of Calabi-Yau pair allows us to use new tools to deal with the study of these groups and (re)interpret some results as statements about the *birational geometry of the pair*. One of these tools is the so-called volume preserving Sarkisov Program, which is a result valid in all dimensions.

This work is concerned with two projects in 2 and 3-dimensional contexts. The first one investigates the decomposition group of a nonsingular plane cubic under the light of the log Calabi-Yau geometry. Using this approach we prove that the standard algorithm of the Sarkisov Program applied to an element of this group is automatically volume preserving. From this, we deduce some properties of the (volume preserving) Sarkisov factorization of its elements.

The second project is a study of the birational geometry of log Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2, where in this case D is an irreducible normal quartic surface with canonical singularities. We completely classify which toric weighted blowups of a point will initiate a volume preserving Sarkisov link from this pair, depending on the singularities of the boundary. These two projects culminated in two preprints [Alv1, Alv2] in preparation.

1.1 A compendium of the results obtained

By means of an approach through log Calabi-Yau geometry, the main result in the first project is the following:

Theorem 1.1.1 (See Theorem 4.2.6)**.** *Let $C \subset \mathbb{P}^2$ be a nonsingular cubic. The standard Sarkisov Program applied to an element of* $\mathrm{Dec}(C)$ *is automatically volume preserving.*

The main fact used in the proof of this result is Theorem 4.2.2 due to Pan [Pan1], which asserts that the base locus of an element $\phi \in \mathrm{Dec}(C) \setminus \mathrm{PGL}(3, \mathbb{C})$ is contained in the curve C. Furthermore, by employing the volume preserving variant of the Sarkisov Program, it becomes feasible to demonstrate a broader fact: all the infinitely near base points of an element of $\mathrm{Dec}(C)$ belong to the strict transforms of C. See Lemma 4.2.5. This will guarantee that when running the (volume preserving) Sarkisov Program, all the surfaces involved, together with the corresponding strict transforms of C, are always Calabi-Yau pairs.

We have the following result, which restricts the possibilities for Mori fibered spaces appearing in a volume preserving factorization of an element of $\mathrm{Dec}(C)$:

Lemma 1.1.2 (See Lemma 4.1.1)**.** *The only (rational) Mori fibered spaces in dimension 2 that admit an irreducible divisor D such that (S, D) is a Calabi-Yau pair are:*

$$\mathbb{P}^2/\,\mathrm{Spec}(\mathbb{C}) \text{ and } \mathbb{F}_n/\mathbb{P}^1 \text{ for } n \in \{0, 1, 2\}.$$

By [Giz, Theorem 6] and [Og2, Theorem 2.2] we have a natural exact sequence induced by the natural action ρ of $\mathrm{Dec}(C)$ on C

$$1 \longrightarrow \mathrm{Ine}(C) \longrightarrow \mathrm{Dec}(C) \xrightarrow{\rho} \mathrm{Bir}(C) \longrightarrow 1, \tag{1.1.1}$$

where the inertia group $\mathrm{Ine}(C)$ is defined as $\ker(\rho)$. Blanc, Pan & Vust [BPV1] asked whether this sequence is split or not. Notice that in particular, C is an elliptic curve and therefore also an algebraic group. One has $\mathrm{Bir}(C) = \mathrm{Aut}(C) = C \rtimes \mathbb{Z}_d$, where C is the group of translations and $d \in \{2, 4, 6\}$, depending on the j invariant of C. In the proof of [Og2, Thorem 2.2], the surjectivity of ρ is proved by exhibiting a set-theoretical section $C \hookrightarrow \mathrm{Dec}(C)$. We show that this set-theoretical section, however, is not a group homomorphism and hence it is not a partial splitting of 1.1.1. From this, we are able to produce a lot of elements in $\mathrm{Ine}(C)$.

More generally, we show the following which negatively answers the question posed in [BPV1]:

Theorem 1.1.3 (See Theorem 4.3.4)**.** *The canonical complex 1.1.1 of the pair* $(\mathbb{P}^2, C)$ *does not admit any splitting at* C *when we write* $\mathrm{Aut}(C) = C \rtimes \mathbb{Z}_d$*.*

Within a similar context in higher dimension, it is natural to ask ourselves about a generalization of the Theorem 4.2.2 and of the Theorem 4.2.6. In dimension 3 we exhibited in detail an interesting counterexample for these both questions arising from the decomposition group of a general quartic surface with a single canonical singularity of type A_1. See Sections 4.4 & 5.4.

The results achieved in the second project embody the essence of explicit birational geometry. In [Gue], Guerreiro studied Sarkisov links initiated by the toric weighted blowup of a point in $\mathbb{P}^3$ or $\mathbb{P}^4$ using a variation of GIT, and gave a complete classification of them with a description of the whole Sarkisov link. In the work [ACM], Araujo, Corti & Massarenti considered irreducible normal quartic surfaces with single canonical singularities of types A_1 and A_2 and solved the same problem in the volume preserving context.

The following result extends the classification given in [ACM] contemplating more types of surface canonical singularities, the so-called Du Val singularities that can be corresponded with simple-laced Dynkin diagrams of type ADE.

Theorem 1.1.4 (See Theorem 5.2.1)**.** *Let* $(\mathbb{P}^3, D)$ *be a log Calabi-Yau pair of coregularity 2 and* $\pi\colon (X, D_X) \to (\mathbb{P}^3, D)$ *be a volume preserving toric* $(1, a, b)$*-weighted blowup of a torus invariant point. Then this point is necessarily a singularity of* D *and, up to permutation, the only possibilities for the weights, depending on the type of singularities, are listed in the following Table 1.1.*

type of singularity	volume preserving weights
A_1	(1,1,1)
A_2	(1,1,1), (1,1,2),
A_3	(1,1,1), (1,1,2), (1,1,3)
A_4	(1,1,1), (1,1,2), (1,1,3), (1,2,3)
A_5	(1,1,1), (1,1,2), (1,1,3), (1,2,3)
A_6	(1,1,1), (1,1,2), (1,1,3), (1,2,3), (1,2,5), (1,3,4)
A_7	(1,1,1), (1,1,2), (1,1,3), (1,2,3), (1,2,5), (1,3,4), (1,3,5)

Table 1.1: Table summarizing volume preserving weights, up to permutation.

The following result is a partial volume preserving version of Theorem 5.2.3 due to Guerreiro [Gue], for the case where the Calabi-Yau pair $(\mathbb{P}^3, D)$ has coregularity two. As a consequence of the toric description of the weighted blowup, all types of strict canonical singularities of type A_n are contemplated.

Theorem 1.1.5 (See Theorem 5.2.4)**.** *Let $(\mathbb{P}^3, D)$ be a log Calabi-Yau pair of coregularity 2 and $\pi\colon (X, D_X) \to (\mathbb{P}^3, D)$ be a volume preserving toric $(1, a, b)$-weighted blowup of a torus invariant point. Then this point is necessarily a singularity of D and, up to permutation, the only possibilities for the weights initiating a volume preserving Sarkisov link, depending on the type of singularities, are listed in the following Table 1.2.*

type of singularity	volume preserving weights
A_1	(1,1,1)
A_2	(1,1,1), (1,1,2)
A_3	(1,1,1), (1,1,2)
A_4	(1,1,1), (1,1,2), (1,2,3)
A_5	(1,1,1), (1,1,2), (1,2,3)
$A_{\geq 6}$	(1,1,1), (1,1,2), (1,2,3), (1,2,5)

Table 1.2: Table summarizing volume preserving weights initiating Sarkisov links, up to permutation.

The last two theorems can be regarded as a first step in the explicit classification of log Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2, up to volume preserving equivalence. Additionally, they signify the initial strides taken in the advancement of a technology designed to explicitly handle volume-preserving birational maps of threefold Mori fibered Calabi-Yau pairs.

1.2 Structure of the thesis

Throughout this thesis, our ground field will be $\mathbb{C}$, or more generally, any algebraically closed field of characteristic zero. Concerning general aspects of birational geometry and singularities of the MMP, we refer the reader to [KM, Kol].

In Chapter 2 we will give an overview of the Sarkisov Program and its standard algorithm, which only exists in dimensions 2 and 3. In Chapter 3 we will introduce some natural classes of singularities of pairs, a collection of results involving the geometry of log Calabi-Yau pairs, and the Sarkisov Program in its volume preserving version. In Chapter 4, we will approach the 2-dimensional case and describe the decomposition group of a nonsingular plane cubic via log Calabi-Yau geometry. In Chapter 5, we will approach the 3-dimensional case and describe the birational geometry of log Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2 making use of tools from toric geometry.

Chapter 2

Cremona groups and the Sarkisov Program

We denote $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x_0, x_1, \ldots, x_n])$ the projective space over the field $\mathbb{C}$ of complex numbers.

A *Cremona map (or transformation)* is simply a birational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$.

With the operation of composition, the set of Cremona transformations forms a group, which is called the *Cremona group* and denoted by $\mathrm{Bir}(\mathbb{P}^n)$. Some authors denote it by $\mathrm{Cr}(n)$ or $\mathrm{Cr}_n(k)$ emphasizing the ground field k.

The investigation of this group constitutes a highly dynamic domain in Algebraic Geometry, with numerous open inquiries surrounding it, such as the description of its generators in higher dimensions as well as the classification of its subgroups.

Algebraically, the elements of the Cremona group correspond to $\mathbb{C}$-automorphisms of $\mathbb{C}(x_1, \ldots, x_n)$, the purely transcendental field extension of $\mathbb{C}$ of transcendence degree n. This correspondence comes from the equivalence of categories between algebraic varieties over $\mathbb{C}$ with dominant rational maps, and finitely generated field extensions of $\mathbb{C}$ with $\mathbb{C}$-algebra homomorphisms. Under this equivalence, we have that

$$\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(x_1, \ldots, x_n)) \simeq \mathrm{Bir}(\mathbb{P}^n).$$

In this work, we are interested in the geometric side from which many interesting questions emerge.

From the MMP point of view, $\mathbb{P}^n$ together with the morphism $\mathbb{P}^n \to \mathrm{Spec}(\mathbb{C})$ has the structure of *Mori fibered space*. See Definition 2.1.1. This feature will allow us to apply all the robust machinery involving the Sarkisov Program to study $\mathrm{Bir}(\mathbb{P}^n)$.

For instance, as previously mentioned, it is an open and hard problem determining explicit generators for the Cremona group in dimension ≥ 3, and recent progress was obtained by Blanc, Lamy & Zimmermann. Using this powerful tool, in [BLZ] they proved that these groups are not generated by linear and Jonquières elements together with any subset with cardinality smaller than that of $\mathbb{C}$.

2.1 The Sarkisov Program

The modern framework for the classification of algebraic varieties up to birational equivalence is by means of the *Minimal Model Program* (MMP for short). Roughly speaking, given a terminal projective variety, the idea is to find a distinguished representative in its birational equivalence class which is simpler in a suitable sense.

Such "simpler" representatives consist of varieties X whose canonical class K_X is nef or whose anticanonical class $-K_X$ is relatively ample for an appropriate fibration. These two classes of objects are the possible outcomes of the MMP. The former ones are denominated *minimal models* and the latter ones are the so-called *Mori fibered spaces*.

Definition 2.1.1. A *Mori fibered space* is a normal projective variety X together with a morphism $f\colon X \to S$, to a lower dimensional normal variety S, such that

1. $f_*\mathcal{O}_X = \mathcal{O}_S$,
2. $-K_X$ is f-ample, and
3. $\rho(X/S) \coloneqq \rho(X) - \rho(S) = 1$.

We denote such a structure by X/S.

We point out that the condition $f_*\mathcal{O}_X = \mathcal{O}_S$ implies connectedness of the fibers by a theorem on formal functions. See [Har, Corollary III.11.4]. Such a result is known as *Zariski's Main Theorem*.

The condition $-K_X$ is f-ample means that $-K_X$ restricted to the fibers is ample, that is, the fibers are Fano varieties.

The number $\rho(X/S) \coloneqq \rho(X) - \rho(S)$ is called the *relative Picard number of f*.

Mori fibered spaces are the outcomes when we run the MMP on a *uniruled variety*. A variety X is *uniruled* if there exists a variety Z and a generically finite dominant map $Z \times \mathbb{P}^1 \dashrightarrow X$. Such a condition implies that X is covered by rational curves.

Starting with a uniruled variety and depending on the choices made along the process, one may obtain two different outputs Y and Y', connected by a birational transformation $Y \dashrightarrow Y'$.

The philosophy of the Sarkisov Program is to give a factorization of $Y \dashrightarrow Y'$ into elementary birational maps called *Sarkisov links*.

The Sarkisov Program in dimension 3 was established by Corti [Cor1, Theorem 3.7] whereas for higher dimensions by Hacon & McKernan [HK1, Theorem 1.1]. As a historical remark, the surface case was considered and reworked several times before Corti's work. The statement is originally due to Max Noether (1870), with further approaches by Castelnuovo (1901), Nagata (1960), and Iskovskikh (1979). See [CKS, Historical remark 2.21].

Before stating the precise notions of Sarkisov links, let us give some definitions from the MMP.

Definition 2.1.2. A *divisorial contraction* is a birational morphism $f\colon Z \to X$ between $\mathbb{Q}$-Gorenstein varieties (canonical divisor is $\mathbb{Q}$-Cartier) such that

1. the exceptional locus $\mathrm{Exc}(f)$ of f is a prime divisor,
2. $-K_Z$ is f-ample and
3. the relative Picard number of f is 1.

Definition 2.1.3. A *Mori divisorial contraction* is a divisorial contraction $f\colon Z \to X$ from a $\mathbb{Q}$-factorial terminal variety Z, associated to an extremal ray $R \subset \overline{NE}(Z)$ such that $K_Z \cdot R < 0$. In particular, X also has $\mathbb{Q}$-factorial terminal singularities.

A *Mori flip* is a flip $\varphi\colon Z \dashrightarrow Z'$ from a $\mathbb{Q}$-factorial terminal variety Z, associated to an extremal ray $R \subset \overline{NE}(Z)$ such that $K_Z \cdot R < 0$. In particular, Z' also has $\mathbb{Q}$-factorial terminal singularities. An *antiflip* is the inverse of a Mori flip. A *Mori flop* is a flop $\varphi\colon Z \dashrightarrow Z'$ between $\mathbb{Q}$-factorial terminal varieties, associated to an extremal ray $R \subset \overline{NE}(Z)$ such that $K_Z \cdot R = 0$.

2.1.1 Sarkisov links

It is shown in [Cor1, HM1] that any birational map between Mori fibered spaces is a composition of *Sarkisov links*:

$$\begin{tikzcd}
X = X_0 \arrow[r, dashed, "\phi_1"] \arrow[rrrr, dashed, bend left, "\phi"] \arrow[d] & X_1 \arrow[r, dashed, "\phi_2"] \arrow[d] & \cdots \arrow[r, dashed, "\phi_{m-1}"] & X_{m-1} \arrow[r, dashed, "\phi_m"] \arrow[d] & X_m = Y \arrow[d] \\
S = Y_0 & Y_1 & & Y_{m-1} & Y_m = T
\end{tikzcd}.$$

Here ϕ stands for a birational map between the Mori fibered spaces X/S and Y/T, and ϕ_i for a Sarkisov link in its decomposition.

We recall now the definition of the 4 types of Sarkisov links. Observe that in the following description, we dispose the varieties of same Picard rank at the same height, and $X \to S$ and $X' \to S'$ always stand for Mori fibered spaces.

1. A *Sarkisov link of type I* is a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[r, dashed] & X' \arrow[d] \\
X \arrow[d] & & S' \arrow[dll] \\
S & &
\end{tikzcd}$$

   where $Z \to X$ is a Mori divisorial contraction, and $Z \dashrightarrow X'$ is a sequence of Mori flips, flops and antiflips. Notice that $\rho(S'/S) = 1$.

2. A *Sarkisov link of type II* is a commutative diagram

$$
\begin{tikzcd}
& Z \arrow[r, dashed] \arrow[dl] & Z' \arrow[dr] & \\
X \arrow[d] & & & X' \arrow[d] \\
S \arrow[rrr, equal] & & & S'
\end{tikzcd}
$$

where $Z \to X$ and $Z' \to X'$ are Mori divisorial contractions, and $Z \dashrightarrow X'$ is a sequence of Mori flips, flops and antiflips. Notice here that $S = S'$.

3. A *Sarkisov link of type III* is a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, dashed] \arrow[d] & Z \arrow[dr] & \\
S \arrow[drr] & & X' \arrow[d] \\
& & S'
\end{tikzcd}
$$

where $X \dashrightarrow Z$ is a sequence of Mori flips, flops and antiflips, and $Z \to X'$ is a Mori divisorial contraction. Notice that $\rho(S/S') = 1$.

4. A *Sarkisov link of type IV* is a commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d] & & X' \arrow[d] \\
S \arrow[dr] & & S' \arrow[dl] \\
& T &
\end{tikzcd}
$$

where $X \dashrightarrow X'$ is a sequence of Mori flips, flops and antiflips, and $S \to T$ and $S' \to T$ are Mori contractions. Hence we have that $\rho(S/T) = \rho(S'/T) = 1$.

We point out that the maps $S' \to S$ (Sarkisov link of type I), $S \to S'$ (Sarkisov link of type III), and $S \to T$ and $S' \to T$ (Sarkisov link of type IV) do not need to give a Mori fibered space structure. They can also be divisorial contractions.

It is important to notice that the Sarkisov Program in dimensions 2 and 3 is algorithmic whereas for higher dimensions it is existential in nature. See [CKS, Theorem 2.24] concerning the 2-dimensional case and see [Mat, Flowcharts 1-8-12 & 13-1-9] for explicit flowcharts in dimensions 2 and 3, respectively.

2.2 Sarkisov algorithm in dimension 2

In this section, we will give an overview of the Sarkisov algorithm in dimension 2. We refer the reader to [Bea, CKS] for generalities on the birational geometry of surfaces.

2.2.1 Linear systems on surfaces

Let S be a nonsingular projective surface. We have the following well-known correspondence of sets:

$$\left\{\begin{array}{c}\text{rational maps } \phi\colon S \dashrightarrow \mathbb{P}^n \\ \text{such that } \phi(S) \text{ is contained in no hyperplane,} \\ \text{up to composition with an element of } \mathrm{Aut}(\mathbb{P}^n)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}n\text{-dimensional} \\ \text{linear systems on } S \\ \text{without fixed part}\end{array}\right\}.$$

Actually, this correspondence is a particular instance of a much more general property. See [Har, Theorem II.7.1].

Given Γ a linear system on S, its *base locus*, denoted by $\mathrm{Bs}(\Gamma)$, is the intersection of all its members. If Γ has no fixed part, Γ is called *mobile* and $\mathrm{Bs}(\Gamma)$ is simply the finite set of points where the corresponding rational map ϕ_Γ is not well defined.

The *multiplicity* m_P of Γ at a point $P \in S$ is the multiplicity of a general member there, that is,

$$m_P \coloneqq \min\{\mathrm{mult}_P(C);\ C \in \Gamma\}.$$

For any birational morphism $f\colon S' \to S$, the *birational transform* of a mobile linear system Γ on S under f is the linear system Γ' on S' obtained by the pullback of Γ and depriving of its possible fixed components.

Consider the case where f is the blowup at a point P belonging to $\mathrm{Bs}(\Gamma)$. From the geometrical interpretation of the blowup, the strict transform C' of a general member C of Γ intersects the exceptional divisor $E \coloneqq \mathrm{Exc}(f)$ at the points corresponding to tangent directions to the curve C at P.

Notice that if all members of Γ share one or more tangent directions at P, then all members of Γ' will intersect E at the corresponding points. Hence such points lie in $\mathrm{Bs}(\Gamma')$. They are denominated *base points of* Γ *infinitely near to* P. Furthermore, they are such that the map birational map $\phi_{\Gamma'} = \phi_\Gamma \circ f$ is not well defined there. Thus, it may happen that a single point blowup is not enough to resolve the indeterminacy of a rational map locally.

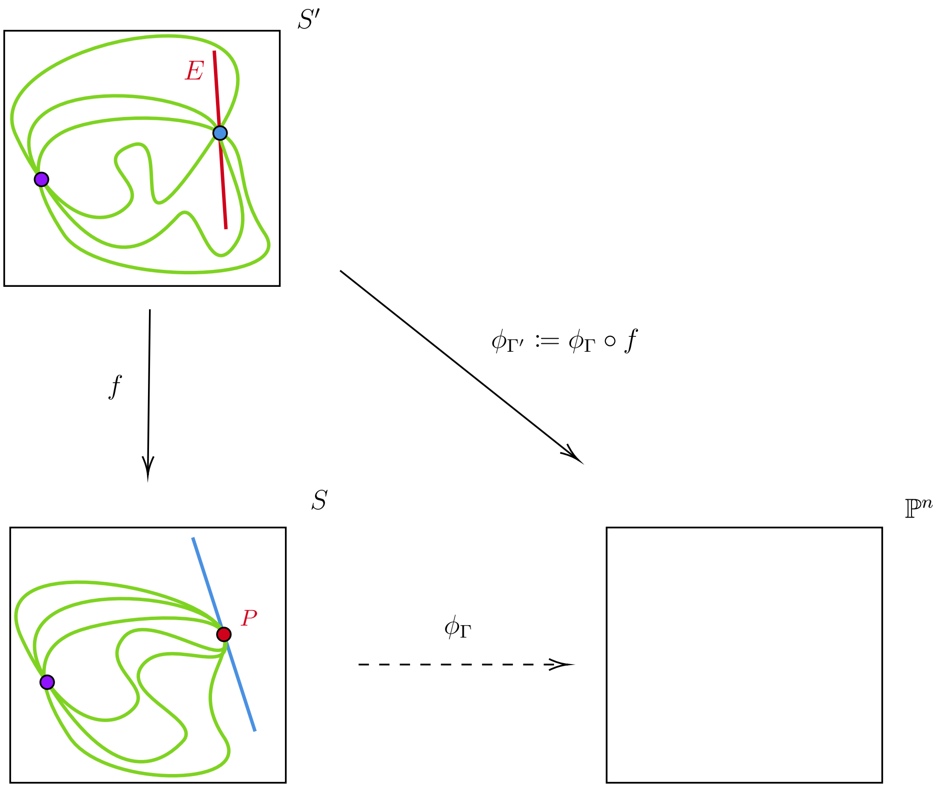


Figure 2.1: Linear system on surface S and rational maps.

When blowing up the base points of Γ' lying in E, it may happen that the birational transform of Γ' has base points lying in the new exceptional divisor. Such points are also base points of Γ infinitely near to P. Geometrically, these ones correspond to higher order tangent directions of the members of Γ.

This motivates the following definition:

Definition 2.2.1 (Infinitesimal neighborhoods)**.** Let P be a point in a nonsingular projective surface S and let E be the exceptional divisor of the blowup of S at P. We will call E the *first infinitesimal neighborhood of* P. For $i > 0$, an i*-th infinitesimal neighborhood of* P, is defined inductively as the collection of points on the first infinitesimal neighborhood of a point in some $(i-1)$-th infinitesimal neighborhood of P. The points that are on an i-th infinitesimal neighborhood of P, for some $i > 0$, are called *points infinitely near to* P. To distinguish the points of S from the infinitely near to S, we will call the points of S *proper*. This notion yields a natural partial ordering of the points proper or infinitely near to S: we write $P \prec Q$ if Q is infinitely near to P.

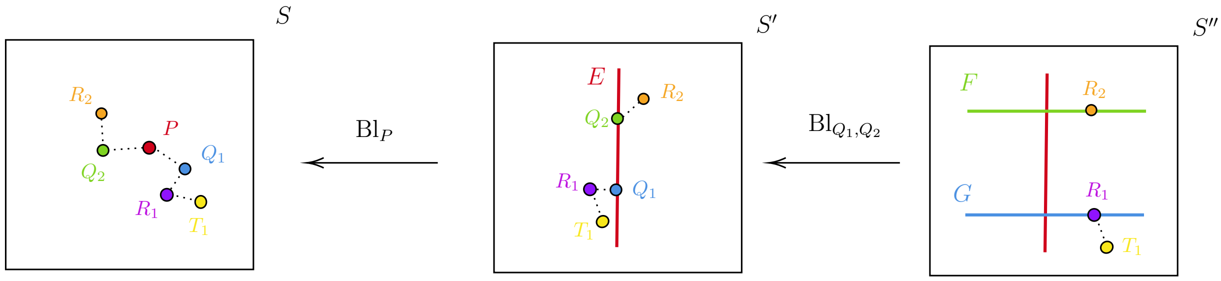


Figure 2.2: Infinitely near points to P.

2.2.2 Plane Cremona maps

An appropriate reference for this topic is [Alb]. Given a plane Cremona map f, by abuse of notation one can associate to it the linear system $f^*|\mathcal{O}_{\mathbb{P}^2}(1)| = f^*|H|$, where H denotes a general line of $\mathbb{P}^2$. Since $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^2}(1) = \mathbb{Z} \cdot [H]$, it follows that $f^*|\mathcal{O}_{\mathbb{P}^2}(1)|$ is contained in some complete linear system $|\mathcal{O}_{\mathbb{P}^2}(d)|$, for some d nonnegative integer. We call such a d the degree of f.

There exist numerous other ways to define it in the literature. For instance, see [Alb, CKS, Des1, Des2, Lam].

Definition 2.2.2. Let f be a plane Cremona map of degree d. Consider $m_1 \geq \cdots \geq m_r$ the multiplicities of the base points of f including the infinitely near ones. The *characteristic* or *homaloidal type* of f is defined to be the list $(d; m_1, \ldots, m_r)$.

Sometimes we will use exponent notation to indicate a repetition in some m_i. For instance, $(8; 4^2, 3^2, 2^3, 1)$ means $(8; 4, 4, 3, 3, 2, 2, 2, 1)$.

A very interesting property of the homaloidal type of a plane Cremona map is expressed by some relations between its components. Such relations are the so-called *Noether-Fano equations* or *equations of condition* as defined in [Alb, Definition 2.5.1].

Theorem 2.2.3 (cf. [Alb] Section 2.5 & [CKS] Theorem 2.9)**.** *Let $\phi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map of degree d with $\mathrm{Bs}(\phi) = \{P_1, \ldots, P_r\}$ including the infinitely near base points. Consider $m_1, \ldots, m_r$ the respective multiplicities. Then they satisfy the equations*

$$\sum_{i=1}^{r} m_i^2 = d^2 - 1, \tag{2.2.1}$$

$$\sum_{i=1}^{r} m_i = 3d - 3. \tag{2.2.2}$$

These formulas have geometrical interpretations and are valid in a more general context involving rational maps from an arbitrary surface to $\mathbb{P}^n$.

It turns out that not all lists of positive integers satisfying these relations come from an actual plane Cremona map. This leads to the notion of *proper characteristic*, that is, a list of positive integers representing the homaloidal type of a plane Cremona map. There exists a test to detect such a property. See [Alb, Section 5.3] and [Lam, Chapter 8] for more details.

2.2.3 Mori fibered surfaces

Mori fibered surfaces are the outputs of the MMP when we start with a uniruled surface.

Let S be a nonsingular projective surface. By the Enriques-Kodaira classification of surfaces, one can show the following:

$$S \text{ is uniruled} \Leftrightarrow S \text{ is ruled} \Leftrightarrow \kappa(S) = -\infty,$$

where $\kappa(S)$ denotes the Kodaira dimension of S.

By definition, S is *ruled* if it is birational to $C \times \mathbb{P}^1$, where C is a nonsingular curve. Observe that $C \times \mathbb{P}^1$ comes with the natural structure morphism $C \times \mathbb{P}^1 \to C$ with fibers isomorphic to $\mathbb{P}^1$.

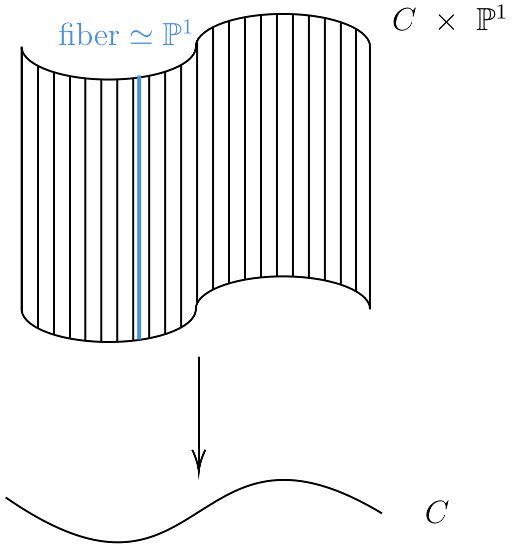


Figure 2.3: A ruled surface.

It turns out that the isomorphism class of C will determine the birational equivalence class of a ruled surface S. One fact corroborating such property is that the irregularity $q = q(C \times \mathbb{P}^1)$, which is a birational invariant, is precisely g. See [Bea, Propositions III.20 & III.21].

Let C be a nonrational and nonsingular curve ($g(C) \geq 1$). By [Bea, Theorem III.10], the minimal models of $C \times \mathbb{P}^1$ are the geometrically ruled surfaces over C, that is, the projective bundles $\mathbb{P}_C(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 vector bundle over the curve C.

Notice that such ruled surfaces are not rational. By a sequence of exercises in [Bea, Exercises III.24], any birational map between them can be decomposed into isomorphisms and some elementary transformations.

The rational ruled surfaces occur when the curve C is rational, that is, $g = 0$. It is well known that their minimal models are $\mathbb{P}^2$ and the geometrically ruled surfaces $\mathbb{F}_n$ with $n \neq 1$, as known as Hirzebruch surfaces. See [Bea, Theorem V.10].

Such surfaces $\mathbb{F}_n$ are defined to be the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ given as the projectivization of the rank two vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. We always consider the Grothendieck notion of a projectivization of a vector bundle, that is, the Proj of the symmetric algebra of its locally free sheaf of sections. In particular, one can show that $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up at a point.

Seen as Mori fibered spaces, all these surfaces carry a structure morphism as follows: $\mathbb{P}^2 \to \mathrm{Spec}(\mathbb{C})$, the projections on the two factors $p_1 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and $p_2 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, $\mathbb{F}_n \to \mathbb{P}^1$.

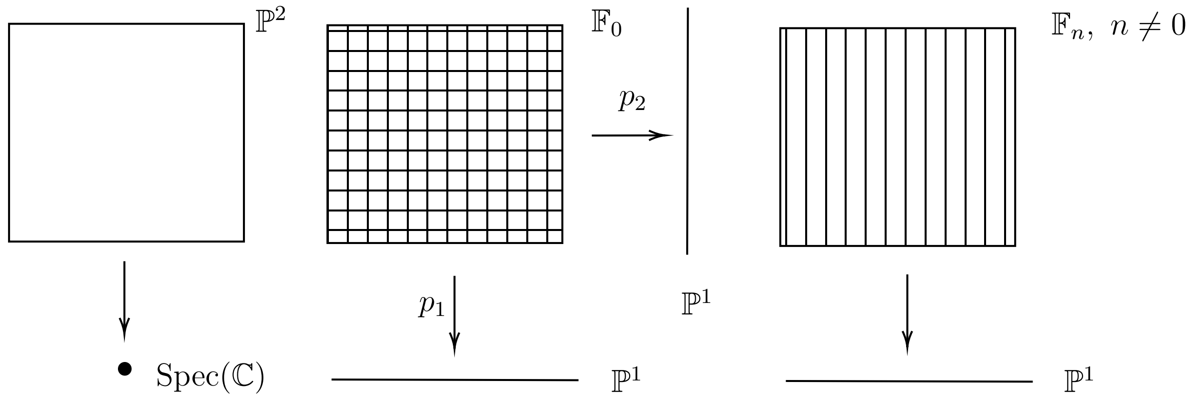


Figure 2.4: Rational Mori fibered surfaces.

Geometry of the Hirzebruch surfaces $\mathbb{F}_n$. The Picard group of the surface $\mathbb{F}_n$ is isomorphic to $\mathbb{Z} \cdot [F] \oplus \mathbb{Z} \cdot [E]$, where F is a fiber of the structure morphism and

$$E = \begin{cases} \text{the negative section}, & \text{if } n \geq 1 \\ \text{any section with self-intersection } 0, & \text{if } n = 0 \end{cases}.$$

To simplify the notation, sometimes we will identify divisors with their corresponding classes in the Picard group. Suppressing and abusing notation, we will denote $\mathbb{Z} \cdot F \oplus \mathbb{Z} \cdot E$ as $\langle F, E \rangle$. Furthermore, for $n = 0$ and also abusing notation, we will call any section with 0 self-intersection a "negative section".

The intersection theory on $\mathbb{F}_n$ is given by

$$\begin{cases} F^2 = 0 \\ F \cdot E = 1 \\ E^2 = -n \end{cases}.$$

Moreover, the canonical class of $\mathbb{F}_n$ is given by $K_{\mathbb{F}_n} = -(2+n)F - 2E$.

2.2.4 Sarkisov links in dimension 2

Sarkisov links in the surface case have a very explicit description. In what follows, we dispose the varieties of same Picard rank at the same height. Every birational self-map of the projective plane or minimal Hirzebruch surfaces is a composition of the following elementary maps:

1. A *Sarkisov link of type I* is a commutative diagram:

$$\begin{tikzcd}
& \mathbb{F}_1 \arrow[dl, "\pi"'] \arrow[d] \\
\mathbb{P}^2 \arrow[d] & \mathbb{P}^1 \arrow[dl] \\
\mathrm{Spec}(\mathbb{C}) &
\end{tikzcd}$$

   where $\pi^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ is a point blowup.

2. A *Sarkisov link of type II* is a commutative diagram:

$$\begin{tikzcd}
\mathbb{F}_n \arrow[r, dashed, "\alpha_P"] \arrow[d] & \mathbb{F}_{n\pm1} \arrow[d] \\
\mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1
\end{tikzcd}$$

   This is an elementary transformation $\alpha_P \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n\pm1}$, by which we mean the blowup of a point $P \in \mathbb{F}_n$, followed by the contraction of the strict transform of the fiber through P (Castelnuovo Contractibility Theorem).

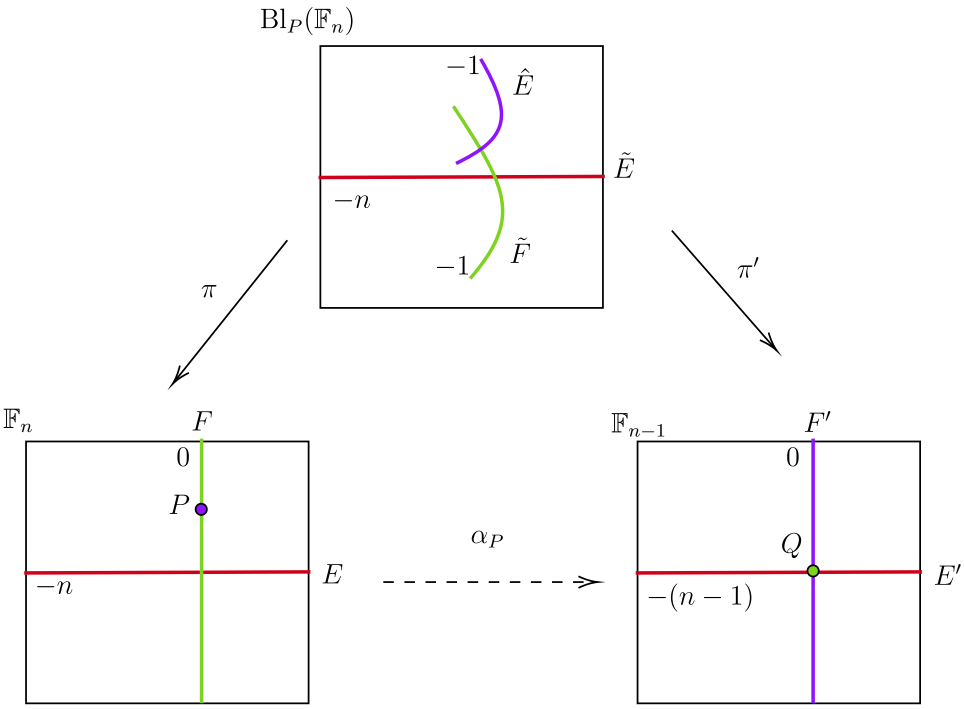


Figure 2.5: Sarkisov link of type II: elementary transformation $\alpha_P \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1}$.

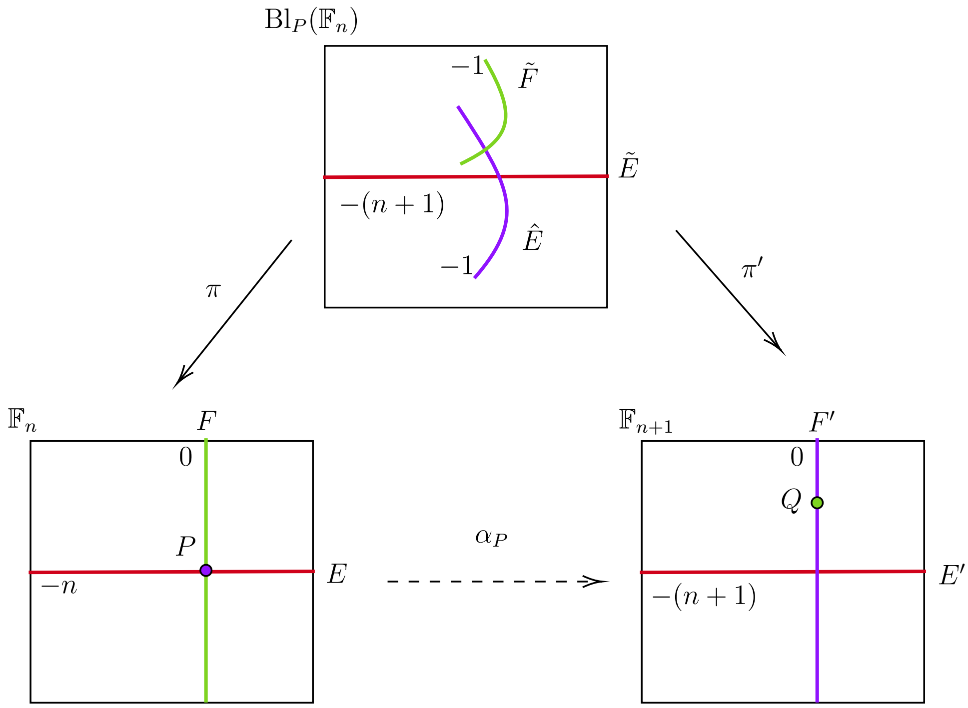


Figure 2.6: Sarkisov link of type II: elementary transformation $\alpha_P \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$.

3. A *Sarkisov link of type III* (the inverse of a link of type I) is a commutative diagram:

$$\begin{tikzcd}
\mathbb{F}_1 \arrow[d] \arrow[dr, "\pi"] & \\
\mathbb{P}^1 \arrow[dr] & \mathbb{P}^2 \arrow[d] \\
& \mathrm{Spec}(\mathbb{C})
\end{tikzcd}$$

where $\pi \colon \mathbb{F}_1 \to \mathbb{P}^2$ is the blowdown of the negative section of $\mathbb{F}_1$.

4. A *Sarkisov link of type IV* is a commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr, "\tau"] \arrow[d, "p_1"'] & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "p_2"] \\
\mathbb{P}^1 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
& \mathrm{Spec}(\mathbb{C}) &
\end{tikzcd}$$

where $\tau \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the involution which exchanges the two factors.

The proof of the Sarkisov Program in this case is based on the untwisting of the *Sarkisov degree*. The aim of this data is to measure the complexity of a birational map between the surfaces involved by comparing the linear system associated with the canonical class of the source.

In what follows, let $\mathbf{F}$ denote either $\mathbb{P}^2$ or some Hirzebruch surface $\mathbb{F}_n$, for $n \geq 0$.

Fixed a birational map

$$\begin{tikzcd} \mathbf{F} \arrow[r, dashed, "\phi"] \arrow[d] & \mathbb{P}^2 \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \text{ or } \mathbb{P}^1 & \mathrm{Spec}(\mathbb{C}) \end{tikzcd},$$

the idea is to construct a Sarkisov link

$$\begin{tikzcd} \mathbf{F} \arrow[r, dashed, "\phi_1"] \arrow[rr, dashed, bend left, "\phi"] \arrow[d] & \mathbf{F}_1 \arrow[d] & \mathbb{P}^2 \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \text{ or } \mathbb{P}^1 & \mathrm{Spec}(\mathbb{C}) \text{ or } \mathbb{P}^1 & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

such that the induced birational map $\phi \circ \phi_1^{-1}$ has a smaller Sarkisov degree than the Sarkisov degree of ϕ.

Definition 2.2.4. The *Sarkisov degree* of a rational map $\mathbf{F} \dashrightarrow \mathbb{P}^2$ given by the mobile linear system Γ is defined as

1. $\dfrac{d}{3}$ in case $\mathbf{F} = \mathbb{P}^2$ and $\Gamma \subset |dH|$, where H is a general line of $\mathbb{P}^2$; or
2. $\dfrac{b}{2}$ in case $\mathbf{F} = \mathbb{F}_n$ and $\Gamma \subset |aF + bE|$. (Notice that if $\mathbf{F} = \mathbb{P}^1 \times \mathbb{P}^1$, the Sarkisov degree is only defined in terms of a choice of one of the two projections $\mathbf{F} \to \mathbb{P}^1$.)

We denote the Sarkisov degree of a rational map ϕ by s-deg(ϕ). One can easily check that necessarily $d, b \geq 1$ and therefore the set of all possible Sarkisov degrees is contained in $\dfrac{1}{6}\mathbb{N} = \dfrac{1}{3!}\mathbb{N}$. Hence such a set is countable and discrete. The two lowest possible values for a Sarkisov degree s-deg are $\dfrac{1}{3}$ and $\dfrac{1}{2}$.

The following two results will be the key ingredients that will allow us to untwist the Sarkisov degree of a birational map $\mathbf{F} \dashrightarrow \mathbb{P}^2$. The proof exposed in [CKS, Theorem 2.24] already details the algorithm. We will follow it throughout this work.

We note that there exist slightly different algorithms employing varying terminology, but fundamentally, they are equivalent. For instance, see [Lam, Mat]. In particular, the notion of Sarkisov degree in [CKS] is the notion of quasi-effective threshold in [Cor1, Mat].

Lemma 2.2.5 (cf. [CKS] Lemma 2.26)**.** *Let $\phi \colon \mathbf{F} \dashrightarrow \mathbb{P}^2$ be a birational map given by some mobile linear system Γ, and assume that ϕ is not an isomorphism. Then Γ has a base point of multiplicity strictly greater than the Sarkisov degree of ϕ except in the following two cases:*

1. $\mathbf{F} = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$*, and* $\Gamma \subset |aF + bE|$ *for* $a < b$*; or*

2. $\mathbf{F} = \mathbb{F}_1$ *and* $\Gamma \subset |aF + bE|$ *where* $\frac{a}{3} < \frac{b}{2}$.

In case (1), Γ *has a base point of multiplicity greater than* $\frac{a}{2}$.

Lemma 2.2.6 (cf. [CKS] Lemma 2.27)**.** *Let* $\phi\colon \mathbf{F} \dashrightarrow \mathbb{P}^2$ *be a rational map given by a mobile linear system* Γ*, where* $\mathbf{F}$ *is either* $\mathbb{P}^2$ *or a rational ruled surface* $\mathbb{F}_n$*. Suppose that* Γ *has a base point* P *of multiplicity greater than the Sarkisov degree of* ϕ*.*

1. *If* $\mathbf{F} = \mathbb{P}^2$*, then the Sarkisov degree of* $\phi \circ \pi$ *is strictly less than the Sarkisov degree of* ϕ*, where* $\pi\colon \mathbb{F}_1 \to \mathbb{P}^2$ *is the blowup of* $\mathbb{P}^2$ *at* P*.*

2. *If* $\mathbf{F} = \mathbb{F}_n$*, then the Sarkisov degree of* ϕ *is equal to the Sarkisov degree of* $\phi \circ \alpha_P^{-1}$*, where* $\alpha_P\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n\pm 1}$ *is the elementary transformation described in the Subsection 2.2.4. In this case, the invariant of* $\phi \circ \alpha_P^{-1}$ *simpler than of* ϕ *is the sum of the multiplicities of the base points, including the infinitely near ones.*

Concise overview of the Sarkisov algorithm in dimension 2. Let us briefly explain how the algorithm proceeds. Consider $\phi\colon \mathbf{F} \dashrightarrow \mathbb{P}^2$ a birational map that is not a morphism. Suppose that ϕ is given by the mobile linear system Γ.

Start by verifying whether Γ has base points with a multiplicity exceeding the Sarkisov degree of ϕ. If not, by Lemma 2.2.5, $\mathbf{F}$ is either $\mathbb{F}_0$ or $\mathbb{F}_1$.

If $\mathbf{F} = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then $a < b$ by item 1 of Lemma 2.2.5. Composing ϕ with the Sarkisov link τ interchanging the two factors, the Sarkisov degree drops from $\frac{b}{2}$ to $\frac{a}{2}$, that is,

$$\text{s-deg}(\phi \circ \tau) = \frac{a}{2} < \frac{b}{2} = \text{s-deg}(\phi).$$

If $\mathbf{F} = \mathbb{F}_1$, then consider $\phi \circ \pi^{-1}$, where π is the Sarkisov link $\mathbb{F}_1 \to \mathbb{P}^2$ contracting the negative section. One has $\text{s-deg}(\phi \circ \pi^{-1}) = \frac{a}{3}$. By item 2 of Lemma 2.2.5, it follows that

$$\text{s-deg}(\phi \circ \pi^{-1}) = \frac{a}{3} < \frac{b}{2} = \text{s-deg}(\phi).$$

If Γ has a base point P with a multiplicity m greater than the Sarkisov degree of ϕ, two cases need to be taken into account.

If $\mathbf{F} = \mathbb{P}^2$, then the composition of ϕ with the Sarkisov link given by the blowup π of P is given by a linear subsystem of $|dF + (d-m)E|$. Since $m > \frac{d}{3} = \text{s-deg}(\phi)$, we have

$$\text{s-deg}(\phi \circ \pi) = \frac{d-m}{2} < \frac{d}{3} = \text{s-deg}(\phi).$$

If $\mathbf{F} = \mathbb{F}_n$, by Lemma 2.2.6, the composition of ϕ with the Sarkisov link given by the inverse of the elementary transformation $\alpha_P\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n\pm 1}$ is such that $\text{s-deg}(\phi \circ \alpha_P^{-1}) = \text{s-deg}(\phi)$. In this case, by item 2 of Lemma 2.2.6, the sum of the multiplicities of the base points of the mobile linear system Γ' describing the map $\phi \circ \alpha_P^{-1}$ is strictly less than the sum of the multiplicities of the base points of Γ.

An interesting application of the Sarkisov Program in the surface case is the classical Noether-Castelnuovo Theorem, which asserts that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by the projective linear transformations (that is, the automorphisms of $\mathbb{P}^2$) and the standard quadratic transformation $(x:y:z) \mapsto (yz:xz:xy)$. See [CKS, Theorem 2.20].

This is a consequence of the following feature: with some extra steps, if necessary, the Sarkisov Program in dimension 2 also gives a factorization of elements of $\mathrm{Bir}(\mathbb{P}^2)$ into *de Jonquières maps*. See [CKS, Theorem 2.30]. Such maps are elements of $\mathrm{Bir}(\mathbb{P}^2)$ that preserve a pencil of lines. In other words, $J \in \mathrm{Bir}(\mathbb{P}^2)$ is a *de Jonquières map* if there exist $P, Q \in \mathbb{P}^2$ such that J sends all the lines through P to lines through Q, up to a finite number. We will call such P, Q the centers of de Jonquières map J.

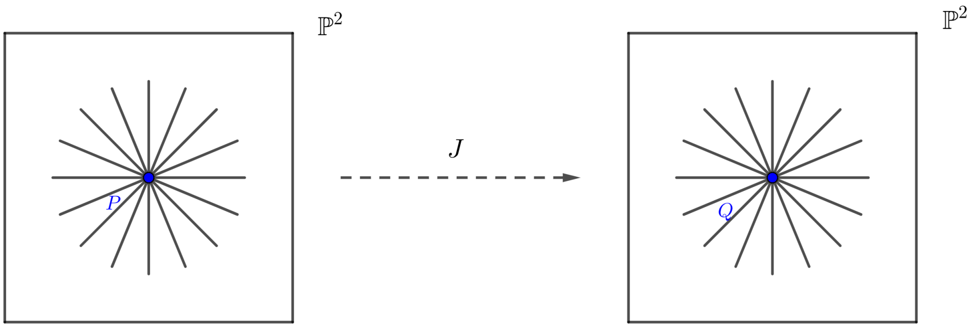


Figure 2.7: de Jonquières map.

Example 2.2.7. The standard quadratic transformation $(x:y:z) \mapsto (yz:xz:xy)$ is a de Jonquières map. Indeed, take $P = Q = (1:0:0)$. One can show the image of any line through P distinct from $\{y=0\}$ and $\{z=0\}$ is another line through P.

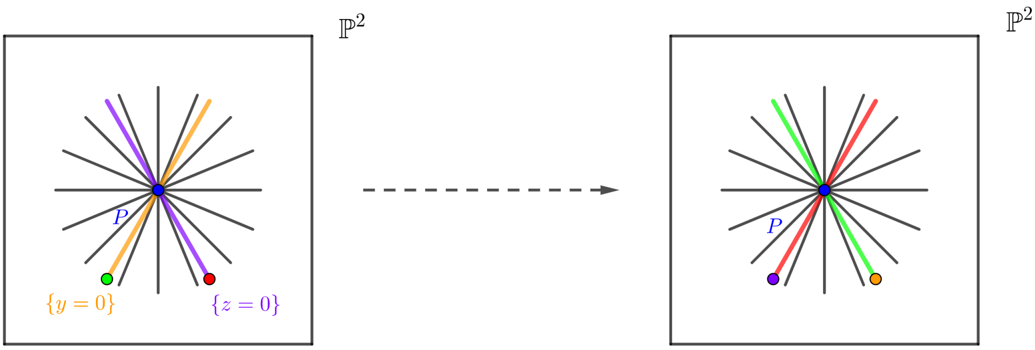


Figure 2.8: Standard quadratic transformation.

We mention two equivalent definitions of a de Jonquières map.

Definition 2.2.8. A birational map $J \in \mathrm{Bir}(\mathbb{P}^2)$ is de Jonquières if one of the following equivalent conditions holds:

1. After composing with a suitable element $L \in \mathrm{PGL}(3, \mathbb{C})$ the map $J' = L \circ J$ admits a factorization

$$\mathbb{P}^2 \overset{\pi^{-1}}{\dashrightarrow} \mathbb{F}_1 \overset{\alpha}{\dashrightarrow} \mathbb{F}_1 \xrightarrow{\pi} \mathbb{P}^2 ,$$

   where α is a *square* birational map, that is, α commutes with the structure morphism $\mathbb{F}_1 \to \mathbb{P}^1$.

2. The map J has a unique proper base point of multiplicity $n-1$ and all the remaining ones (possibly infinitely near) have multiplicity 1, where $n = \deg(J)$. The homaloidal type of J is necessarily $(n; n-1, 1^{2n-2})$ due to the Noether-Fano equations 2.2.3.

Remark 2.2.9. The decomposition obtained in the Sarkisov Program is far from being unique, that is, the algorithm is not deterministic. It depends on the choices made along the process. For instance and when that is the case, the choice of a base point of the induced birational map to $\mathbb{P}^2$ with multiplicity greater than its Sarkisov degree. We may have more than one base point realizing this maximal multiplicity, which leads to "different" Sarkisov links of type I and II.

2.3 Sarkisov algorithm in dimension 3

The notion of Sarkisov degree of a birational map between Mori fibered surfaces is equivalent to the notion of quasi-effective threshold in [Cor1, Mat].

The essence of the Sarkisov algorithm in dimension 3 is practically the same in dimension 2 with the difference that only the quasi-effective threshold will not be enough to guarantee the untwisting of the Sarkisov degree. Thus, it was in fact necessary to resort to more data in order to have this property.

Throughout this section, we will fix a birational map between threefold Mori fibered spaces:

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S & S' \end{tikzcd} .$$

Our aim is to define the Sarkisov degree s-deg(ϕ) of ϕ, which will consist of a triple of values and not a single one as in the surface case. We will briefly explain this notion, referring the reader to [Cor1, Mat] for more explicit and precise definitions.

First, we choose and fix a sufficiently large and divisible positive integer μ' and an ample divisor A' on S' such that the linear system

$$\mathcal{H}' = |-\mu' K_{X'} + (f')^* A'|$$

is very ample on X'. Indeed, this is possible due to [Ii, Theorem 7.11] or [Har, Proposition II.7.10(b)]. We remark that the latter result holds with any $\mathcal{L}$ f-ample playing the role of $\mathcal{O}_X(1)$ in its statement.

This choice is the starting point of the factorization process and $\mathcal{H}'$ remains unchanged throughout it, since the Mori fibered space X'/S' will always be the target space of the induced birational maps. The definition of Sarkisov degree depends upon this choice.

Consider a resolution of indeterminacy

$$\begin{tikzcd} & Y \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd},$$

where Y is a nonsingular projective variety and σ and σ' are birational morphisms.

The *homaloidal transform* $\mathcal{H}$ on X of $\mathcal{H}'$ is defined as

$$\mathcal{H} := \sigma_*(\sigma')^*\mathcal{H}'.$$

We point out that the homaloidal transform does not depend on the choice of Y. If $X' \xrightarrow{f'} S'$ is $\mathbb{P}^3 \to \operatorname{Spec}(\mathbb{C})$, then $\mathcal{H}$ can be seen as a general member of the linear system associated to $\phi\colon X \dashrightarrow \mathbb{P}^3$.

Now we are ready for the definition of Sarkisov degree in dimension 3.

Definition 2.3.1 (cf. [Cor1] Definition 5.1)**.** The *Sarkisov degree* of $(\mathcal{H},\ f\colon X \to S)$ is the triple (μ, c, e) where

1. μ is the *quasi-effective threshold*, defined to be the positive rational number such that

   $$\mathcal{H} + \mu K_X \equiv 0 \text{ over } S.$$

   By means of the Projection Formula, this is equivalent to

   $$(\mathcal{H} + \mu K_X) \cdot F = 0,$$

   for any curve F in a fiber of f. Since f gives a Mori fibered space structure, one has

   $$\rho(X/S) = 1 \Rightarrow \overline{NE}(X/S) \text{ is 1-dimensional.}$$

   Thus, all the curves contracted by f are numerically proportional, and hence only one of them, let us say F, will be enough to solve for μ the equation $(\mathcal{H} + \mu K_X) \cdot F = 0$. Therefore, $\mu = -\dfrac{\mathcal{H} \cdot F}{K_X \cdot F} \in \mathbb{Q}^*_+$. Indeed, it is well-defined since $-K_X \cdot F > 0$ because $-K_X$ is f-ample, and $\mathcal{H} \cdot F > 0$ because $\mathcal{H}$ is a homaloidal transform.

   Notice that in fact μ only depends on μ'.

   As well as in dimension 2, the quasi-effective thresholds form a countable and discrete set in the whole set of rational numbers. The verification of this fact is much more subtle in this case. It involves certain boundedness results due to Kawamata [Kaw2] with respect to $\mathbb{Q}$-Fano varieties with Picard number one.

2. c is the *canonical threshold* of the pair $(X, \mathcal{H})$ if $\mathcal{H}$ is not base point free and ∞, otherwise. The former is by definition

$$c := \max\{t \in \mathbb{Q}_+ \mid (X, t\mathcal{H}) \text{ has canonical singularities}\}.$$

3. e is the number of *crepant exceptional divisors* with respect to the pair $(X, c\mathcal{H})$ if $\mathcal{H}$ is not base point free. In this case, more precisely

$$e := \sharp\{E \mid E \text{ is exceptional over } X \text{ and } a(E, X, c\mathcal{H}) = 0\}.$$

   If $\mathcal{H}$ is base point free, then e is not defined. We will write $e = *$ when such a situation occurs.

For the notions of *canonical singularities* and *discrepancies* $a(E, X, c\mathcal{H})$, we refer the reader to the next Chapter 3.

On the set of triples (μ, c, e) we introduce a *partial ordering* as follows:

$$(\mu, c, e) > (\mu_1, c_1, e_1) \text{ if either}$$

1. $\mu > \mu_1$, or
2. $\mu = \mu_1$ and $c < c_1$ (no mistype here), or
3. $\mu = \mu_1, c = c_1$ and $e > e_1$.

For the sake of completeness, we will mention the *Noether-Fano-Iskovskikh inequalities or criterion* that will allow us to decide whether ϕ is an isomorphism of Mori fibered spaces in terms of s-deg(ϕ) and K_X.

Theorem 2.3.2 (cf. [Cor1] Noether-Fano-Iskovskikh inequalities or criterion, Theorem 4.2)**.** *Keeping the notation of this section, one has:*

1. *$\mu \geq \mu'$, and equality implies that ϕ induces a rational map $S \dashrightarrow S'$.*

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, dashed] & S' \end{tikzcd}$$

2. *If $K_X + \dfrac{1}{\mu}\mathcal{H}$ is canonical and nef, ϕ is an isomorphism, and it induces an isomorphism $S \simeq S'$. In particular $\mu = \mu'$.*

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "\simeq"] & S' \end{tikzcd}$$

We have the following straightforward corollary from the Noether-Fano-Iskovskikh inequalities:

Corollary 2.3.3. *If ϕ is not an isomorphism of Mori fibered spaces, then either*

1. $\left(K_X, \dfrac{1}{\mu}\mathcal{H}\right)$ *is not canonical, or*

2. $\left(K_X, \dfrac{1}{\mu}\mathcal{H}\right)$ *is canonical, but not nef.*

Concise overview of the Sarkisov algorithm in dimension 3. Let us briefly explain how to construct a Sarkisov link ϕ_1 that untwists the Sarkisov degree of ϕ. We omit the rather delicate verification of such a phenomenon. We refer the reader to [Cor1, Mat] for more thorough details.

Two distinct approaches are employed for the construction of ϕ_1, depending on whether we are in Case 1 or 2 of Corollary 2.3.3.

Case 1: In this case, necessarily we have $c < \dfrac{1}{\mu}$. There exists then an *extremal blowup* [Cor1, Proposition-Definition 2.10] $\sigma\colon Z \to X$ with

$$K_Z + c\mathcal{H}_Z = \sigma^*(K_X + c\mathcal{H}),$$

where $\mathcal{H}_Z$ denotes the strict transform of $\mathcal{H}$ on Z.

The next step is to run the $(K_Z + c\mathcal{H}_Z)$-MMP over S. It is a special kind of MMP, which is called *2-ray game*. See [Cor2, Section 2.2] or [Mat, Chapter 13]. The winner of this game leads to an untwisting Sarkisov link of type I or II.

Case 2: In this case, necessarily we have $c \geq \dfrac{1}{\mu}$. One constructs a suitable contraction $S \to T$, where T is a normal projective variety. Running the $\left(K_Z + \dfrac{1}{\mu}\mathcal{H}_Z\right)$-MMP over T, we obtain an untwisting Sarkisov link of type III or IV.

Remark 2.3.4. The complexity of the Sarkisov algorithm increases in dimension 3 compared to dimension 2 due to the need to consider additional data and properties. Indeed, the birational geometry of threefolds is harder than it is for surfaces. We refer the reader to [Mat, Chapter 1], which traces a Sarkisov algorithm in dimension 2 analogous to the one explained in this section in dimension 3 by using an MMP point of view. This approach is not taken in the Sarkisov algorithm for dimension 2 as described in [CKS].

Chapter 3

Log Calabi-Yau geometry and the volume preserving Sarkisov Program

In this chapter, we give an overview of the concepts and ideas involving the geometry of log Calabi-Yau pairs and the volume preserving Sarkisov Program.

3.1 Log Calabi-Yau geometry

Definition 3.1.1. Let X be a normal projective variety. A *divisor over* X is a prime divisor E on Y, where Y is any normal projective variety admitting a birational morphism f to X. The *center* of a divisor E over X is the closure of the set-theoretic image $f(E) \subset X$. We denote it by $z_E X$. A divisor is said to be *exceptional over* X if its center on X has codimension at least two.

Definition 3.1.2. Let (X, D) be a pair consisting of a normal projective variety X and a $\mathbb{Q}$-Weil divisor $D = \sum d_i D_i$. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \to X$ be a birational morphism from a normal variety Y with f-exceptional divisors E_i, and write

$$K_Y + f_*^{-1}D \equiv_{\mathbb{Q}} f^*(K_X + D) + \sum_{E_i \ f\text{-exceptional}} a(E_i, X, D)E_i.$$

We recall that the coefficients $a(E_i, X, D)$ are rational numbers which only depend on the discrete valuations ν_{E_i} on $K(X)$ corresponding to each E_i. We call them *discrepancies* of E_i with respect to the pair (X, D). We say that the pair (X, D) is[1]

$$\left.\begin{array}{r} \textit{terminal} \\ \textit{canonical} \\ \textit{Kawamata log terminal (klt)} \\ \textit{purely log terminal (plt)} \\ \textit{log canonical (lc)} \end{array}\right\} \text{ if } a(E, X, D) \begin{cases} > 0, \\ \geq 0, \\ > -1 \text{ and } d_j < 1 \text{ for any } j\text{, that is, } \lfloor D \rfloor = 0, \\ > -1, \\ \geq -1, \end{cases}$$

[1] In accordance with common usage, we will refer to a pair as having terminal singularities if it is terminal, for example.

for any E exceptional divisor over X.

Sometimes we will be identifying a divisor over a variety with its corresponding valuation on its function field. This is well-posed because Zariski showed that a valuation becomes a divisor after a finite number of blowups. See [KM, Lemma 2.45].

A *log canonical center* of (X, D) is the center on X of a divisor E over X (not necessarily exceptional) with discrepancy $a(E, X, D) = -1$ (the notion of discrepancy extends to non-exceptional divisors. See [KM, Definition 2.5]). We will also introduce a sixth class of singularities denominated *divisorial log terminal (dlt)*.

Definition 3.1.3. Consider a pair as in the previous definition and assume that $0 \leq d_i \leq 1$. Such a pair is *dlt* if and only if there exists a closed subset $Z \subset X$ with the following properties:

1. $X \setminus Z$ is nonsingular and $D|_{X \setminus Z}$ is a simple normal crossing (SNC for short) divisor.

2. If $f\colon Y \to X$ is birational and $E \subset Y$ is an irreducible divisor such that $z_E X \subset Z$, then $a(E, X, D) > -1$.

Example 3.1.4. If X is a nonsingular projective variety and D is a reduced SNC divisor on X, then the pair (X, D) is dlt.

Remark 3.1.5. Given a lc pair (X, D), the divisor D is usually called a *boundary divisor* in the literature. The reason behind this is that the ambient variety X can be seen as a compactification of $U = X \setminus D$. Another common terminology in the literature is to refer to the divisor $K_X + D$ as *log canonical divisor*.

Definition 3.1.6. A *log Calabi-Yau pair* is a log canonical pair (X, D) consisting of a normal projective variety X and a reduced Weil divisor D on X such that $K_X + D \sim 0$. This condition implies the existence of a top degree rational differential form $\omega = \omega_{X,D} \in \Omega^n_X$, unique up to nonzero scaling, such that $D + \operatorname{div}(\omega) = 0$. By abuse of language, we call this differential the *volume form*.

From now on, we will call a log Calabi-Yau pair simply a Calabi-Yau pair. Sometimes in a more general context, it is admitted that D is a $\mathbb{Q}$-divisor and that $K_X + D \sim_{\mathbb{Q}} 0$, but we will not need this generality here. Since $K_X + D \sim 0$, we have that $K_X + D$ is readily Cartier, and hence all the discrepancies with respect to the pair (X, D) are integer numbers.

Let us give now some examples of Calabi-Yau pairs.

Example 3.1.7. Let X be a Fano or weak Fano variety (K_X nef and big in the second case). The linear system $|-K_X|$ has plenty of sections. For $D \in |-K_X|$ reduced, one has (X, D) a Calabi-Yau pair.

Example 3.1.8. For $X = \mathbb{P}^2$, we have four possibilities for the boundary divisor.

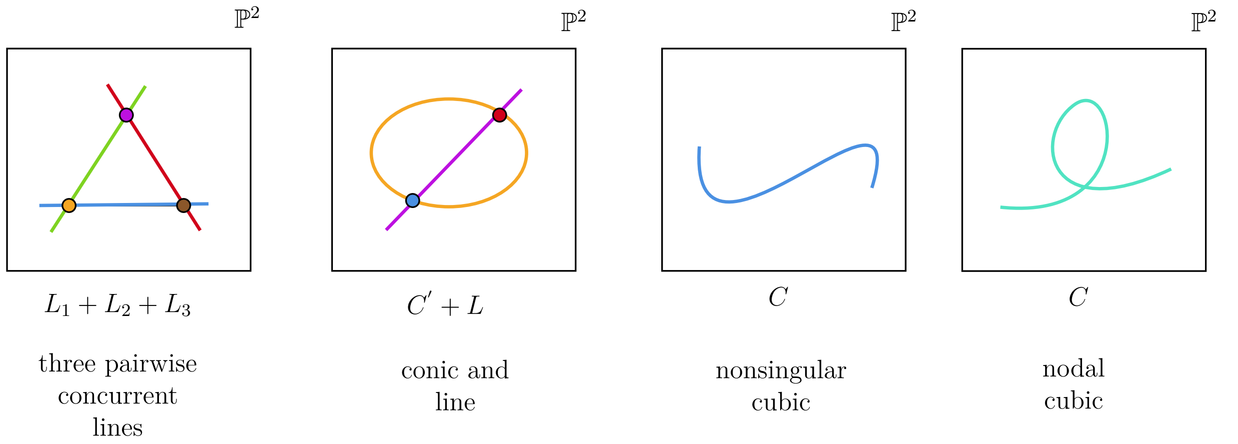


Figure 3.1: Calabi-Yau pair structures on $\mathbb{P}^2$.

The first three possibilities give dlt Calabi-Yau pairs and the fourth one a lc Calabi-Yau pair.

Recall that the definition of Calabi-Yau pair requires log-canonicity. The boundary divisor cannot be realized by a cuspidal cubic C because the resultant pair is no longer log canonical. In fact, one can check that a log resolution of the singularity will create an exceptional divisor E_3 with discrepancy -2 with respect to the pair.

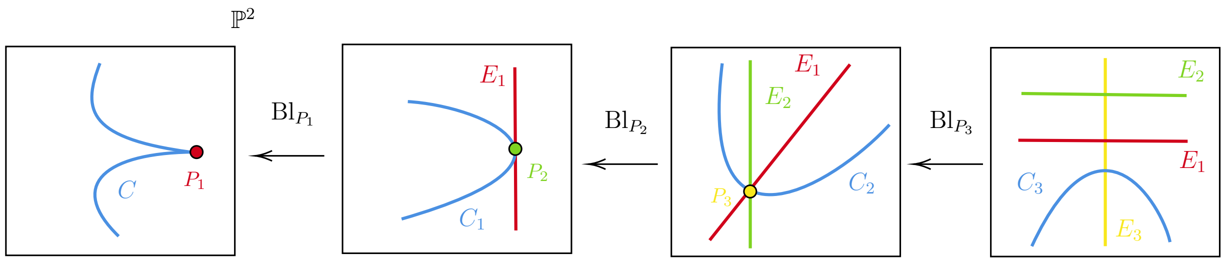


Figure 3.2: Log resolution of the cuspidal cubic.

Example 3.1.9. For $X = \mathbb{F}_n$, consider D the cycle of rational curves consisting of two fibers of the structure morphism in addition to the negative and positive sections.

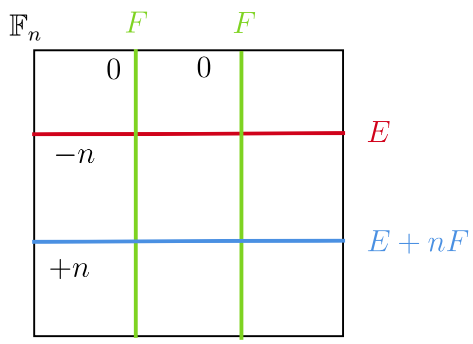


Figure 3.3: Calabi-Yau pair structure on $\mathbb{F}_n$.

Example 3.1.10. Let X be a nonsingular toric variety. Take $D = \sum_\rho D_\rho$, where the sum runs over the torus-invariant divisors. One can check that (X, D) is a dlt Calabi-Yau pair.

We now introduce some classes of pairs taking into account the singularities of the ambient varieties and the divisors.

Definition 3.1.11. We say that a pair (X, D) is (t, c), respectively, (t, lc), if X has terminal singularities and the pair (X, D) has canonical, respectively log canonical singularities. We say that a pair (X, D) is $\mathbb{Q}$-factorial if X is $\mathbb{Q}$-factorial.

If (X, D) is (t,lc), then $a(E, X, D_X) \leq 0$ implies $a(E, X, D_X) = -1$ or 0.

Proposition–Definition 3.1.12 (cf. [KM] Lemma 2.30)**.** *A proper birational morphism $f\colon (Z, D_Z) \to (X, D_X)$ is called crepant if $f_* D_Z = D_X$ and $f^*(K_X + D_X) \sim K_Z + D_Z$. The term "crepant" (coined by Reid) refers to the fact that every f-exceptional divisor E has discrepancy $a(E, X, D_X) = 0$. Furthermore, for every divisor E over X and Z, one has $a(E, Z, D_Z) = a(E, X, D_X)$.*

Definition 3.1.13. A birational map of pairs $\phi\colon (X, D_X) \dashrightarrow (Y, D_Y)$ is called *crepant* if it admits a resolution

$$\begin{tikzcd}
 & (Z, D_Z) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
(X, D_X) \arrow[rr, dashed, "\phi"] & & (Y, D_Y)
\end{tikzcd}$$

in such a way that p and q are crepant birational morphisms.

This definition is equivalent to asking that $a(E, X, D_X) = a(E, Y, D_Y)$ for every valuation E of $K(X) \simeq K(Y)$ as in item 2 in Proposition 3.1.14.

For Calabi-Yau pairs, the notion of *crepant birational equivalence* becomes *volume preserving equivalence*, since $\phi^* \omega_{Y, D_Y} = \lambda \omega_{X, D_X}$, for some $\lambda \in \mathbb{C}^*$. In this case, we call such ϕ a *volume preserving* map.

As a consequence of these equivalences, we have the following:

Proposition 3.1.14 (cf. [CK] Remark 1.7)**.** *Let (X, D_X) and (Y, D_Y) be Calabi-Yau pairs and $\phi\colon X \dashrightarrow Y$ an arbitrary birational map. The following conditions are equivalent:*

1. *The map $\phi\colon (X, D_X) \dashrightarrow (Y, D_Y)$ is volume preserving.*

2. *For all geometric valuations E with center on both X and Y, the discrepancies of E with respect to the pairs (X, D_X) and (Y, D_Y) are equal: $a(E, X, D_X) = a(E, Y, D_Y)$.*

3. *Let*

$$\begin{tikzcd}
 & (Z, D_Z) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
(X, D_X) \arrow[rr, dashed, "\phi"] & & (Y, D_Y)
\end{tikzcd}$$

   be a common log resolution of the pairs (X, D_X) and (Y, D_Y). The birational map ϕ induces an identification $\phi_*\colon \Omega^n_X \xrightarrow{\sim} \Omega^n_Y$, where $n = \dim(X) = \dim(Y)$. By abuse of notation, we write

$$p^*(K_X + D_X) = q^*(K_Y + D_Y)$$

to mean that for all $\omega \in \Omega^n_X$, we have

$$p^*(D_X + \operatorname{div}(\omega)) = q^*(D_Y + \operatorname{div}(\phi_*(\omega))).$$

The condition is: for some (or equivalently for any) common log resolution as above, we have

$$p^*(K_X + D_X) = q^*(K_Y + D_Y).$$

Remark 3.1.15. As an immediate consequence of the definition, a composition of volume preserving maps is volume preserving. So the set of volume preserving self-maps of a given Calabi-Yau pair (X, D) forms a group, denoted by $\mathrm{Bir}^{\mathrm{vp}}(X, D)$. In particular, this group is a subgroup of $\mathrm{Bir}(X)$.

Definition 3.1.16. A *Mori fibered Calabi-Yau pair* is a $\mathbb{Q}$-factorial (t,lc) Calabi-Yau pair (X, D) with a Mori fibered space structure on X. We denote the Calabi-Yau pair (X, D) together with a Mori fibered structure by $(X, D)/S$.

If (Z, D_Z) and (X, D_X) are (t,lc) Calabi-Yau pairs, then a Mori divisorial contraction $f\colon Z \to X$ is volume preserving as a map of Calabi-Yau pairs if and only if $K_Z + D_Z = f^*(K_X + D_X)$, in the sense of Proposition 3.1.12. In this case, we have $D_X = f_* D_Z$.

Let (Z, D_Z) and $(Z', D_{Z'})$ be (t,lc) Calabi-Yau pairs, and $\varphi\colon Z \dashrightarrow Z'$ a Mori flip, flop or antiflip. Then $\varphi\colon (Z, D_Z) \dashrightarrow (Z', D_{Z'})$ is volume preserving if and only if $D_{Z'} = \varphi_* D_Z$. Indeed, φ is an isomorphism in codimension 1, that is, it does not contract divisors.

Notice that all the boundary divisors all well-determined in these instances. More generally, we have the following definition:

Definition 3.1.17. A *volume preserving Sarkisov link* is a Sarkisov link as previously described with the following additional data and property: there exist divisors D_X on X, $D_{X'}$ on X', D_Z on Z, and $D_{Z'}$ on Z', making $(X, D_X), (X', D_{X'}), (Z, D_Z)$ and $(Z', D_{Z'})$ (t,lc) Calabi-Yau pairs, and all the divisorial contractions, Mori flips, flops and antiflips that constitute the Sarkisov link are volume preserving for these Calabi-Yau pairs.

Remark 3.1.18. A Sarkisov link of any type is volume preserving if and only if the corresponding divisorial contractions and extractions, and isomorphisms in codimension 1 involved (when it is the case) are volume preserving.

We end this section stating the result of Corti & Kaloghiros, which holds in all dimensions.

Theorem 3.1.19 (cf. [CK] Theorem 1.1)**.** *Any volume preserving map between Mori fibered Calabi-Yau pairs is a composition of volume preserving Sarkisov links.*

$$\begin{tikzcd}[column sep=small]
(X,D_X)=(X_0,D_0) \arrow[r, dashed, "\phi_1"] \arrow[d] \arrow[rrrrr, dashed, bend left=10, "\phi"] & (X_1,D_1) \arrow[r, dashed, "\phi_2"] \arrow[d] & \cdots \arrow[r, dashed, "\phi_{m-1}"] & (X_{m-1},D_{m-1}) \arrow[r, dashed, "\phi_m"] \arrow[d] & (X_m,D_m)=(Y,D_Y) \arrow[d] \\
S=Y_0 & Y_1 & & Y_{m-1} & Y_m=T
\end{tikzcd}$$

Here ϕ stands for a volume preserving map between the Mori fibered Calabi-Yau pairs $(X,D_X)/S$ and $(Y,D_Y)/T$, and ϕ_i for a volume preserving Sarkisov link in its decomposition.

3.1.1 Framework of birational geometry of Calabi-Yau pairs

In this subsection, we will list a collection of results from [ACM] which will frequently be used in this work. These results constitute a framework for the birational geometry of Calabi-Yau pairs and we will be able to see their manifestations in many parts of this work.

Given a Calabi-Yau pair (X,D), the work of Corti & Kaloghiros [CK] provides an effective tool to investigate $\mathrm{Bir}^{\mathrm{vp}}(X,D)$ with the additional structure of Mori fibered space on (X,D), as exposed in [ACM].

As we have mentioned before in the Introduction, under restrictions on the singularities of $(\mathbb{P}^n,D)$, the decomposition group of the hypersurface D, denoted by $\mathrm{Dec}(D)$ or simply $\mathrm{Bir}(\mathbb{P}^n,D)$, coincides with $\mathrm{Bir}^{\mathrm{vp}}(\mathbb{P}^n,D)$. This is the content of the following result:

Proposition 3.1.20 (cf. [ACM] Proposition 2.6)**.** *Let (X,D_X) and (Y,D_Y) be (t,c) Calabi-Yau pairs, and $f\colon X \dashrightarrow Y$ an arbitrary birational map. Then $f\colon (X,D_X)\dashrightarrow (Y,D_Y)$ is volume preserving if and only if $f_*D_X=D_Y$ and $f_*^{-1}D_Y=D_X$. (This condition is equivalent to asking that the restriction of f to each component of D_X is a birational map to a component of D_Y, and the same for f^{-1}).*

In particular, if (X,D) is a (t,c) Calabi-Yau pair with D irreducible, then $\mathrm{Bir}^{\mathrm{vp}}(X,D)$ coincides with the group of birational self-maps $f\colon X\dashrightarrow X$ that restrict to birational self-maps $f|_D\colon D\dashrightarrow D$.

Remark 3.1.21. For various interesting cases, when considering (X,D) as (t,c), it follows that D is irreducible. This holds, for example, when X is a Fano variety with $\dim X>1$. In fact, under the assumption of (X,D) being (t,c), D becomes normal, and hence, the irreducibility of D is equivalent to its connectedness. A key case when this fails is for $X=\mathbb{P}^1, D=\{0,\infty\}$.

Remark 3.1.22 (Canonicity is necessary)**.** Proposition 3.1.20 does not hold in general for volume preserving maps between (t,lc) Calabi-Yau pairs. Consider the divisor $D=L_1+L_2+L_3$ on $\mathbb{P}^2$ given by the sum of the three coordinate lines. By taking a log resolution of $(\mathbb{P}^2,D)$ given by the blowup of the three coordinate points, one can check that $(\mathbb{P}^2,D)$ is a strict log Calabi-Yau pair. Indeed, the discrepancy of the three corresponding exceptional divisors with respect to $(\mathbb{P}^2,D)$ is equal to -1. This log resolution also shows that the standard quadratic transformation $(x : y : z) \mapsto (yz : xz : xy)$ is volume preserving. It is well known that this birational map is an involution and contracts the components of D to the three coordinate points of $\mathbb{P}^2$.

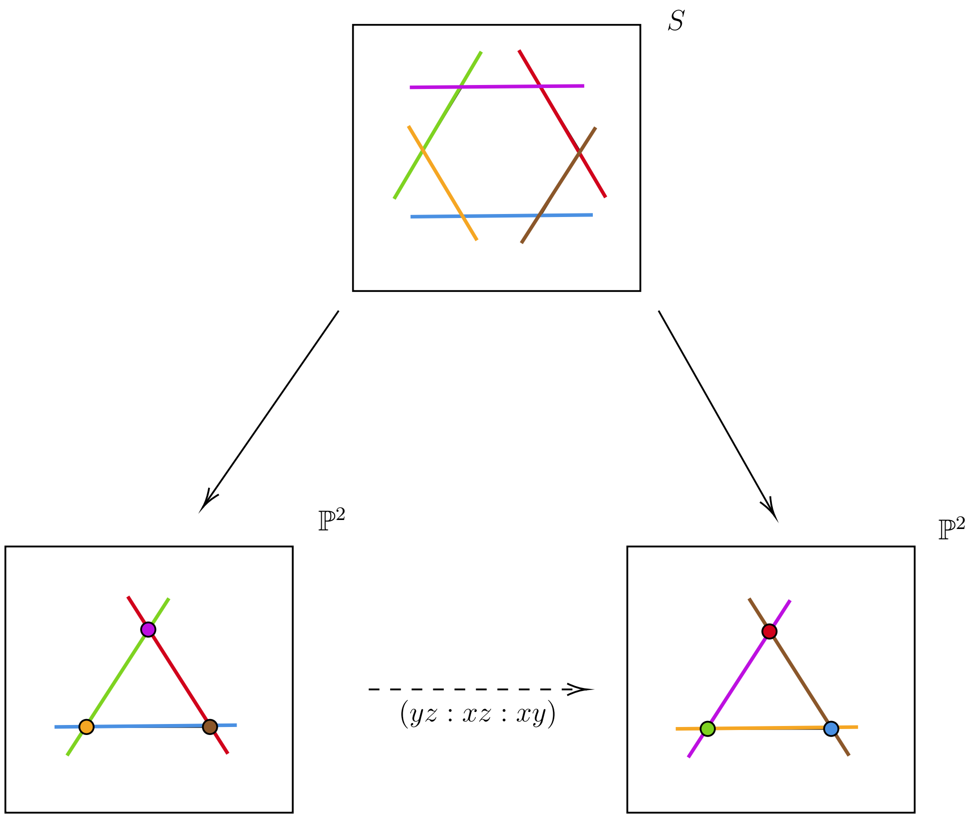


Figure 3.4: Log resolution of the standard quadratic transformation.

The following result ensures that canonicity is preserved when we run a volume preserving Sarkisov Program for canonical Mori fibered Calabi-Yau pairs.

Lemma 3.1.23 (cf. [ACM] Lemma 2.8)**.** *Let $f\colon (X, D_X) \dashrightarrow (Y, D_Y)$ be a volume preserving birational map between (t,lc) Calabi-Yau pairs. Then (X, D_X) is canonical if and only if so is (Y, D_Y).*

In other words, volume preserving maps do not create worse singularities in the canonical case. We point out that the same does not happen in the terminal case. For instance, we have the following:

Example 3.1.24. Inspired by [Rei3, Example 1, Section 1.9] , let $D \subset \mathbb{P}^{n+1}$ be an irreducible hypersurface of degree $n + 2$ having a single isolated singularity at P of multiplicity n. Moreover, suppose that the projectivized tangent cone at P is a nonsingular hypersurface $F \subset \mathbb{P}^n$ of degree n.

Consider $\pi\colon X \to \mathbb{P}^{n+1}$ the blowup at P. By the Adjunction Formula, one can check that π is a resolution of singularities of D and also a volume preserving (crepant) map between the Calabi-Yau pairs (X, D_X) and $(\mathbb{P}^{n+1}, D)$, where $D_X := \pi_*^{-1}D$. Notice that the pair (X, D_X) is terminal whereas the pair $(\mathbb{P}^{n+1}, D)$ has a strict canonical singularity at P.

Proposition 3.1.25 (cf. [ACM] Proposition 3.9)**.** *Let (X, D_X) be a (t,lc) Calabi-Yau pair, and $f\colon (Y, D_Y) \to (X, D_X)$ a volume preserving divisorial contraction with center $Z \subset X$. Then $Z \subset D_X$.*

Suppose moreover that (X, D_X) is canonical, and that D_X is terminal at the generic point of Z. Then $\operatorname{codim}_X Z = 2$, and D_Y is the strict transform of D_X in Y.

We can directly check that, in particular, such a proposition holds for the case of a nonsingular hypersurface $D \subset \mathbb{P}^{n+1}$ of degree $n+2$ and $\pi\colon X \to \mathbb{P}^{n+1}$ a volume preserving blowup along a nonsingular center Z.

Set $c := \operatorname{codim}_{\mathbb{P}^{n+1}} Z$ and $E := \operatorname{Exc}(\pi)$. Since D is nonsingular, its multiplicity along Z is given by

$$m = \begin{cases} 0, & \text{if } Z \not\subset D \\ 1, & \text{if } Z \subset D \end{cases}.$$

By the Adjunction Formula, we can write $K_X = \pi^* K_{\mathbb{P}^{n+1}} + (c-1)E$. Moreover, $\tilde{D} = \pi^* D - mE$. Summing up we

$$K_X + \tilde{D} = \pi^*(K_{\mathbb{P}^{n+1}} + D) + (c - 1 - m)E.$$

Since $\pi_* \tilde{D} = D$, by Proposition 3.1.12, π is volume preserving if and only $c - 1 - m = 0$. Since $c \geq 0$ and $m \in \{0, 1\}$, that happens if and only if $c = 2$ and $m = 1$.

3.2 The coregularity

In our context of log Calabi-Yau geometry, we can always find a *dlt modification* for any Calabi-Yau pair [CK, Theorem 1.7]. More precisely, given a Calabi-Yau pair (X, D_X), there always exists a volume preserving morphism $f\colon (Y, D_Y) \to (X, D_X)$ where (Y, D_Y) is a $\mathbb{Q}$-factorial dlt Calabi-Yau pair and Y has at worst terminal singularities.

For dlt Calabi-Yau pairs, the log canonical centers are well understood via the *strata* of the boundary divisor, which is the collection of irreducible components of all possible finite intersections between the divisors appearing in its support. An element of the strata is called *stratum*.

Theorem 3.2.1 (cf. [Kol] Theorem 4.16)**.** *Let (X, D) be a dlt pair (not necessarily Calabi-Yau) and consider $D_1, \dots, D_r$ the irreducible divisors appearing in D with coefficient 1 (in the Calabi-Yau case this is automatically satisfied by all components of D). Set $I := \{1, \dots, r\}$.*

1. *The log canonical centers of (X, D) are exactly the irreducible components of $D_J := \bigcap_{j \in J} D_j$ for any subset $J \subset I$. For $J = \emptyset$, we define $D_\emptyset := X$.*

2. *Every log canonical center is normal and has pure codimension $|J|$.*

Since volume preserving maps between Calabi-Yau pairs preserve discrepancies, any concept expressed in terms of them will be a volume preserving invariant. In particular, these maps send log canonical centers onto log canonical centers. Using this fact, it is possible to show that the dimension of a minimal log canonical center (with respect to the inclusion) on a dlt modification of a Calabi-Yau pair (X, D_X) is a volume preserving invariant.

Definition 3.2.2. The *coregularity* $\operatorname{coreg}(X, D_X)$ is defined to be the dimension of a minimal log canonical center in a dlt modification $f\colon (Y, D_Y) \to (X, D_X)$. A Calabi-Yau pair (X, D_X) is called *maximal* if $\operatorname{coreg}(X, D_X) = 0$.

Thus we have $0 \leq \operatorname{coreg}(X, D_X) \leq \dim X$ for any Calabi-Yau pair (X, D_X). The pairs with maximum coregularity $\operatorname{coreg}(X, D_X) = \dim X$ are necessarily of the form $(X, 0)$. By definition of Calabi-Yau pair, this implies that X is a Calabi-Yau variety.

It is also possible to define the notion of coregularity in terms of the dimension of the ambient variety and the corresponding *dual complex* $\mathcal{D}(X, D_X)$ of the pair (X, D_X). In a broad sense, this object is a CW-complex that encodes the geometry of the log canonical centers. We have $\operatorname{coreg}(X, D_X) = \dim X - \dim \mathcal{D}(X, D_X) - 1$. Notice that the case of minimum coregularity corresponds to the case where $\dim \mathcal{D}(X, D_X)$ is maximal, which is one justification for the terminology "maximal pair". We refer the reader to [Duc, KX, Mor1, Mor2] for more details.

One can think Calabi-Yau pairs as generalizations of Calabi-Yau varieties. So the coregularity becomes a coarse measure of how far the corresponding *crepant log structure* (see [Kol, Section 4.4]) on a Calabi-Yau pair is from making the ambient variety Calabi-Yau.

Chapter 4

The 2-dimensional case

In this chapter, we study the decomposition group of a nonsingular plane cubic under the light of the log Calabi-Yau geometry.

4.1 Mori fibered Calabi-Yau pairs in dimension 2

In Chapter 3 we already gave some examples of Mori fibered Calabi-Yau pairs in dimension 2. It turns out that not all of them admit an irreducible boundary divisor.

Lemma 4.1.1. *The only (rational) Mori fibered spaces in dimension 2 that admit an irreducible divisor D such that (S, D) is a Calabi-Yau pair are:*

$$\mathbb{P}^2/\operatorname{Spec}(\mathbb{C}) \text{ and } \mathbb{F}_n/\mathbb{P}^1 \text{ for } n \in \{0, 1, 2\}.$$

Moreover, if D is nonsingular, then D is isomorphic to an elliptic curve.

Proof. The case $S = \mathbb{P}^2$ is simply Example 3.1.8. For the remaining cases $\mathbb{F}_n$, this result follows from intersecting D with the negative section.

If $(\mathbb{F}_n, D)$ is a Calabi-Yau pair, then $D \sim (n+2)F + 2E$ because $K_{\mathbb{F}_n} = -(n+2)F - 2E$. Since $\{F, E\}$ is a $\mathbb{Z}$-basis for $\operatorname{Pic}(\mathbb{F}_n)$, it follows that $D \neq E$.

We must then have that $D \cdot E \geq 0$ by the properties of the intersection number. Observe that

$$D \cdot E = ((n+2)F + 2E) \cdot E = (n+2) - 2n = -n + 2 \geq 0 \Leftrightarrow n \leq 2.$$

We have just shown that if $(\mathbb{F}_n, D)$ is a Calabi-Yau pair, then $n \leq 2$. For the converse, it is easy to construct examples of such pairs by applying conveniently the Lemmas 4.2.3 & 4.2.4 to the Calabi-Yau pair $(\mathbb{P}^2, D)$, where D is a nonsingular cubic.

Besides being irreducible, assume D is nonsingular. By the Adjunction Formula for curves, we have that

$$\begin{aligned} D^2 + D \cdot K_S = 2g - 2 &\Rightarrow (-K_S)^2 - K_S \cdot K_S = 2g - 2 \\ &\Rightarrow 0 = 2g - 2 \end{aligned}$$

$$\Rightarrow g = 1 \qquad .$$

Therefore, D is an elliptic curve and the result then follows. □

Remark 4.1.2. The last part is a particular case of the following more general situation: for a Calabi-Yau pair (X, D) with boundary divisor D nonsingular, since $K_X + D \sim 0$, by the Adjunction Formula we have that $K_D \sim 0$. Thus, D has trivial canonical class.

4.1.1 A geometric feature of $\mathbb{F}_1$ and the geometry of Calabi-Yau pairs $(\mathbb{F}_n, C)$

The surface $\mathbb{F}_1$ is isomorphic to the blowup of $\mathbb{P}^2$ at a point. Let $\pi \colon \mathbb{F}_1 \to \mathbb{P}^2$ be such blowup at P. As Mori fibered space, the surface $\mathbb{F}_1$ admits a structure morphism f to $\mathbb{P}^1$, whose fibers are the strict transforms of all lines through P.

Consider $C \subset \mathbb{P}^2$ a nonsingular curve of degree d passing through P. Let L be a line passing through P in $\mathbb{P}^2$. We have $\tilde{L} = \pi^* L - E$ and

$$\begin{aligned}
\tilde{C} \cdot \tilde{L} &= (\pi^* C - E) \cdot (\pi^* L - E) \\
&= \pi^* C \cdot \pi^* L - \pi^* C \cdot E - \pi^* L \cdot E + E^2 \\
&= C \cdot L + E^2 \\
&= d - 1 \qquad .
\end{aligned}$$

This implies that the strict transforms of the lines through P intersect $\tilde{C}$ with multiplicity $d-1$. We point out that the fibers of f are not all transverse to $\tilde{C}$. By Bézout Theorem, a line passing through P may intersect C at another point Q with multiplicity bigger than 1, that is, such line is tangent to C at Q. In this case, the fiber of f through $\pi^{-1}(Q)$ is tangent to $\tilde{C}$ at this point.

The tangent line to C at P may intersect C at P with multiplicity d. In this case, the fiber through $E \cap \tilde{C}$ is tangent to $\tilde{C}$ there if $d \geq 3$.

These geometric features and properties of the blowup induce a $(d-1:1)$ covering morphism from $\tilde{C}$ to $\mathbb{P}^1$ given by the restriction of f to $\tilde{C}$. The ramification points of this morphism correspond to the points of $\tilde{C}$ such that the fibers of f through them are tangent to $\tilde{C}$. By the Riemann-Hurwitz Formula, this morphism has at most $(d-1)d-2$ ramification points.

Indeed, since the (geometric) genus is a birational invariant, we get

$$g(\tilde{C}) = g(C) = \frac{(d-1)(d-2)}{2}.$$

The last equality is due to [Ful1, Proposition 5, Chapter 8]. We have that

$$2g(\tilde{C}) - 2 = \deg(f|_{\tilde{C}}) \cdot (2g(\mathbb{P}^1) - 2) + \deg(R)$$

$$\Rightarrow \deg(R) = 2 \cdot \frac{(d-1)(d-2)}{2} - 2 - (d-1)(2 \cdot 0 - 2)$$

$$= (d-1)(d-2) - 2 + 2(d-1)$$
$$= d(d-1) - 2,$$

where $\deg(R)$ is the degree of the ramification divisor R.

The following picture illustrates the situation for $d = 3$:

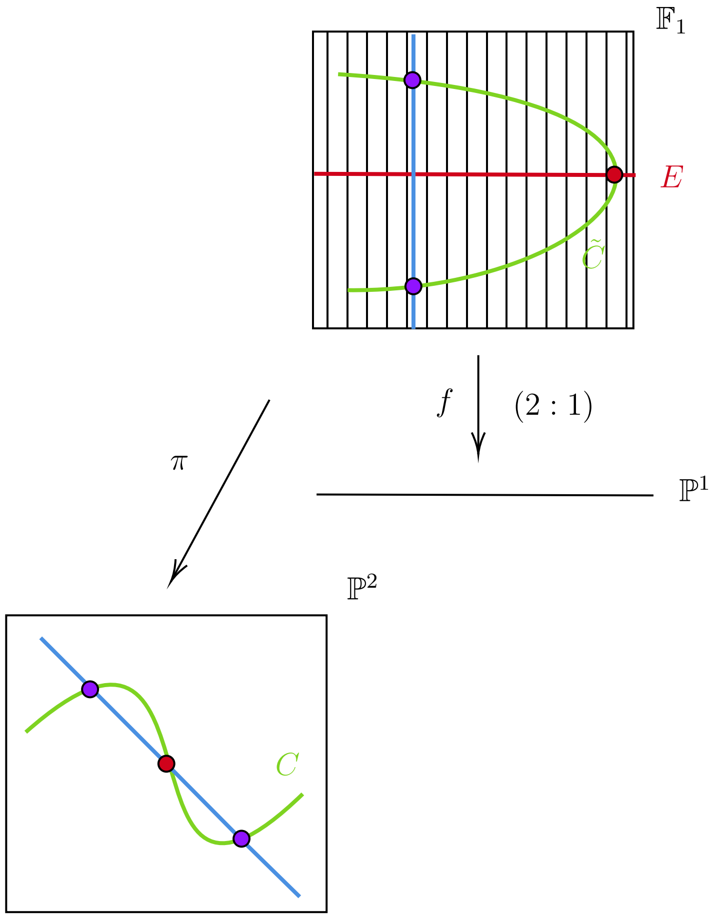


Figure 4.1: $(2:1)$ covering morphism.

Furthermore, the degree of the ramification divisor R at a point $Q \in \mathbb{P}^1$ is exactly

$$\deg(f|_{\tilde{C}}) - \sharp((f|_{\tilde{C}}^{-1})(Q)) = d - 1 - \sharp((f|_{\tilde{C}}^{-1})(Q)).$$

Thus, one can deduce that for $d = 3$, the morphism $f|_{\tilde{C}}$ has 4 ramification points.

Geometry of the Calabi-Yau pairs $(\mathbb{F}_n, C)$. Let $(\mathbb{F}_n, C)$ be a Calabi-Yau pair. Assuming C is irreducible and nonsingular, Lemma 4.1.1 implies that $n \in \{0, 1, 2\}$ and C is an elliptic curve. Since $C \cdot F = ((2+n)F + 2E) \cdot F = 2$, C intersects any fiber of the structure morphism $f\colon \mathbb{F}_n \to \mathbb{P}^1$ in 2 points counted with local multiplicity.

Thus, everything said in this section will analogously hold for the induced $(2:1)$ covering of $\mathbb{P}^1$ given by $f|_C\colon C \to \mathbb{P}^1$.

4.2 The decomposition group of a nonsingular plane cubic

Decomposition and inertia groups. These groups were introduced in [Giz] in a general context involving the language of schemes. This terminology has its origin in concepts of Commutative Algebra with some arithmetic implications. Restricting ourselves to the category of projective varieties and rational maps, these groups have the following definitions:

Definition 4.2.1. Let X be a projective variety and $\mathrm{Bir}(X)$ be its group of birational automorphisms. Given $Y \subset X$ an (irreducible) subvariety, the *decomposition group* of Y in $\mathrm{Bir}(X)$ is the group

$$\mathrm{Bir}(X, Y) = \{\varphi \in \mathrm{Bir}(X) \mid \varphi(Y) \subset Y,\ \varphi|_Y \colon Y \dashrightarrow Y \text{ is birational}\}.$$

The *inertia group* of Y in $\mathrm{Bir}(X)$ is the group

$$\{\varphi \in \mathrm{Bir}(X, Y) \mid \varphi|_Y = \mathrm{Id}_Y \text{ as birational map}\}.$$

When X is normal and Y is a prime divisor D such that (X, D) is a Calabi-Yau pair, one has $\mathrm{Bir}^{\mathrm{vp}}(X, D) = \mathrm{Bir}(X, D)$ provided the pair (X, D) has canonical singularities. See Proposition 3.1.20.

When the ambient variety X is $\mathbb{P}^n$, we denote such groups by $\mathrm{Dec}(Y)$ and $\mathrm{Ine}(Y)$, respectively.

Let $C \subset \mathbb{P}^2$ be an irreducible nonsingular cubic. We have readily that $(\mathbb{P}^2, C)$ is a Calabi-Yau pair with (t,c) singularities according to the Definition 3.1.11. In particular, C is an elliptic curve by Lemma 4.1.1. The following theorem by Pan [Pan1] gives an interesting property of the elements of $\mathrm{Dec}(C)$:

Theorem 4.2.2 (cf. [Pan1] Théorème 1.3)**.** *Let $C \subset \mathbb{P}^2$ be an irreducible, nonsingular and nonrational curve and suppose there exists $\phi \in \mathrm{Dec}(C) \setminus \mathrm{PGL}(3, \mathbb{C})$. Then $\deg(C) = 3$ and $\mathrm{Bs}(\phi) \subset C$, where $\mathrm{Bs}(\phi)$ denotes the set of proper base points of ϕ.*

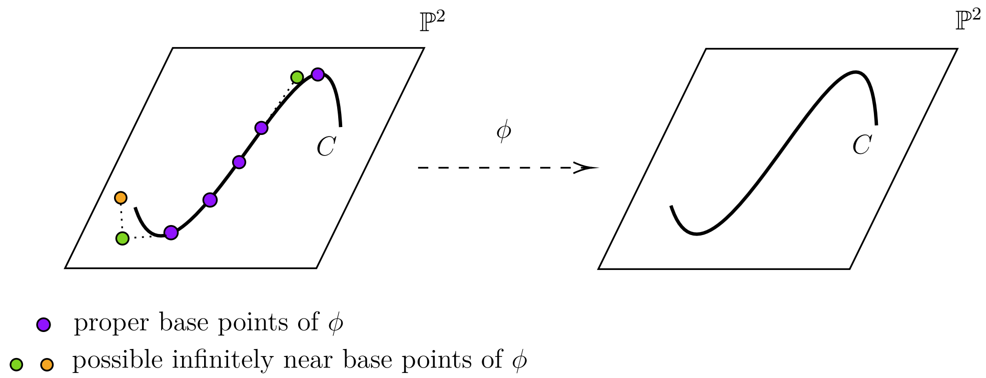


Figure 4.2: Element of $\mathrm{Dec}(C)$.

By Proposition 3.1.20, $\operatorname{Bir}^{\mathrm{vp}}(\mathbb{P}^2, C) = \operatorname{Dec}(C)$, that is, the elements of $\operatorname{Dec}(C)$ are exactly the volume preserving self-maps of the Calabi-Yau pair $(\mathbb{P}^2, C)$ and vice-versa.

Under the point of view of log Calabi-Yau geometry, Theorem 3.1.19 ensures the existence of a volume preserving factorization for any element of $\operatorname{Dec}(C)$. On the other hand, the elements of $\operatorname{Dec}(C)$ can be seen as ordinary maps in $\operatorname{Bir}(\mathbb{P}^2)$ and consequently they admit a Sarkisov factorization. There is no reason at first for this standard Sarkisov factorization to be volume preserving. By the adjective *standard* here, we mean a Sarkisov factorization obtained by running the usual Sarkisov Program without taking into account the volume preserving property. Our result Theorem 4.2.6 says that the Sarkisov algorithm in dimension 2 is automatically volume preserving for an element of $\operatorname{Dec}(C)$.

Notation. We denote by $\operatorname{Bs}(\phi)$ the proper base locus of a rational map $\phi\colon X \dashrightarrow Y$ between projective varieties, and by $\underline{\operatorname{Bs}}(\phi)$ its full base locus, including the infinitely near one.

Before stating and proving Theorem 4.2.6, we will show a couple of lemmas followed by a stronger version of Theorem 4.2.2.

Lemma 4.2.3. *Let S be a nonsingular projective surface admitting a Calabi-Yau pair structure (S, C) with boundary divisor C nonsingular. Consider $P \in S$. Then $f\colon (\operatorname{Bl}_P(S), \tilde{C}) \to (S, C)$ is volume preserving if and only if $P \in C$.*

Proof. Let $E := \operatorname{Exc}(f)$. By the Adjuction Formula, we can write $K_{\operatorname{Bl}_P(S)} = f^*K_S + E$. Since C is nonsingular, we have $\tilde{C} = f^*C - mE$, where $m \in \{0, 1\}$, depending on whether $P \notin C$ or $P \in C$, respectively. Summing up we get

$$K_{\operatorname{Bl}_P(S)} + \tilde{C} = f^*(K_S + C) + (1 - m)E.$$

Since $f_*\tilde{C} = C$, by Proposition 3.1.12, $(\operatorname{Bl}_P(S), \tilde{C})$ is a Calabi-Yau pair and f is volume preserving if and only if $1 - m = 0$, that is, if and only if $P \in C$. $\square$

Lemma 4.2.4. *Let S be a nonsingular projective surface admitting a Calabi-Yau pair structure (S, C) with boundary divisor C nonsingular. Let $E \subset S$ be a (-1)-curve, $f\colon S \to S'$ the contraction of E and $C' = f(C) \subset S'$. Then $f\colon (S, C) \to (S', C')$ is volume preserving if and only if $C \cdot E = 1$ (and therefore C' is nonsingular).*

Proof. By the Castelnuovo Contratibility Theorem, we have $E = \operatorname{Exc}(f)$ and $S \simeq \operatorname{Bl}_P(S')$ for $\{P\} = f(E)$. Observe that C' is indeed a curve, since $C \neq E$. In fact, using that C is nonsingular and (S, C) is a Calabi-Yau pair, the Adjunction Formula for curves gives us $g(C) = 1$, which is different from $g(E) = 0$. Moreover, we have $C = \tilde{C}'$.

Set $m := m_P(C')$. Observe that $C \sim f^*C' - mE$ and so

$$C \cdot E = (f^*C' - mE) \cdot E = -mE^2 = m.$$

Notice that

$$0 = f_*(K_S + C) = K_{S'} + C',$$

and therefore (S', C') is a Calabi-Yau pair.

By analogous computations to the previous lemma, we get

$$K_S + C = f^*(K_{S'} + C') + (1-m)E.$$

Since $f_*C = C'$, by Proposition 3.1.12, f is volume preserving if and only if $1 - m = 0$, that is, if and only if $P \in C'$. Thus, f is volume preserving if and only if $C \cdot E = 1$.

Furthermore, this also shows that C' is nonsingular at P and therefore C' is nonsingular. $\square$

Let $C \subset \mathbb{P}^2$ be a nonsingular cubic. Recall that Pan [Pan1] showed that if $\phi \in \mathrm{Dec}(C) \setminus \mathrm{PGL}(3,\mathbb{C})$, then $\mathrm{Bs}(\phi) \subset C$. Using the volume preserving version of the Sarkisov Program, it is possible to show that more is true: $\underline{\mathrm{Bs}}(\phi)$ is contained in C. Of course, this is an abuse of language since the points in $\underline{\mathrm{Bs}}(\phi)$ do not lie on $\mathbb{P}^2$, but on some infinitesimal neighborhood of the points in $\mathrm{Bs}(\phi)$. Thus, $\underline{\mathrm{Bs}}(\phi)$ is contained in C means that the points in $\underline{\mathrm{Bs}}(\phi)$ belong to the strict transforms of C intersected with the infinitesimal neighborhoods of the points in $\mathrm{Bs}(\phi)$. The subsequent lemma presents a stronger form of Theorem 4.2.2.

Lemma 4.2.5. *Let $C \subset \mathbb{P}^2$ be a nonsingular cubic. Consider $\phi \in \mathrm{Dec}(C) \setminus \mathrm{PGL}(3,\mathbb{C})$. Then $\underline{\mathrm{Bs}}(\phi) \subset C$.*

Proof. Let $P_1 \prec \cdots \prec P_l$ be any maximal increasing sequence of base points in $\underline{\mathrm{Bs}}(\phi)$, starting with a base point $P_1 \in \mathrm{Bs}(\phi)$. We will show by increasing induction on i that all $P_i \in C$. The base case $i = 1$ follows from Theorem 4.2.2, which shows that $\mathrm{Bs}(\phi) \subset C$.

Consider a volume preserving Sarkisov factorization of ϕ:

$$\begin{tikzcd}
(\mathbb{P}^2, C) = (S_0, C_0) \arrow[r, dashed, "\phi_0"] \arrow[rrr, dashed, bend left, "\phi"] & (S_1, C_1) \arrow[r, dashed, "\phi_1"] & \cdots \arrow[r, dashed, "\phi_{k-1}"] & (S_k, C_k) = (\mathbb{P}^2, C)\ .
\end{tikzcd}$$

Recall that each $S_j \simeq \mathbb{P}^2$ or $\mathbb{F}_n$ for $n \in \{0,1,2\}$ by Lemma 4.1.1. Lemma 3.1.23 establishes that canonicity is retained for volume preserving maps between Mori fibered Calabi-Yau pairs. Since $(\mathbb{P}^2, C)$ is a (t,c) Calabi-Yau pair, this implies that all the intermediate ones (S_j, C_j) are also (t,c). By Proposition 3.1.20, we have that each C_j is the strict transform of C on S_j. Since each (S_j, C_j) is, in particular, a dlt Calabi-Yau pair, by [KM, Proposition 5.51] or [ACM, Remark 1.3(1)], we have that each C_i is normal, therefore nonsingular. Thus, it follows that $C_j \simeq C$ for all $j \in \{0, 1, \ldots, k-1\}$.

According to the Sarkisov algorithm described in [CKS] and summarized in Section 2.2, notice that a point in $\underline{\mathrm{Bs}}(\phi)$ appears in the proper base locus of some induced birational map after the blowup of a base point of the previous one with multiplicity higher than its Sarkisov degree. see Lemma 2.2.5. This only occurs for Sarkisov links of type I and II.

Thus, the key observation is that for all $i \in \{0, 1, \ldots, l\}$, there exists j_i such that the Sarkisov link of type I or II $\phi_{j_i} \colon S_{j_i} \dashrightarrow S_{j_i+1}$ starts with the blowup of the image of P_i on S_{j_i}.

Since the Sarkisov link is volume preserving, in particular so is the blowup of P_i that initiates the link by Remark 3.1.18. Conclude from Lemma 4.2.3.

$\square$

Theorem 4.2.6. *Let $C \subset \mathbb{P}^2$ be a nonsingular cubic. The standard Sarkisov Program applied to an element of* $\mathrm{Dec}(C)$ *is automatically volume preserving.*

Proof. Given $\phi\colon (\mathbb{P}^2, C) \dashrightarrow (\mathbb{P}^2, C)$ volume preserving, consider a Sarkisov decomposition of $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by the Sarkisov algorithm in dimension 2 explained in Section 2.2:

$$\begin{tikzcd}
(\mathbb{P}^2, C) = (S_0, C_0) \arrow[r, dashed, "\phi_0"] \arrow[ddrrrrr, dashed, "\phi"] & S_1 = \mathbb{F}_1 \arrow[r, dashed, "\phi_1"] & S_2 = \mathbb{F}_{1\pm 1} \arrow[r, dashed, "\phi_2"] & \cdots \arrow[r, dashed, "\phi_{i-1}"] & S_i \arrow[r, dashed, "\phi_i"] \arrow[ddr, dashed, "\psi_i"] & S_{i+1} \arrow[d, dashed, "\phi_{i+1}"] \\
 & & & & & \vdots \arrow[d, dashed, "\phi_{k-1}"] \\
 & & & & & (S_k, C_k) = (\mathbb{P}^2, C)
\end{tikzcd}.$$

The proof is by increasing induction on i. We will show the following:

- for each $i \in \{0, 1, \ldots, k-1\}$ the strict transform $C_i \subset S_i$ of C is nonsingular and makes (S_i, C_i) a Calabi-Yau pair,
- the base locus of the induced birational map

$$\psi_i := \phi \circ \phi_0^{-1} \circ \cdots \circ \phi_{i-1}^{-1}\colon S_i \dashrightarrow \mathbb{P}^2$$

  is contained in C_i, and
- ϕ_i is volume preserving for $i \in \{1, ..., k-1\}$.

The basis of induction is $i = 0$. In this case, $\psi_0 = \phi$ and we are set by assumption and Theorem 4.2.2. Suppose that the statement holds for $i \in \{1, \ldots, k-1\}$. Let us show that it also holds for $i + 1$.

Consider $\mathrm{Bs}(\phi_i) = \{P_1, \ldots, P_r\} \subset C_i$ with nonincreasing multiplicities $m_1 \geq \cdots \geq m_r$, defined in Section 2.2. By Proposition 3.1.20 combined with Lemma 4.1.1, we have that $S_i = \mathbb{P}^2$ or $\mathbb{F}_n$ for $n \in \{0, 1, 2\}$ and $C_i = (\phi_{i-1} \circ \cdots \circ \phi_0)_* C$.

We will check the induction step for all four types of Sarkisov links.

Sarkisov link of type I: By Lemma 2.2.5, the base point P_1 has multiplicity m_1 greater than the Sarkisov degree of ψ_i. According to the Sarkisov algorithm, ϕ_i is the blowup of $\mathbb{P}^2$ at P_1.

Since $P_1 \in C_i$, by Lemma 4.2.3 we get that ϕ_i^{-1} is volume preserving, consequently ϕ_i, and $(S_{i+1}, \tilde{C}_i)$ is a Calabi-Yau pair. Taking $C_{i+1} := \tilde{C}_i$, we have that C_{i+1} is nonsingular since the restriction of the blowup of a point of a nonsingular curve is an isomorphism between the curve and its strict transform.

Observe that $\phi_i(P_i) \in \mathrm{Bs}(\psi_{i+1})$ for $i \in \{2, \dots, r\}$ because the blowup ϕ_i is an isomorphism between $S_i \setminus \{P_i\}$ and $S_{i+1} \setminus E$, where $E := \mathrm{Exc}(\phi_i^{-1})$.

If ψ_{i+1} is well defined along E, one has $\mathrm{Bs}(\psi_{i+1}) \subset C_{i+1}$. By [Pan1, Corollaire 2.1], the members of the linear system Γ_i associated to ψ_i may share only one tangent direction at P_1. If that is the case, Lemma 4.2.5 guarantees that the corresponding infinitely near base point $P_1' \in E$ of ψ_i belongs to C_{i+1}.

In this scenario, we have the situation illustrated in Figure 4.3, where by abuse of notation, we write $\phi_i(P_j) = P_j$ for $j \neq 1$.

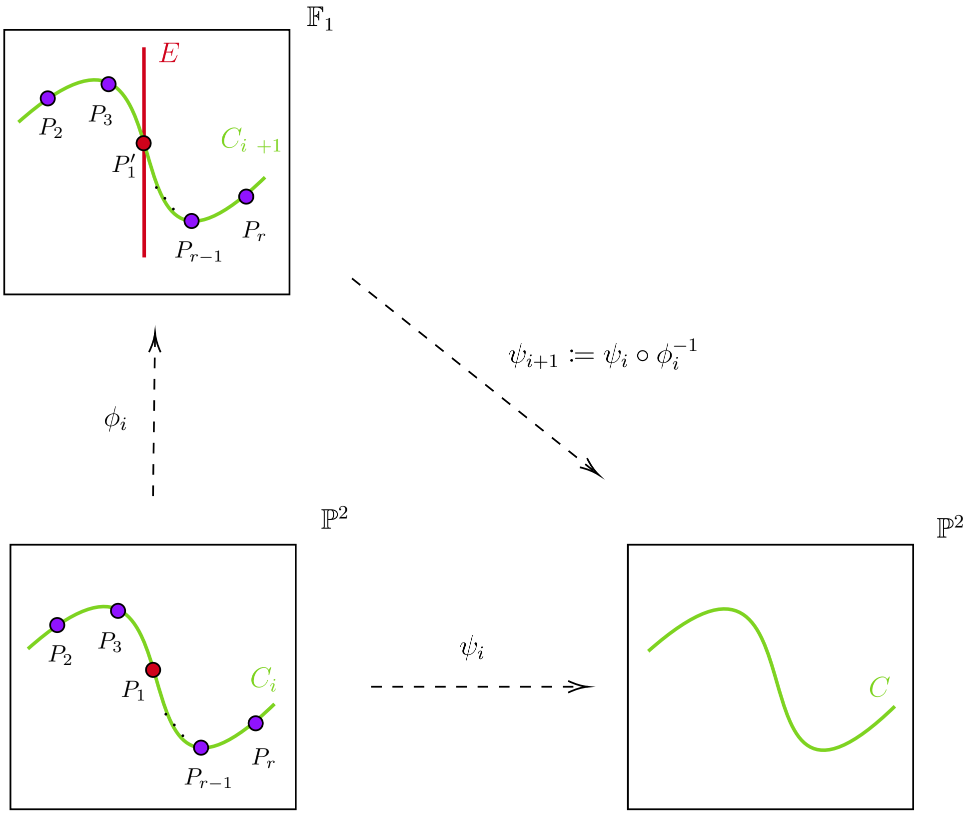


Figure 4.3: Step i of the Sarkisov Program.

Sarkisov link of type II: According to the Sarkisov algorithm, ϕ_i is the elementary transformation $\alpha_{P_1} : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n\pm1}$ centered at the base point P_1 with maximum multiplicity greater than the Sarkisov degree of ψ_i.

From the discussion in Subsection 4.1.1, we are led to four cases depending on the behavior of this base point with respect to the induced $(2:1)$ covering morphism $C_i \to \mathbb{P}^1$ obtained by restriction of $f : \mathbb{F}_n \to \mathbb{P}^1$.

- *Case 1*: P_1 belongs to the negative section of f and the fiber of f through P_1 is transverse to C_i.

- *Case 2*: P_1 belongs to the negative section of f and the fiber of f through P_1 is tangent to C_i.
- *Case 3*: P_1 does not belong to the negative section of f and the fiber of f through P_1 is tranverse to C_i.
- *Case 4*: P_1 does not belong to the negative section of f and the fiber of f through P_1 is tangent to C_i.

Let $F_1 \subset \mathbb{F}_n$ be the fiber of f through P_1. Let E_1 be the negative section of f.

Let $\sigma\colon S \to \mathbb{F}_n$ be the blowup of $\mathbb{F}_n$ at P_1. Set $F_1' \coloneqq \mathrm{Exc}(\sigma)$ and $\check{C}$ the strict transform of C_i. We have the following:

$$\begin{aligned} &\sigma^* F_1 = \tilde{F}_1 + F_1' \Rightarrow \tilde{F}_1 = \sigma^* F_1 - F_1' \\ &\sigma^* C_i = \check{C} + F_1' \Rightarrow \check{C} = \sigma^* C_i - F_1' \end{aligned} \quad ,$$

which implies

$$\begin{aligned} \check{C} \cdot \tilde{F}_1 &= (\sigma^* C_i - F_1') \cdot (\sigma^* F_1 - F_1') \\ &= \sigma^* C_i \cdot \sigma^* F_1 - \sigma^* C_i \cdot F_1' - F_1' \cdot \sigma^* F_1 + (F_1')^2 \\ &= C_i \cdot F_1 + (F_1')^2 \\ &= 2 - 1 = 1. \end{aligned}$$

The strict transform of the fiber of f through P_1 is transverse to $\check{C}$. All the strict transforms of the remaining fibers of f intersect $\check{C}$ with multiplicity 2.

For $\mathbb{F}_n$ we have $\mathrm{Pic}(\mathbb{F}_n) = \langle F_1, E_1 \rangle$. Since $(\mathbb{F}_n, C_i)$ is a Calabi-Yau pair, with this notation, we have $C_i = (n+2)F_1 + 2E_1$.

Case 1: Both F_1' and $\tilde{F}_1$ are transverse to $\check{C}$. Note that F_1' and $\tilde{F}_1$ have a transversal intersection since $F_1' \cdot \tilde{F}_1 = 1$. Let σ' be the blowdown of $\tilde{F}_1$.

Since $\check{C} \cdot \tilde{F}_1 = 1$, by Lemma 4.2.4 we have that σ' is volume preserving. Then so is α_{P_1} because a composition of volume preserving maps is also volume preserving.

Set $C_{i+1} \coloneqq \sigma'_* \check{C}$. We have the following picture:

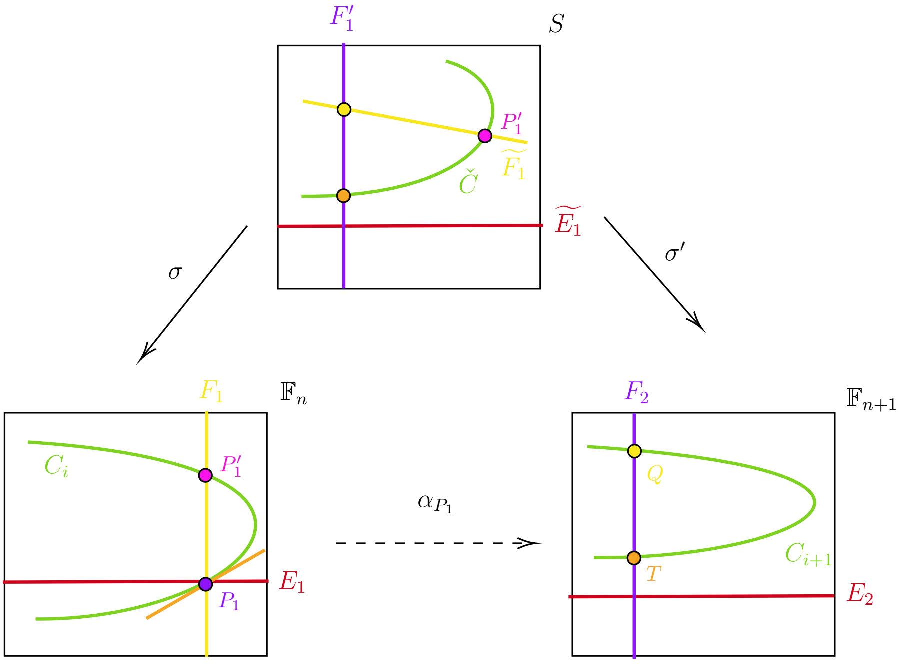


Figure 4.4: Case 1 - Sarkisov link of type II.

We have that $\mathrm{Pic}(S) = \langle \tilde{F}_1, \tilde{E}_1, F_1' \rangle$ and $\mathrm{Pic}(\mathbb{F}_{n+1}) = \langle F_2, E_2 \rangle$, where $F_2 \coloneqq \sigma'_* F_1'$ and $E_2 \coloneqq \sigma'_* \tilde{E}_1$. We have $\check{C} = (n+2)\tilde{F}_1 + (n+3)F_1' + 2\tilde{E}_1$ which implies

$$C_{i+1} = \sigma'_* \check{C} = (n+3)F_2 + 2E_2 \sim -K_{\mathbb{F}_{n+1}},$$

since $\sigma'_* \tilde{F}_1 = 0$. Therefore $(\mathbb{F}_{n+1}, C_{i+1})$ is a Calabi-Yau pair. Notice that C_{i+1} remains nonsingular and $\tilde{F}_1$ is contracted to a point $Q \in C_{i+1}$. The last two properties are consequences of the Lemma 4.2.4.

Observe that $\phi_i(P_j) \in \mathrm{Bs}(\psi_{i+1})$ for $j \in \{2, \ldots, r\}$ such that $P_j \in \mathbb{F}_n \setminus F_1$ because ϕ_i is an isomorphism between $\mathbb{F}_n \setminus F_1$ and $\mathbb{F}_{n+1} \setminus F_2$.

Furthermore, notice that the blowdown σ' may introduce a new base point $Q \in C_{i+1}$. According to the Sarkisov algorithm, one has $m_Q < m_1$. Therefore, we have $\mathrm{Bs}(\psi_{i+1}) \subset C_{i+1}$.

The same observations in the case of the Sarkisov link of type I will hold here and henceforth accordingly for the remaining instances of the Sarkisov link of type II.

Case 2: We will basically imitate the proof of the previous case with the proper modifications. The curves $F_1', \tilde{F}_1$ and $\check{C}$ are pairwise transverse. We have the following picture:

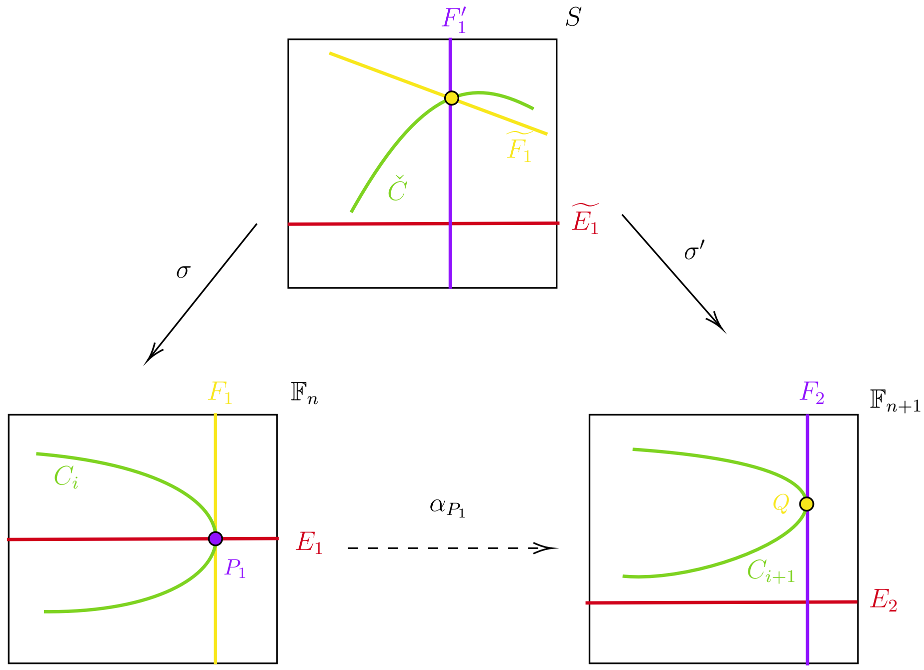


Figure 4.5: Case 2 - Sarkisov link of type II.

It also follows that σ' is volume preserving by Lemma 4.2.4. Hence, once again $(\mathbb{F}_{n+1}, C_{i+1})$ is a Calabi-Yau pair, C_{i+1} remains nonsingular and $\tilde{F}_1$ is contracted to a point $Q \in C_{i+1}$. The difference here is that the intersection number $C_{i+1} \cdot F_2 = 2$ implies that F_2 is tangent to C_{i+1} at Q. Otherwise, $\check{C}$ and F_1' would be separated in S by σ', what does not occur.

The discussion about $\mathrm{Bs}(\psi_{i+1})$ is the same.

The proof of the remaining cases is completely analogous to the previous ones with the proper modifications. We only exhibit pictures that illustrate the geometry behind them. The reader should be convinced of their proof based on them.

Case 3:

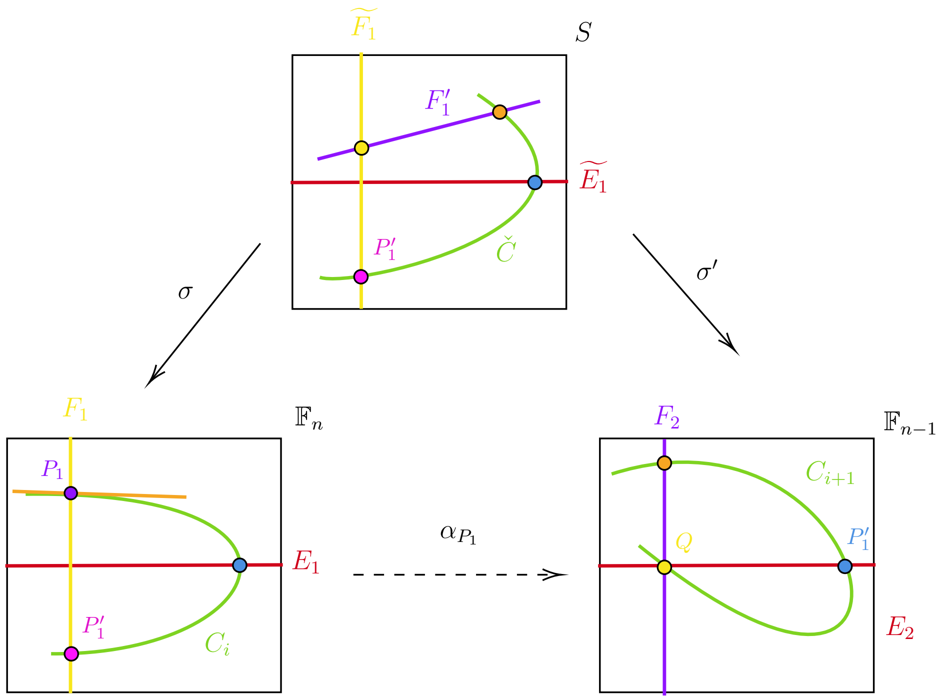


Figure 4.6: Case 3 - Sarkisov link of type II.

Case 4:

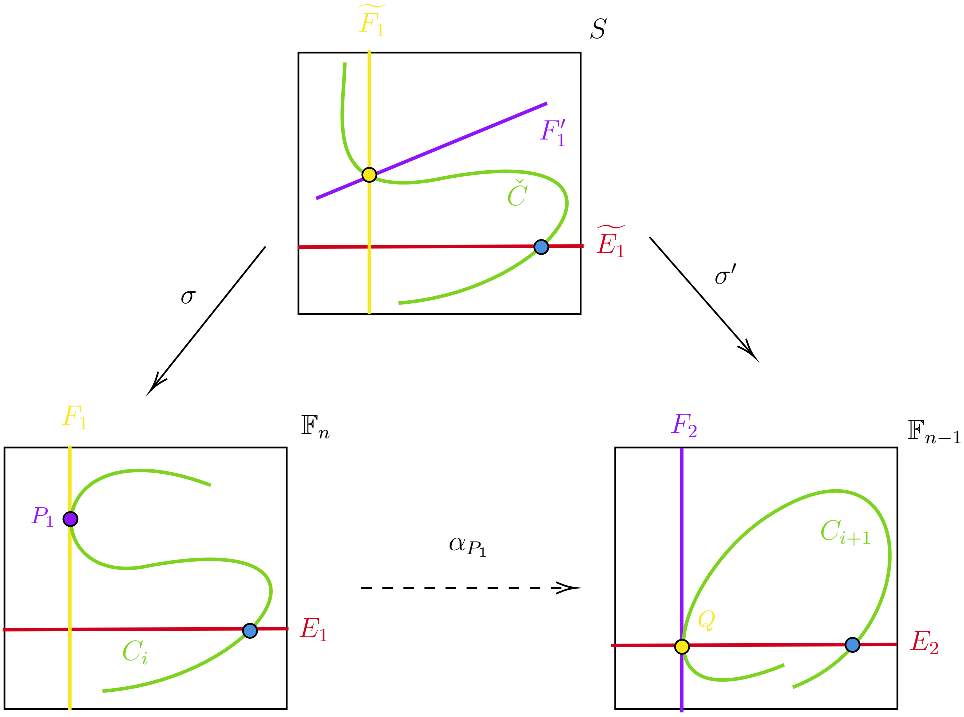


Figure 4.7: Case 4 - Sarkisov link of type II.

Sarkisov link of type III: A Sarkisov link of this type is necessarily preceded by a link of type II. A Sarkisov link of type III occurs when there exists no base point in $\mathbb{F}_1$ with multiplicity greater than the Sarkisov degree of ψ_i. If it is preceded by a Sarkisov link of type I, one can show we will have a contradiction with this fact.

By the induction hypothesis we have that $(S_i, C_i) = (\mathbb{F}_1, C_i)$ is a Calabi-Yau pair with C_i nonsingular. Thus, $C_i = 3F + 2E$ in $\text{Pic}(\mathbb{F}_1) = \langle F, E \rangle$ and so $C_i \cdot E = 3 - 2 = 1$, which implies that E is transverse to C_i.

Let σ be the blowdown of E and $C_{i+1} := \sigma_* C_i$. By Lemma 4.2.3, we have that σ is volume preserving. Therefore, $(\mathbb{P}^2, C_{i+1})$ is a Calabi-Yau pair, C_{i+1} is nonsingular and E is contracted to a point $Q \in C_{i+1}$.

By the properties of the blowup, it is immediate that $\text{Bs}(\psi_{i+1}) \subset C_{i+1}$. Furthermore, notice that σ may introduce Q as a new base point belonging to C_{i+1}.

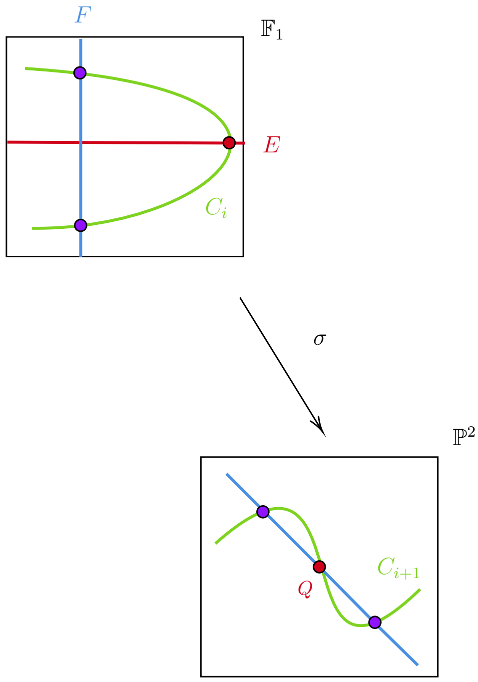


Figure 4.8: Sarkisov link of type III.

Just an interesting remark in a more general situation, the irreducible curves in $\mathbb{F}_1$ such that E is tangent to them have a singular pushforward in $\mathbb{P}^2$ with Q a nonordinary multiple point. See [Har, Example 3.9.5]. Thus, the fact that $(\mathbb{F}_1, C_i)$ is a Calabi-Yau pair by the induction hypothesis is really important.

Sarkisov link of type IV: A Sarkisov link of this type is also necessarily preceded by a link of type II. The previous arguments ensure that $(S_i, C_i) = (\mathbb{F}_0, C_i)$ is a Calabi-Yau pair with C_i nonsingular. The involution $\tau\colon \mathbb{F}_0 \to \mathbb{F}_0$ is clearly an automorphism that changes the structure morphism of $\mathbb{F}_0$. So it is immediate that τ is volume preserving. Just to avoid confusion, denote $\mathbb{F}_0'$ as the codomain of τ and set $C_{i+1} \coloneqq \tau_* C_i$, $E' \coloneqq \tau_* E$, $F' \coloneqq \tau_* F$. It is straightforward that $(K_{\mathbb{F}_0'}, C_{i+1})$ is a Calabi-Yau pair with C_{i+1} nonsingular and $\tau^*(K_{\mathbb{F}_0'} + C_{i+1}) = K_{\mathbb{F}_0} + C_i$.

Moreover, it is clear that $\mathrm{Bs}(\psi_{i+1}) \subset C_{i+1}$.

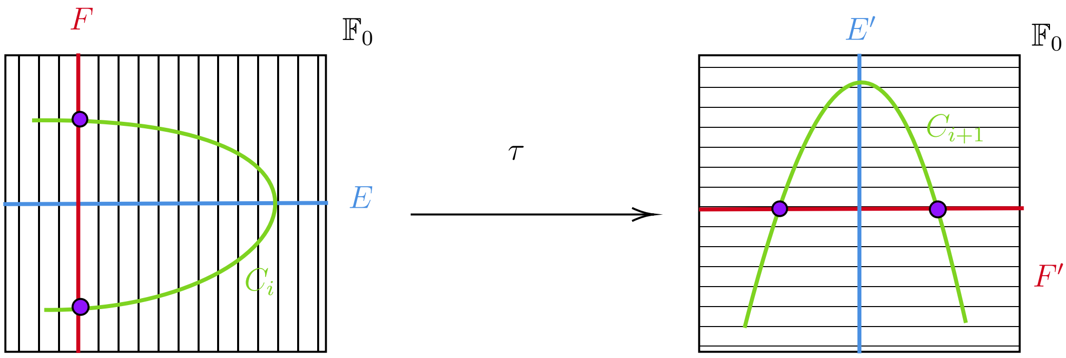


Figure 4.9: Sarkisov link of type IV.

□

Remark 4.2.7. By Remark 3.1.22, the standard quadratic transformation is volume preserving with respect to the boundary divisor given by the sum of the three coordinate lines L_i. Seen as an ordinary map in $\mathrm{Bir}(\mathbb{P}^2)$, the standard Sarkisov decomposition is not volume preserving with respect to the strict transform of $L_1 + L_2 + L_3$. On the other hand, seen as a volume preserving map of $(\mathbb{P}^2, D)$, where $D = L_1 + L_2 + L_3$, the intermediate Calabi-Yau pairs appearing in its decomposition into volume preserving Sarkisov links will not be of the form (X, D_X), where D_X is the strict transform of $L_1 + L_2 + L_3$ on X.

In a general sense, we are allowed to choose freely an anticanonical divisor of X so that we have a Calabi-Yau pair, that is, we may add more prime divisors to D_X in order to make $(X, D_X + F)$ a Calabi-Yau pair. In our case, for example, in the first step of the standard Sarkisov Program we have that $(\mathbb{F}_1, D_{\mathbb{F}_1})$ is not a Calabi-Yau pair and we need to add to $D_{\mathbb{F}_1}$ necessarily the negative section of $\mathbb{F}_1$.

The point is that when our initial Calabi-Yau pair is (t,c), the Calabi-Yau pairs appearing in a factorization of a self-volume preserving map have the form (X, D_X), where D_X is the strict transform of the initial boundary divisor. This is a consequence of Proposition 3.1.20.

The Sarkisov Program in dimension 2 can also yield a factorization of elements in $\mathrm{Bir}(\mathbb{P}^2)$ into de Jonquières maps, incorporating additional steps. See [CKS, Theorem 2.30]. Recall that such maps are elements of $\mathrm{Bir}(\mathbb{P}^2)$ that preserve a pencil of lines.

Let us brief the ideas behind [CKS, Theorem 2.30] in our context, obtaining the first de Jonquières map appearing in the factorization into such maps. Given $\phi \in \mathrm{Dec}(C)$, we have a volume preserving factorization initiated by a point blowup followed by a chain of elementary transformations $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n\pm1}$. If this chain ends with the blowdown $\mathbb{F}_1 \to \mathbb{P}^2$, we have the following picture:

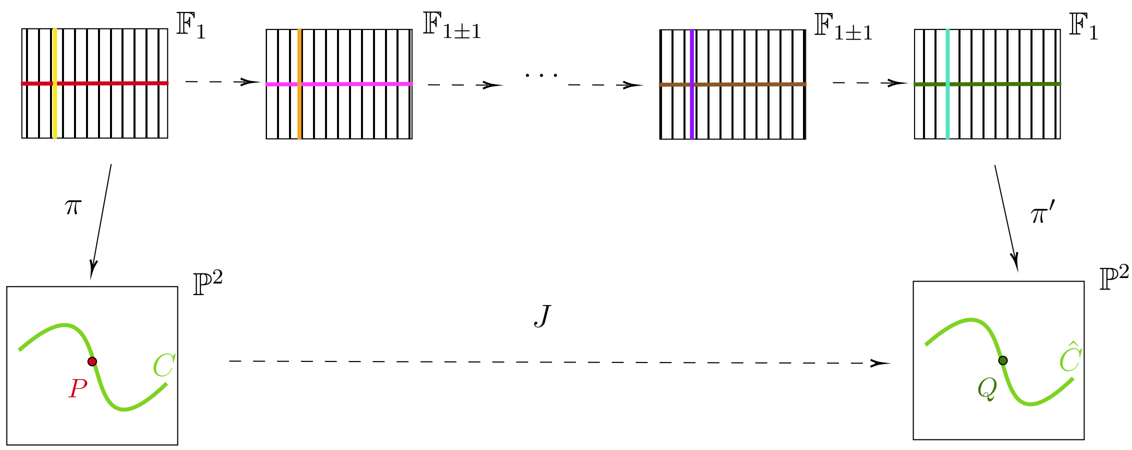


Figure 4.10: de Jonquières map induced by a Sarkisov factorization.

Observe that in each step we switch only one fiber of the fibrations in the Hirzebruch surfaces. In this case, the composition $J\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1 \dashrightarrow \cdots \dashrightarrow \mathbb{F}_1 \to \mathbb{P}^2$ is such that it induces an isomorphism between the generic fibers of the structures morphisms $\mathbb{F}_1 \to \mathbb{P}^1$, $\mathbb{F}_1' \to \mathbb{P}^1$, where $\mathbb{F}_1'$ denotes the last $\mathbb{F}_1$ to avoid confusion. Therefore J satisfies item 1 of Definition 2.2.8 and hence it is a de Jonquières map. In particular, J is volume preserving and its centers belong to the cubic and its strict transform. We point out that P is not necessarily equal to Q.

If the chain of elementary transformations $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n\pm1}$ ends with $\mathbb{F}_0$, the algorithm of the Sarkisov Program says that we must continue with a link of type IV. Instead of doing that, we will proceed with an elementary transformation $\mathbb{F}_0 \dashrightarrow \mathbb{F}_1$ centered at a base point P' of the induced birational map $\mathbb{F}_0 \dashrightarrow \mathbb{P}^2$, followed by the blowdown $\mathbb{F}_1 \to \mathbb{P}^2$. We have the following (commutative) scenario:

$$\begin{tikzcd}
\mathbb{F}_{1\pm1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & \mathbb{F}_{1\pm1} \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & & S \arrow[dl, swap, "\text{vol. pres.}"] \arrow[dr, "\text{vol. pres.}"] & & \\
\mathbb{F}_1 \arrow[u, dashed] & & & & \mathbb{F}_0 \arrow[rr, dashed, "\alpha_{P'}"] & & \mathbb{F}_1 \arrow[r, "\text{vol. pres.}"] & \mathbb{P}^2 \arrow[d, dashed, "\phi\circ J^{-1}"] \\
& \mathbb{P}^2 \arrow[ul, dashed] \arrow[urrr, dashed, "\text{vol. pres.}"] \arrow[urrrrrr, dashed, "J"] \arrow[rrrrrr, dashed, "\phi"] & & & & & & \mathbb{P}^2
\end{tikzcd}$$

By similar arguments as in the proof of the Theorem 4.2.6, the composition

$$J\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1 \dashrightarrow \cdots \dashrightarrow \mathbb{F}_1 \to \mathbb{F}_0 \overset{\alpha_{P'}}{\dashrightarrow} \mathbb{F}_1 \to \mathbb{P}^2$$

is a volume preserving de Jonquières map whose centers belong to the cubic and its strict transform.

Furthermore, in both cases the Sarkisov degree of $\phi \circ J^{-1}$ is smaller than the Sarkisov degree of ϕ. These arguments show the following corollary.

Corollary 4.2.8. *The centers of de Jonquières transformations obtained from the (volume preserving) Sarkisov Program applied to any element of* $\mathrm{Dec}(C)$ *belong to the cubic and its strict transform.*

We recall that Lemma 4.1.1 restricts possibilities for the Mori fibered spaces appearing in a volume preserving factorization of $\phi \in \mathrm{Dec}(C)$. This will be illustrated by examples in the next section.

4.3 The canonical complex of $C \subset \mathbb{P}^2$

Let $C \subset \mathbb{P}^2$ be an irreducible plane curve (not necessarily nonsingular). We have a complex (not necessarily exact) induced by the natural action ρ of $\mathrm{Dec}(C)$ on C

$$1 \longrightarrow \mathrm{Ine}(C) \longrightarrow \mathrm{Dec}(C) \xrightarrow{\rho} \mathrm{Bir}(C) \longrightarrow 1, \tag{4.3.1}$$

where $\mathrm{Ine}(C)$ is identified with $\ker(\rho)$. This complex is called the *canonical complex* of the pair $(\mathbb{P}^2, C)$ and the obstruction to its exactness is the surjectivity of ρ.

In [BPV1], Blanc, Pan & Vust studied canonical complexes under the usual trichotomy: genera $g \geq 2, g = 1$ and $g = 0$. See [BPV1] for interesting examples in which the map $\mathrm{Dec}(C) \to \mathrm{Bir}(C)$ is not surjective. In the case where C is nonsingular, one has $\mathrm{Bir}(C) = \mathrm{Aut}(C)$.

If $g \geq 2$, one can easily check that C has degree > 3. See [Ful1, Proposition 5, Chapter 8]. By the first part in the proof of the Theorem 4.2.2 or [Pan1, Corollaire 3.6], in this case, the group $\mathrm{Dec}(C)$ is trivial in the sense that it is given by the automorphisms of $\mathbb{P}^2$ that preserve C. That is, if $g \geq 2$, then

$$\mathrm{Dec}(C) = \mathrm{Aut}(\mathbb{P}^2, C) := \mathrm{Dec}(C) \cap \mathrm{Aut}(\mathbb{P}^2).$$

This fact together with the following result due to Matsumura & Monsky [MM] when $n \geq 2$ and Chang [Ch] when $n = 1$ implies that the canonical complex of the pair $(\mathbb{P}^2, C)$ is exact.

Theorem 4.3.1 (cf. [MM] Theorem 2 and [Ch] Theorem 1)**.** *Let n and d be positive integers and X be a nonsingular hypersurface of degree d in $\mathbb{P}^{n+1}$. If $(n, d) \neq (2, 4), (1, 3)$, then the natural group homomorphism* $\mathrm{Aut}(\mathbb{P}^n, X) \to \mathrm{Aut}(X)$ *is surjective.*

In the case of $d > n$, this result can also be obtained through adjoint systems [Pan1, Remarque 3.7]. When $(n, d) = (1, 3)$, then $X = C$ is a nonsingular plane cubic. The result by Pan about its decomposition group indicates the existence of nonlinear maps inducing automorphisms of C by restriction. So $\mathrm{Dec}(C)$ is larger than $\mathrm{Aut}(\mathbb{P}^2, C)$.

In [Bl1], Blanc showed that the inertia group of C is generated by its elements of degree 3, and except for the identity, such elements are the ones with lowest degree. By [Giz, Theorem 6] and [Og2, Theorem 2.2] we have that $\mathrm{Aut}(C)$ is no longer derived from $\mathrm{Aut}(\mathbb{P}^2, C)$, but from $\mathrm{Dec}(C)$. Consequently, this implies the exactness of the canonical complex of $(\mathbb{P}^2, C)$. From this, we can therefore also describe $\mathrm{Aut}(C)$ as the quotient group $\mathrm{Dec}(C)/\operatorname{Ine}(C)$.

Once we know that a given short complex is exact, we may ask if it splits or not. In [BPV1], Blanc, Pan & Vust posed the problem of splitting or non-splitting of the canonical complex of $(\mathbb{P}^2, C)$.

Question by Blanc, Pan & Vust. C is an elliptic curve and therefore also an algebraic group. One has $\mathrm{Aut}(C) = C \rtimes \mathbb{Z}_d$, where C is identified with its group of translations and $d \in \{2, 4, 6\}$, depending on the j invariant of C. More precisely, $\mathbb{Z}_d \simeq \mathrm{Aut}(C, O)$, the group of automorphisms of C which fix the neutral element O of the group operation of C.

Denote by $\oplus$ the group law on C and fix $O \in C$ a neutral element. Given $P \in C \setminus \{O\}$, consider a map $\varphi_P \in \mathrm{Dec}(C)$ which by restriction to C induces the translation T_P by P, that is, $(\varphi_P)|_C(Q) = Q \oplus P$ for all $Q \in C$.

We observe that there are infinitely many maps in $\mathrm{Dec}(C)$ which will induce the same translation T_P on C. Any composition with an element of $\operatorname{Ine}(C)$ plays the same role. We point out that $\operatorname{Ine}(C)$ as well as $\mathrm{Dec}(C)$ are infinite uncountable groups. This is based on the existence of a free subgroup of the former, their descriptions in terms of presentations and the cardinality of our ground field $\mathbb{C}$. See [Bl1, Theorem 6] and [Pan1, Théorèm 1.4].

One can explicitly obtain such a map φ_P, for example, by homogenizing the expressions of the group law on C with affine coordinates in its Weierstrass normal form, and extending them to $\mathbb{P}^2$ as a rational map. In what follows we make this explicit.

After a suitable change of coordinates, we can write the equation of C in the Weierstrass normal form as

$$y^2 = x^3 + px + q,$$

where $p, q \in \mathbb{C}$ are not mutually zero, and (x, y) are affine coordinates of $\mathbb{A}^2_{(x,y)} = \{z \neq 0\} \subset \mathbb{P}^2$. The neutral element O of $\oplus$ is the unique intersection of C with the line at infinity: $O = (0 : 1 : 0)$, which is an inflection point.

Let us consider now the translation in C by $(a, b) \in C$. In these coordinates, the group law can be explicitly described as $(x', y') = (x, y) \oplus (a, b)$ if and only if

$$x' = \left(\frac{y-b}{x-a}\right)^2 - x - a, \;\; y' = \frac{y-b}{x-a}(x'-a) + b.$$

Let $\mathbb{P}^2$ have homogeneous coordinates $(x : y : z)$. Consider the rational map $\varphi_P \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by the same equations above in the open subset $\{z \neq 0\}$. One can verify birationality and that indeed $\varphi \in \mathrm{Dec}(C)$. In this expression, φ_P is not defined when $x = a$.

Let us extend φ_P to the largest possible open subset of $\mathbb{P}^2$. Performing some algebraic manipulations and homogenizing, the extension obtained, also denoted by φ_P, becomes

$$\begin{aligned} \varphi_P \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\ (x : y : z) &\longmapsto (F_1(x,y,z) : F_2(x,y,z) : F_3(x,y,z)), \end{aligned}$$

with

$$\begin{aligned}
F_1(x,y,z) &= z(y-bz)^2(x-az) - (x^2-a^2z^2)(x-az)^2,\\
F_2(x,y,z) &= z(y-bz)^3 - (y-bz)(x+2az)(x-az)^2,\\
F_3(x,y,z) &= z(x-az)^3
\end{aligned}$$

for all $(x:y:z) \in \mathrm{Dom}(\varphi_P) \coloneqq \mathbb{P}^2 \setminus \mathrm{Bs}(\varphi_P)$. One can check that $\mathrm{Bs}(\varphi_P) = V(F_1, F_2, F_3) = \{P, O\} \subset C$, as predicted by Theorem 4.2.2.

Let $\Gamma \subset |4H|$ be the linear system associated to φ_P, where H denotes a general line of $\mathbb{P}^2$. We have that Γ is contained in the linear system of plane quartics passing through P and O with certain multiplicities m_P and m_O, respectively. To compute them, let us make use of the Noether-Fano equations or equations of condition from Theorem 2.2.3.

These equations imply that we have two possibilities for the multiplicities of a plane birational map of degree 4 in nonincreasing order, namely, $(3,1,1,1,1,1,1)$ or $(2,2,2,1,1,1)$. Both cases include the multiplicities of the infinitely near base points.

A careful analysis of the general member of Γ shows that we are in the first case, with $m_P = 3$ and $m_O = 1$. More precisely, Γ is contained in the linear system of plane quartics passing through P with multiplicity 3 and passing through O with multiplicity 1 and sharing one tangent direction at O and higher order Taylor terms up to order 5. The shared tangent direction is exactly the tangent direction to C at O.

Its homaloidal type is $(4; 3, 1^6)$, where the coordinate before the semicolon indicates its degree and the further ones in nonincreasing order represent the multiplicities of all base points including the infinitely near ones. See Definition 2.2.2.

According to Definition 2.2.8(2), such homaloidal type makes φ_P a de Jonquières map and the configuration of the seven base points of φ_P (including the infinitely near ones) is as follows

$$\begin{gathered}
P,\\
O \prec P_1 \prec P_2 \prec P_3 \prec P_4 \prec P_5,
\end{gathered}$$

where the notation $P_i \prec P_{i+1}$ indicates that P_{i+1} is infinitely near to P_i, that is, each P_i belongs to the i-th infinitesimal neighborhood of O. See Definition 2.2.1.

By blowing up five times consecutively, we can check that the infinitely near base points $P_1 \prec P_2 \prec P_3 \prec P_4 \prec P_5$ over O are independent of P, and the shared tangent directions in the infinitely near points are exactly the tangent directions to the strict transforms of C at them.

We may ask ourselves if $\varphi_{Q\oplus P} = \varphi_Q \circ \varphi_P$, for all $P, Q \in C \setminus \{O\}$. If this relation is confirmed, it would yield a splitting of the canonical complex of the pair $(\mathbb{P}^2, C)$ at C, coming from a set-theoretical section $\eta\colon C \hookrightarrow \mathrm{Dec}(C)$. In general, the splitting property of a given short exact sequence can be relative to some subgroup and not global. In our context, this subgroup can be continuous or discrete.

Let us investigate the possible set-theoretical section $\eta\colon C \hookrightarrow \mathrm{Dec}(C)$ given by

$$\eta(P) = \begin{cases} \varphi_P, & \text{if } P \neq O \\ \mathrm{Id}_{\mathbb{P}^2}, & \text{otherwise} \end{cases}.$$

If η is also a group homomorphism, we would have $\varphi_P^{-1} = \varphi_{\ominus P}$ for all $P \in C$, where $\ominus P$ denotes the inverse of P under the group law $\oplus$ of C.

For all $P, Q \in C \setminus \{O\}$ with $P \neq \ominus Q$, let us compare the degrees of $\varphi_{Q\oplus P}$ and $\varphi_Q \circ \varphi_P$. We already know that $\deg(\varphi_{Q\oplus P}) = 4$. The following result will allow us to compute the second degree.

Corollary 4.3.2 (cf. [Alb] Corollary 4.2.12)**.** *Let f be a plane Cremona map of homaloidal type $(d; m_1, \dots, m_r)$, and let g be a plane Cremona map of homaloidal type $(e; \ell_1, \dots, \ell_s)$ satisfying that the first k base points of f coincide with those of g and no further coincidence. Then the composite map $g \circ f^{-1}$ has degree*

$$de - \sum_{i=1}^{k} m_i \ell_i.$$

Since $\underline{\mathrm{Bs}}(\varphi_{\ominus P}) = \{\ominus P, O \prec P_1 \prec P_2 \prec P_3 \prec P_4 \prec P_5\}$, we have

$$\underline{\mathrm{Bs}}(\varphi_P) \cap \underline{\mathrm{Bs}}(\varphi_{\ominus Q}) = \{O \prec P_1 \prec P_2 \prec P_3 \prec P_4 \prec P_5\}.$$

Therefore, the previous result tells us that

$$\deg(\varphi_Q \circ \varphi_P) = 4 \cdot 4 - 6 \cdot 1 \cdot 1 = 10 \neq 4 = \deg(\varphi_{Q\oplus P}),$$

which implies that the maps $\varphi_{Q\oplus P}$ and $\varphi_Q \circ \varphi_P$ are distinct and do not make η a group homomorphism.

Thus, our candidate η does not yield a splitting of the canonical complex of $(\mathbb{P}^2, C)$ at C. We therefore conclude that we have $\varphi_Q \circ \varphi_P, \varphi_{Q\oplus P} \in \rho^{-1}(T_{Q\oplus P})$ with $\varphi_Q \circ \varphi_P \neq \varphi_{Q\oplus P}$. This allows us to produce many elements in $\mathrm{Ine}(C)$. For instance, given $P, Q, R \in C$ such that $P \oplus Q \oplus R = O$, then $\varphi_{P\oplus Q} \circ \varphi_R$ and $\varphi_P \circ \varphi_{Q\oplus R}$ belong to $\mathrm{Ine}(C)$.

In [BF], Blanc & Furter examined topologies and structures of the Cremona groups.

Definition 4.3.3 (cf. [BF] Definition 2.1)**.** Let A and X be irreducible algebraic varieties, and let f be an A-birational self-map of the A-variety $A \times X$ satisfying the following:

1. f induces an isomorphism $U \xrightarrow{\simeq} V$, where U and V are open subsets of $A \times X$, whose projections on A are surjective,

2. $f(a, x) = (a, \mathrm{pr}_2(f(a, x)))$, where pr_2 denotes the second projection. Hence for each point $a \in A$, the birational map $x \dashrightarrow \mathrm{pr}_2(f(a, x))$ corresponds to an element $f_a \in \mathrm{Bir}(X)$.

The map $a \mapsto f_a$ represents a map from A to $\mathrm{Bir}(X)$ and it is called a *morphism* from A to $\mathrm{Bir}(X)$.

These notions yield the following topology on $\mathrm{Bir}(X)$ called the *Zariski topology*: a subset $F \subset \mathrm{Bir}(X)$ is closed in this topology if for any algebraic variety A and any morphism $A \to \mathrm{Bir}(X)$, its preimage is closed.

In [BF], Blanc & Furter studied the case where $X = \mathbb{P}^n$. Very recently and using more tools, Hassanzadeh & Mostafazadehfard [HM2] investigated similar aspects of $\mathrm{Bir}(X)$ when X is an arbitrary projective variety over an infinite field k, of any characteristic and not necessarily algebraically closed.

Observe that a section $\eta\colon C \hookrightarrow \mathrm{Dec}(C)$ induces a C-birational self-map f of the C-variety $C \times \mathbb{P}^2$ in the following way

$$\begin{aligned} f\colon C \times \mathbb{P}^2 &\dashrightarrow C \times \mathbb{P}^2 \\ (P, x) &\longmapsto (P, \eta_P(x)), \end{aligned}$$

where $\eta_P := \eta(P)$, for all $P \in C$.

Indeed, for all $P \in C$, the map η_P is birational. Since $\mathrm{Bs}(\eta_P) \subset C$ for all $P \in C$ by Theorem 4.2.2, f determines an isomorphism of $U = V = C \times (\mathbb{P}^2 \setminus C)$ onto $C \times (\mathbb{P}^2 \setminus C)$ and therefore satisfies item 1 of Definition 4.3.3.

This implies that η is a morphism from C to $\mathrm{Bir}(\mathbb{P}^2)$, whose image is contained in $\mathrm{Dec}(C)$.

More generally, we will show the following which negatively answers the question posed in [BPV1]:

Theorem 4.3.4. *The canonical complex 4.3.1 of the pair* $(\mathbb{P}^2, C)$ *does not admit any splitting at* C *when we write* $\mathrm{Aut}(C) = C \rtimes \mathbb{Z}_d$.

Proof. For the sake of contradiction, suppose that we have a splitting given by a section $\eta\colon C \hookrightarrow \mathrm{Dec}(C)$. From the above discussion, it follows that η is also a morphism from C to $\mathrm{Bir}(\mathbb{P}^2)$ with respect to the Zariski topology. By [BF, Lemma 2.19], the image $\eta(C)$ of η is a closed subgroup of $\mathrm{Bir}(\mathbb{P}^2)$, which has bounded degree. Thus $C \simeq \eta(C)$ as algebraic varieties, and therefore $\eta(C)$ is a projective algebraic group inside $\mathrm{Bir}(\mathbb{P}^2)$. However, this violates the fact that any algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is affine [BF, Remark 2.21]. Hence, there does not exist any section η, which implies the result. $\square$

4.3.1 (Volume preserving) Sarkisov factorizations of φ_P

Under the isomorphism $\mathrm{Aut}(C) \simeq C \rtimes \mathbb{Z}_d$, we have that $\mathrm{Aut}(C, O) \simeq \mathbb{Z}_d$ comes from $\mathrm{Aut}(\mathbb{P}^2, C)$ whereas C comes from $\mathrm{Dec}(C)$. One can also check that there exist finitely many elements in $\mathrm{Aut}(C)$ derived from $\mathrm{Aut}(\mathbb{P}^2, C)$.

Let us describe the Sarkisov factorization of φ_P for $P \in C \setminus \{O\}$. It will strongly depend on the nature of the intersection $L \cap C$, where L is the line through P and O. We will use the same notation as before for $\mathrm{Bs}(\varphi_P)$.

The first step in the Sarkisov Program is the blowup of $\mathbb{P}^2$ at the base point P with maximum multiplicity. Notice that $\tilde{L}$ corresponds to a fiber of the structure morphism $\mathbb{F}_1 \to \mathbb{P}^1$. The second step will be the elementary transformation $\alpha_O\colon \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$. The type of intersection of the strict transform of C with the zero section (horizontal fiber of $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$) through P_1, infinitely near to O, will determine whether the next infinitely near base point P_2 will belong to the negative section of $\mathbb{F}_1$, the codomain of the elementary transformation $\alpha_{P_1} : \mathbb{F}_0 \dashrightarrow \mathbb{F}_1$. We may have either $\alpha_{P_2} : \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$ or $\alpha_{P_2} : \mathbb{F}_1 \dashrightarrow \mathbb{F}_2$, where P_2 is infinitely near to P_1, and does not belong, respectively belongs, to the negative section of $\mathbb{F}_1$.

Continuing with this analysis, we reach three possibilities for the application of the (volume preserving) Sarkisov Program on φ_P. In the following tables, φ_i is the composition of the inverse maps of the previous Sarkisov links with φ_P, Γ_i is the associated linear system, and s- deg is the corresponding Sarkisov degree.

step	map	linear system	s-deg	$\sum m_i$
0	$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$	$\Gamma \subset \lvert 4H \rvert$	4/3	9
1	$\varphi_1 : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_1 \subset \lvert 4F_1 + E_1 \rvert$	1/2	6
2	$\varphi_2 : \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$	$\Gamma_2 \subset \lvert 3F_2 + E_2 \rvert$	1/2	5
3	$\varphi_3 : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_3 \subset \lvert 3F_3 + E_3 \rvert$	1/2	4
4	$\varphi_4 : \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$	$\Gamma_4 \subset \lvert 2F_4 + E_4 \rvert$	1/2	3
5	$\varphi_5 : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_5 \subset \lvert 2F_5 + E_5 \rvert$	1/2	2
6	$\varphi_6 : \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$	$\Gamma_6 \subset \lvert F_6 + E_6 \rvert$	1/2	1
7	$\varphi_7 : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_7 \subset \lvert F_7 + E_7 \rvert$	1/2	0

Table 4.1: First possibility for the application of the Sarkisov Program on φ_P.

step	map	linear system	s-deg	$\sum m_i$
0	$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$	$\Gamma \subset \lvert 4H \rvert$	4/3	9
1	$\varphi_1 : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_1 \subset \lvert 4F_1 + E_1 \rvert$	1/2	6
2	$\varphi_2 : \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$	$\Gamma_2 \subset \lvert 3F_2 + E_2 \rvert$	1/2	5
3	$\varphi_3 : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_3 \subset \lvert 3F_3 + E_3 \rvert$	1/2	4
4	$\varphi_4 : \mathbb{F}_2 \dashrightarrow \mathbb{P}^2$	$\Gamma_4 \subset \lvert 3F_4 + E_4 \rvert$	1/2	3
5	$\varphi_5 : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_5 \subset \lvert 2F_5 + E_5 \rvert$	1/2	2
6	$\varphi_6 : \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$	$\Gamma_6 \subset \lvert F_6 + E_6 \rvert$	1/2	1
7	$\varphi_7 : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_7 \subset \lvert F_7 + E_7 \rvert$	1/2	0

Table 4.2: Second possibility for the application of the Sarkisov Program on φ_P.

step	map	linear system	s-deg	$\sum m_i$
0	$\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$	$\Gamma \subset \lvert 4H\rvert$	4/3	9
1	$\varphi_1\colon \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_1 \subset \lvert 4F_1 + E_1\rvert$	1/2	6
2	$\varphi_2\colon \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$	$\Gamma_2 \subset \lvert 3F_2 + E_2\rvert$	1/2	5
3	$\varphi_3\colon \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_3 \subset \lvert 3F_3 + E_3\rvert$	1/2	4
4	$\varphi_4\colon \mathbb{F}_2 \dashrightarrow \mathbb{P}^2$	$\Gamma_4 \subset \lvert 3F_4 + E_4\rvert$	1/2	3
5	$\varphi_5\colon \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_5 \subset \lvert 2F_5 + E_5\rvert$	1/2	2
6	$\varphi_6\colon \mathbb{F}_2 \dashrightarrow \mathbb{P}^2$	$\Gamma_6 \subset \lvert 2F_6 + E_6\rvert$	1/2	1
7	$\varphi_7\colon \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$	$\Gamma_7 \subset \lvert F_7 + E_7\rvert$	1/2	0

Table 4.3: Third possibility for the application of the Sarkisov Program on φ_P.

Notice that from these Sarkisov factorizations, we can deduce that φ_P is a de Jonquières map. In fact, after blowing up the base point P, the series of elementary transformations terminates with $\mathbb{F}_1$ and also ends the factorization. This satisfies item 1 of Definition 2.2.8. This is already followed from the configuration of base points of φ_P.

4.4 Higher dimensions

This chapter addresses the 2-dimensional case. Within a similar framework in higher dimension, it is natural to ask ourselves about generalizations of Theorem 4.2.2 and Theorem 4.2.6, namely,

1. *Let $D \subset \mathbb{P}^n$ be a hypersurface of degree $n+1$ with canonical singularities and consider $\phi \in \mathrm{Dec}(D) \setminus \mathrm{Aut}(\mathbb{P}^n, D)$. Does it hold that $\mathrm{Bs}(\phi) \subset D$?*
2. *In dimension 3 and under the same assumptions, is the Sarkisov algorithm applied to ϕ automatically volume preserving?*

In dimension 3, we are dealing with a quartic surface $D \subset \mathbb{P}^3$. If D is nonsingular, then it is a K3 surface and $\mathrm{Bir}(D) = \mathrm{Aut}(D)$. In [Og2, PQ], there are produced examples of such quartic surfaces for which no nontrivial automorphism is derived from $\mathrm{Bir}(\mathbb{P}^3)$ by restriction. In these examples, we have $\mathrm{Dec}(D) = \{\mathrm{Id}_{\mathbb{P}^3}\}$. Thus, neither of those questions is meaningful in such circumstances. We remark that in this case, $(\mathbb{P}^3, D)$ is a (t,c) Calabi-Yau pair. The situation changes if we allow strict canonical singularities on D. Recall that in this case, by Proposition 3.1.20, we have $\mathrm{Bir}^{\mathrm{vp}}(\mathbb{P}^3, D) = \mathrm{Dec}(D)$.

Based on the minimal resolution process, the simplest canonical surface singularity is of type A_1 [KM, Theorem 4.22]. Consider $D \subset \mathbb{P}^3$ a general irreducible normal quartic surface having such a type of singularity at $P = (1:0:0:0)$. After suitable coordinate change, one can show that the equation of D is of the form $x_0^2 A + x_0 B + C$, where $A, B, C \in \mathbb{C}[x_1, x_2, x_3]$ are general homogeneous polynomials of degrees 2, 3, 4, respectively. Moreover, A is a quadratic form of rank 3.

In the proof of [ACM, Claim 5.8], Araujo, Corti & Massarenti show that the birational involution

$$\phi\colon (x_0 : x_1 : x_2 : x_3) \mapsto (-Ax_0 - B : Ax_1 : Ax_2 : Ax_3)$$

belongs to $\mathrm{Dec}(D)$. One can check that $\mathrm{Bs}(\phi) = V(A, B)$ and it consists of the union of six pairwise distinct lines through P if we take B general enough. This implies that $\mathrm{Bs}(\phi) \not\subset D$ as D does not contain lines. This is in contrast with Theorem 4.2.2 in dimension 2, which asserts that the base locus is contained in the boundary divisor. This fact will allow us to construct a Sarkisov factorization that is not volume preserving, which shows that a generalization of the Theorem 4.2.6 does not hold in higher dimensions.

Thus, the answer to both initial questions in this section is no. In the next chapter, we will exhibit the details of this example in Section 5.4.

Chapter 5

The 3-dimensional case

In this chapter, we will focus on explicit birational geometry, a subject of significant importance as emphasized in various works by Corti and Reid. This branch presents numerous open problems that await explicit constructions. In particular, the problem of describing divisorial extractions is studied in many contexts. For instance, see the works [CPR, Gue, Pae1, Pae2].

In some cases, it is possible to give an explicit description of all volume preserving Sarkisov links between given Calabi-Yau pairs. Even if we know the types of links, there do not exist explicit descriptions of all of them. So describing all possible links becomes very relevant for classification purposes. This is the main motivation of this chapter in the case of log Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2.

5.1 Birational geometry of Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2

Concerning the geometry of a log Calabi-Yau pair (X, D_X), the most important discrete volume preserving invariant is the coregularity that varies between 0 and $\dim X$. See Definition 3.2.2.

In the recent work [Duc], Ducat classified all pairs of the form $(\mathbb{P}^3, D)$ with coregularity less than or equal to one, up to volume preserving equivalence. For those with minimum possible coregularity, called *maximal pairs*, he showed that they are all volume preserving equivalent to toric pairs (T, D_T). This in particular shows the following conjecture originated in the work of Shokurov for the case where $X = \mathbb{P}^3$.

Conjecture 5.1.1. *Suppose that (X, D_X) is a maximal log Calabi–Yau pair and X is a rational threefold. Then (X, D_X) has a toric model.*

The case of $\operatorname{coreg}(\mathbb{P}^3, D) = 2$ occurs if and only if D is an irreducible normal quartic surface with canonical singularities. See Lemma 5.1.6. As pointed out by Ducat, this case is the hardest one in terms of an explicit classification of volume preserving equivalence classes. To illustrate this fact, Oguiso [Og1] exhibited two nonsingular isomorphic quartic surfaces $D, D' \subset \mathbb{P}^3$ such that there is no $\varphi \in \mathrm{Bir}(\mathbb{P}^3)$ mapping D birationally onto D'. By Proposition 3.1.20 any map having this property would automatically be volume preserving for the pairs $(\mathbb{P}^3, D)$ and $(\mathbb{P}^3, D')$.

The first step in a Sarkisov decomposition of a birational map $X/\operatorname{Spec}(\mathbb{C}) \dashrightarrow Y/T$ from a Fano variety of Picard rank 1 to a Mori fibered space always consists of a divisorial extraction with center in X. In the case of $\dim X = 3$ and 0-dimensional center $P \in X$, by [Kaw1, Theorem 1.1], in suitable analytic coordinates this divisorial extraction can be described as the weighted blowup of P with weights $(1, a, b)$, where $\mathrm{GCD}(a, b) = 1$. We call this map a *Kawakita blowup* of P.

In [Gue], Guerreiro studied Sarkisov links initiated by the toric weighted blowup of a point in $\mathbb{P}^3$ or $\mathbb{P}^4$ using variation of GIT, and gave a complete classification of them with a description of the whole Sarkisov link. In the work [ACM], Araujo, Corti & Massarenti considered irreducible normal quartic surfaces with single canonical singularities of types A_1 and A_2 and solved the same problem in the volume preserving context.

The main result of this chapter extends the classification given in [ACM] contemplating more types of surface canonical singularities, the so-called Du Val singularities, which can be corresponded with simple-laced Dynkin diagrams of type ADE. In our context of $\mathrm{coreg}(\mathbb{P}^3, D) = 2$, we have the following:

Theorem 5.1.2 (See Theorem 5.2.1)**.** *Let $(\mathbb{P}^3, D)$ be a log Calabi-Yau pair of coregularity 2 and $\pi\colon (X, D_X) \to (\mathbb{P}^3, D)$ be a volume preserving toric $(1, a, b)$-weighted blowup of a torus invariant point. Then this point is necessarily a singularity of D and, up to permutation, the only possibilities for the weights, depending on the type of singularities, are listed in the following Table 5.1.*

type of singularity	volume preserving weights
A_1	(1,1,1)
A_2	(1,1,1), (1,1,2),
A_3	(1,1,1), (1,1,2), (1,1,3)
A_4	(1,1,1), (1,1,2), (1,1,3), (1,2,3)
A_5	(1,1,1), (1,1,2), (1,1,3), (1,2,3)
A_6	(1,1,1), (1,1,2), (1,1,3), (1,2,3), (1,2,5), (1,3,4)
A_7	(1,1,1), (1,1,2), (1,1,3), (1,2,3), (1,2,5), (1,3,4), (1,3,5)

Table 5.1: Table summarizing volume preserving weights, up to permutation.

The following result is a partial volume preserving version of [Gue, Theorem 1.1] for the case where $\mathrm{coreg}(\mathbb{P}^3, D) = 2$. The toric description of the weighted blowup allows us to encompass all types of strict canonical singularities of type A_n.

Theorem 5.1.3 (See Theorem 5.2.4)**.** *Let $(\mathbb{P}^3, D)$ be a log Calabi-Yau pair of coregularity 2 and $\pi\colon (X, D_X) \to (\mathbb{P}^3, D)$ be a volume preserving toric $(1, a, b)$-weighted blowup of a torus invariant point. Then this point is necessarily a singularity of D and, up to permutation, the* *only possibilities for the weights initiating a volume preserving Sarkisov link, depending on the type of singularities, are listed in the following Table 5.2.*

type of singularity	volume preserving weights
A_1	(1,1,1)
A_2	(1,1,1), (1,1,2)
A_3	(1,1,1), (1,1,2)
A_4	(1,1,1), (1,1,2), (1,2,3)
A_5	(1,1,1), (1,1,2), (1,2,3)
$A_{\geq 6}$	(1,1,1), (1,1,2), (1,2,3), (1,2,5)

Table 5.2: Table summarizing volume preserving weights initiating Sarkisov links, up to permutation.

Remark 5.1.4 (Important)**.** The colorful weights displayed in the Tables 5.1 & 5.2 are not volume preserving for a generic quartic having that type of singularity. By this, we mean that some closed conditions exist on the coefficients of the equation of D in order for these weights to satisfy the volume preserving property. These conditions will be stated later on in this work together with criteria to detect such singularities. See 5.3.2.1.

Remark 5.1.5. Even if the weights $(1,1,3)$ yield a volume preserving divisorial extraction for some D', it will not generate a Sarkisov link. See [Gue, Lemma 3.2] or notice that after computing the whole link through variation of GIT, we will get a codomain with worse than terminal singularities, which is not allowed in the Sarkisov Program [Cor1, HM1]. For the complete description of the Sarkisov links initiated by the remaining weights. See [Gue, Table 1] which we will reproduce later on in this work for the reader's convenience, see Table 5.5.

The classification problem of Calabi-Yau pairs up to volume preserving equivalence can be refined through the coregularity. Few results are known, even for interesting cases of Calabi-Yau pairs such as $(\mathbb{P}^n, D)$, where $D \subset \mathbb{P}^n$ is a hypersurface of degree $n+1$.

The following lemma explains the context of the chapter:

Lemma 5.1.6. *Let $(\mathbb{P}^3, D)$ be a Calabi-Yau pair. One has* $\operatorname{coreg}(\mathbb{P}^3, D) = 2$ *if and only if D is an irreducible normal quartic surface with at worst canonical singularities*

Proof. ($\Rightarrow$) Consider $f\colon (X, D_X) \to (\mathbb{P}^3, D)$ a dlt modification of $(\mathbb{P}^3, D)$. By Theorem 3.2.1 combined with [CK, Theorem 1.7] we have that X is terminal and D_X is irreducible and normal. Since (X, D_X) is a dlt pair, by [KM, Theorem 2.44] there exists a log resolution $g\colon X' \to X$ of (X, D_X) such that $a(E_i, X, D_X) > -1$ for every exceptional divisor $E_i \subset X'$.

Discrepancies with respect to Calabi-Yau pairs are always integer numbers because the log canonical divisor is Cartier. This implies that $a(E_i, X, D_X) \geq 0$ for every exceptional divisor $E_i \subset X'$, and hence (X, D_X) is a (t,c) pair. Lemma 3.1.23 ensures that canonicity is preserved under volume preserving maps, which guarantees that $(\mathbb{P}^3, D)$ is also (t,c). Making use of the Adjunction Formula, then D has canonical singularities and, in particular, it is normal. By Proposition 3.1.20, one has the irreducibility of D.

($\Leftarrow$) Let us show that $(\mathbb{P}^3, D)$ is already a dlt Calabi-Yau pair. Making use of the Adjunction Formula, we have that $(\mathbb{P}^3, D)$ is (t,c). Taking $Z = \mathrm{Sing}(D)$ in Definition 3.1.3 or by [KM, Theorem 2.44], we have that $(\mathbb{P}^3, D)$ is dlt. Then $\mathrm{coreg}(\mathbb{P}^3, D) = 2$ is immediate by Theorem 3.2.1. In particular, notice that $\mathrm{Sing}(D) = \emptyset$ implies that the pair $(\mathbb{P}^3, D)$ is terminal. $\square$

In this chapter, we will study the birational geometry of Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2, which in dimension 3 are such that the ambient varieties have the "closest" *crepant log structure* (see [Kol, Section 4.4]) to being a Calabi-Yau variety. Roughly speaking and based on Ducat's work [Duc], it seems that the bigger the coregularity is, the harder the classification problem up to volume preserving equivalence will be.

5.1.1 A short compendium on singularities of quartics surfaces

To understand the birational geometry of Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2, Lemma 5.1.6 indicates that we must study canonical singularities realizable by irreducible normal quartic surfaces in $\mathbb{P}^3$. In this subsection and the following parts of this chapter, we will refer to such varieties as simply quartic surfaces.

We recall now some facts about singularities of quartic surfaces which will be useful in this work. It is well known that normality implies absence of singularities in codimension 1. Thus the singular locus in this surface case consists of a finite set of points, and therefore the singularities are all isolated.

A first natural problem is to find an upper bound for the cardinality of $\mathrm{Sing}(D)$, that is, the biggest possible number of singular points realizable by D. This can be answered by taking into account the multiplicity of a singular point belonging to $\mathrm{Sing}(D)$.

- Suppose there exists $P \in \mathrm{Sing}(D)$ such that $m_P(D) = 2$, that is, P is a double point. One can show that the projection away from P induces a correspondence between the remaining (possibly) singular points of D and the singular points of an irreducible plane sextic C, which are at most 15 by [Ful1, Theorem 2, Chapter 5]. Therefore, if $\mathrm{Sing}(D)$ contains a double point, then $\sharp(\mathrm{Sing}(D)) \leq 16$ [1].

- Suppose there exists $P \in \mathrm{Sing}(D)$ such that $m_P(D) = 3$, that is, P is a triple point. By [Cat, Proposition 1], we have in this case that $\sharp(\mathrm{Sing}(D)) \leq 7$. More generally, if $X \subset \mathbb{P}^n$ is an irreducible hypersurface of degree d with $\mathrm{Sing}(X)$ a finite set and containing a point of multiplicity $d-1$, then $\sharp(\mathrm{Sing}(X)) \leq 1 + \dfrac{d(d-1)}{2}$ [2].

- Suppose there exists $P \in \mathrm{Sing}(D)$ such that $m_P(D) = 4$. In this case, D is geometrically the cone over an irreducible plane quartic with vertex P. Consider C such quartic

[1]This holds over any ground field of characteristic $\neq 2$.
[2]This holds over any characteristic and only requires the ground field to be algebraically closed.

which lies in a plane section of $\mathbb{P}^3$ not passing through P. Let us fix a such plane section isomorphic to $\mathbb{P}^2$. One can check the following:

$$\operatorname{Sing}(D) = \begin{cases} \{P\}, & \text{if } C \subset \mathbb{P}^2 \text{ is nonsingular} \\ \{\text{lines through } P \text{ and singularities of } C \subset \mathbb{P}^2\}, & \text{otherwise} \end{cases}.$$

In the second case, D is no longer normal, since $\operatorname{codim} \operatorname{Sing}(D) = 1$.

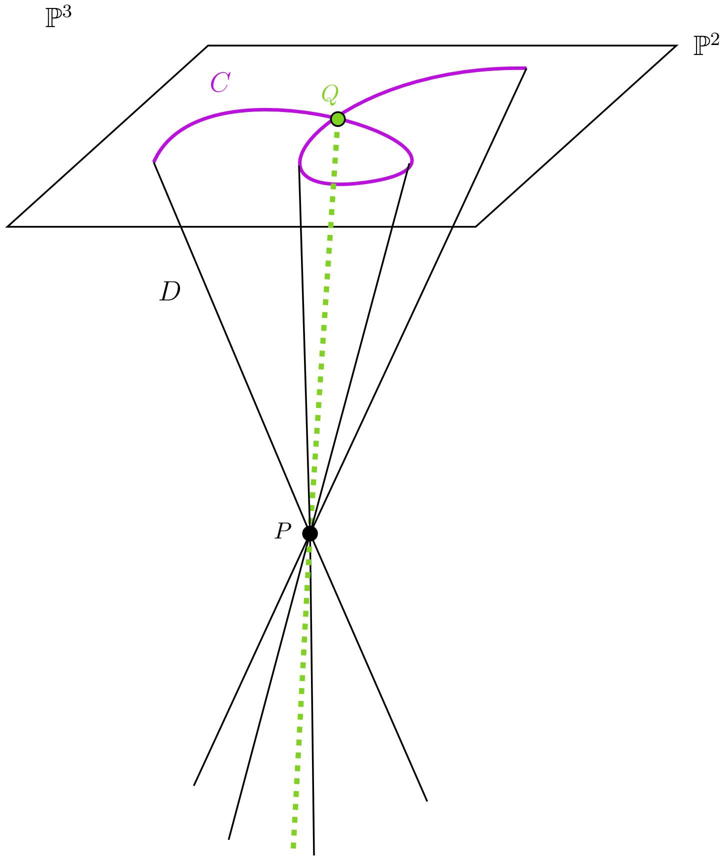


Figure 5.1: Nonnormal quartic.

Still thinking in terms of multiplicities, let us move on to an MMP point of view and allow $P \in D$ to be arbitrary. Set $m := m_P(D)$.

Let $\pi\colon X \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ at P and $E = \operatorname{Exc}(\pi)$. We can think of π as a possible first step in the desingularization process of D at P if $P \in \operatorname{Sing}(D)$. One computes $a(E, \mathbb{P}^3, D) = 2 - m$. This is an indication of the following equivalences: the singularity at P is

$$\left.\begin{array}{r} \text{terminal} \\ \text{canonical} \\ \text{log canonical} \\ \text{worse than log canonical} \end{array}\right\} \text{if and only if } m = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \end{cases}.$$

More generally, terminality for normal surfaces is equivalent to the absence of singularities. See [KM, Theorem 4.5] or [Mat, Theorem 4-6-5]. Since the context of our work leads to quartic surfaces with at worst canonical singularities, we will explicitly describe them later on.

The minimal requirement for singularities from the MMP perspective is log canonical. By the equivalence shown in Lemma 5.1.6, the case of $m = 3$ leads to a Calabi-Yau pair $(\mathbb{P}^3, D)$ with $\operatorname{coreg}(\mathbb{P}^3, D) \leq 1$. In this situation, which in particular includes cases where D is non-normal and not necessarily irreducible, we refer the reader to [Duc, Section 3], which contains a nice summary of the corresponding classification of singularities on quartic surfaces.

5.1.2 Irreducible quartic surfaces with canonical singularities

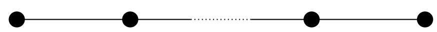
It is well known that canonical surface singularities are precisely nonsingular points together with *Du Val surface singularities*. See for instance [Rei3]. As a matter of fact, there exist several characterizations of such singularities as one can see in the survey by Durfee [Dur].

Definition 5.1.7. A normal surface singularity $P \in S$ is called a *Du Val* surface singularity if X has a minimal resolution of singularities $\phi\colon \tilde{S} \to S$ such that $K_{\tilde{S}} \cdot E = 0$ for every exceptional curve $E \subset \tilde{S}$.

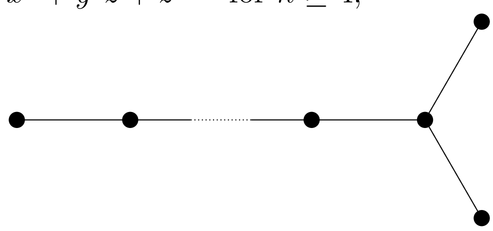
It turns out that the resolution graph of a Du Val singularity can be corresponded to a finite $\mathbb{Z}$-module admitting a symmetric bilinear form with certain properties. See [Sha1, Section 4.3]. In this case, the configuration of a such resolution graph is determined by a simple-laced Dynkin diagram of type ADE.

It can be proved that a Du Val surface singularity $P \in S$ is analytically isomorphic to a surface singularity $0 \in \{f = 0\} \subset \mathbb{A}^3_{(x,y,z)}$, where f is one of the following equations together with the corresponding resolution graph:

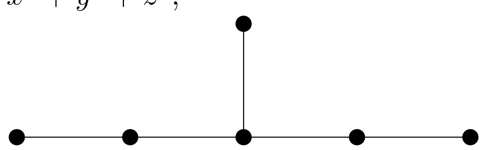
A_n: $x^2 + y^2 + z^{n+1}$ for $n \geq 1$,



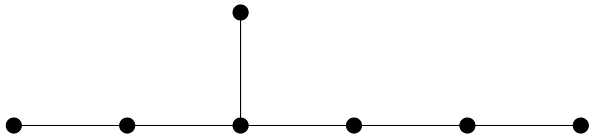
D_n: $x^2 + y^2 z + z^{n-1}$ for $n \geq 4$,



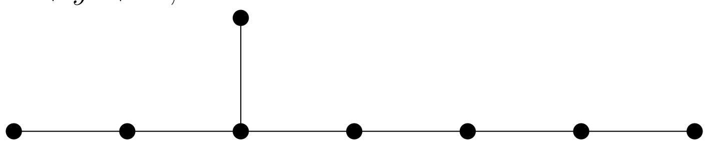
E_6: $x^2 + y^3 + z^4$,



E_7: $x^2 + y^3 + yz^3$,



E_8: $x^2 + y^3 + z^5$,



Notice that we can distinguish classes of Du Val singularities in terms of the projectivized tangent cone at the singularity. Indeed, we have three different behaviors

type of singularity	projectivized tangent cone
A_1	irreducible (and nonsingular) conic
$A_n, n \geq 2$	two concurrent lines
D-E	double line

Table 5.3: Du Val singularities and projectivized tangent cone.

Du Val surface singularities appear in many different contexts. They are also denominated *rational double points* because, as we have just seen, they are double points in certain suitable analytic coordinates.

Just to emphasize the numerous and diverse proprieties enjoyed by these singularities, let us mention two of them whose proofs can be found in [Dur]:

- All Du Val surface singularities appear as singularities of quotient surfaces $\mathbb{C}^2/G$, where G is a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. The group corresponding to the Dynkin diagram A_n is the cyclic group of order n and the groups corresponding to the remaining ones are not abelian.

- Following the previous notation and as a consequence of the Adjunction Formula, there exists a minimal resolution $\phi \colon \tilde{S} \to S$ such that $\mathrm{Exc}(\phi)$ consists of a tree of (-2)-curves having a configuration given by the corresponding Dynkin diagram, and such that $K_{\tilde{S}} = \phi^* K_S$. In other words, $K_{\tilde{S}}$ is trivial in a neighborhood of $\mathrm{Exc}(\phi)$. In this case, all the discrepancies are zero and ϕ is called a *crepant resolution.*

In [IN, Um1, Um2] minimal desingularizations of a normal quartic surface are classified. The case of nonrational singularities leads us to more possibilities and that of rational singularities (or equivalently canonical singularities) leads us to K3 surfaces.

Lemma 5.1.8. *Let $S \subset \mathbb{P}^3$ be a normal quartic surface with canonical singularities. Denote $\phi \colon \tilde{S} \to S$ to be its minimal resolution of singularities. Then $\tilde{S}$ is a K3 surface.*

Proof. Such minimal resolution can be taken crepant, that is, $K_{\tilde{S}} = \phi^* K_S$. By the Adjunction Formula, one can deduce that $K_S \sim 0$ and hence $K_{\tilde{S}} \sim 0$. Since S is a complete intersection of dimension 2, [Bea, Lemma VIII.9] guarantees that $q = h^1(S, \mathcal{O}_S) = 0$. The irregularity q is a birational invariant, so $q = 0$ also for $\tilde{S}$. By the Enriques-Kodaira classification of surfaces, we get that $\tilde{S}$ is a K3 surface. $\square$

Keeping the same notation of Lemma 5.1.8, one can check that the line bundle $\mathcal{L} := \phi^*(\mathcal{O}_{\mathbb{P}^3}(1))$ is nef, $\deg(\mathcal{L}) = B^2 = 4$ and $h^0(\tilde{S}, \mathcal{L}) = 4$, where $B \in \phi^*|H|$ is a Weil divisor such that $\mathcal{L} \simeq \mathcal{O}_{\tilde{S}}(B)$ and $|H|$ denotes the complete linear system of planes in $\mathbb{P}^3$. See [Ur2]. From the properties of linear systems and associated rational maps, it is immediate that ϕ is induced by the complete linear system $\phi^*|H|$.

Conversely, a nef line bundle $\mathcal{L} \simeq \mathcal{O}_{\tilde{S}}(B)$ of degree 4 on $\tilde{S}$, where B is a Weil divisor, under some suitable conditions explained in [Ur2], defines a morphism $\phi_{|B|} \colon \tilde{S} \to \mathbb{P}^3$ such that the image is a quartic surface having canonical singularities.

That is an idea to find some minimal models for canonical singularities, but the hard part is to find an explicit description of the quartic surface $\phi_{|B|}(\tilde{S})$ in terms of its equation.

Notice that for a K3 surface to be a nonsingular model of a normal quartic surface with strict canonical singularities, it must contain (-2)-curves. A normal quartic surface with terminal singularities is necessarily nonsingular and actually, it is already a K3 surface. In this case, the quartic itself is its minimal model since K3 surfaces do not have (-1)-curves due to the Adjuction Formula for curves.

As an interesting fact, conversely, the canonical models of K3 surfaces are precisely normal surfaces with canonical singularities. Usually, these canonical models are obtained by following an MMP perspective contracting the (-1)-curves first, followed by the (-2)-curves. The contractions of the (-2)-curves exist due to the Contraction Theorem by Artin. See [Rei1, Section 4.15].

5.1.3 Configuration of canonical singularities on quartic surfaces

The classification of singularities on varieties is not an easy task, and even worse is to determine the possible configurations of them. Some satisfactory results were established in certain cases, but the existence of varieties realizing many of these configurations remains open so far.

In our case of quartic surfaces, it is not practical to deal with the problem by direct analysis of its homogeneous equation. There exist many papers on this question in the last decades. In the case where the quartic surface only has canonical singularities, some answers were given by Kato & Naruki and Urabe [KN, Ur1, Ur2]. Some years after their works, an explicit enumeration of configurations of such singularities was obtained by Yang [Ya]. We warn the reader that the list in [Ya] is insanely enormous.

All those authors based their works on the fact given by Lemma 5.1.8 and the very rich geometry of K3 surfaces, as we now explain.

A *Dynkin graph* is a disjoint finite union of connected Dynkin graphs (diagrams) of types A, D or E. Let S be a normal quartic surface with canonical singularities. For each integer $k \geq 1$ and each integer $l \geq 4$, let a_k and b_l denote the numbers of singularities of types A_k and D_l on S, respectively. For $m \in \{6, 7, 8\}$, let c_m denote the number of singularities of type E_m.

One can show that a minimal resolution of S can be obtained by a composition

$$S_s \to S_{s-1} \to \cdots \to S_1 \to S_0 =: S$$

where each map $S_i \to S_{i-1}$ is the blowup of a rational double point over another one in S and $\tilde{S} := S_s$ is a K3 surface. Denote by ϕ such a composition.

It is immediate to see that the resolution graph, in this case, is the Dynkin graph $G = \sum a_k A_k + b_l D_l + c_m E_m$. The number $r = r(G) = \sum a_k k + b_l l + c_m m$ is called the *rank* of G. This number is bounded above by 19 and this turns out to be a necessary condition for a Dyinkin graph to correspond to the resolution graph of a quartic surface having only canonical singularities.

Indeed, the classes of all components of the exceptional divisor $\mathrm{Exc}(\phi)$ are $\mathbb{Q}$-linearly independent in the Picard group of $\tilde{S}$ over $\mathbb{Q}$. We can see this by noticing that each one of them has self-intersection $-2 < 0$ and therefore the Contraction Theorem by Artin [Rei1, Section 4.15] implies in the existence of a contraction $\pi\colon \tilde{S} \to S'$. Let $C := \mathrm{Exc}(\pi)$ be such a (-2)-curve.

We have that $\mathrm{Pic}_{\mathbb{Q}}(\tilde{S}) = \pi^* \mathrm{Pic}_{\mathbb{Q}}(S') \oplus \mathbb{Q} \cdot [C]$, where $\mathrm{Pic}_{\mathbb{Q}}(S) := \mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, each component (vertex of G) contributes with 1 to the Picard number of $\tilde{S}$. Hence, we have the relation $\rho(S) + r = \rho(\tilde{S})$.

It is well-known that for K3 surfaces, the Picard number is bounded above by 20. Given a K3 surface Z, there exists an injection of lattices $\mathrm{Pic}(Z) \hookrightarrow H^2(Z, \mathbb{Z})$ via the induced long exact sequence in cohomology of the exponential sequence of Z. Such injection induces an isometry $\mathrm{Pic}(Z) \simeq H^{1,1}(Z) \cap H^2(Z, \mathbb{Z})$. By Hodge Theory computations, one knows $h^{1,1}(Z) = 20$ whence the upper bound is obtained. See [Huy, Subsections 1.3.2 & 1.3.3].

Thus the problem of determining when a Dynkin graph G with $r \leq 19$ occurs as a resolution graph in our context becomes a purely lattice-theoretic problem. Computational methods allowed Kato & Naruki, Urabe and Yang to obtain their results.

By [Ya, Theorem 2.1] all the possible Dynkin graphs G with $r(G) \leq 14$ are realizable as resolution graphs in our context whereas some of the remaining ones with $15 \leq r(G) \leq 19$ are not. See [Ya] for explicit descriptions.

Furthermore, the very interesting connection between $\rho(S)$ and $\rho(\tilde{S})$ allows us to give an upper bound for $\rho(S)$ which must be at most $20 - r$.

Example 5.1.9. Let $S \subset \mathbb{P}^3$ be a quartic surface having a single singularity of type A_n or D_n. It follows that n is at most 19 in both cases by the previous paragraphs. We have the existence, respectively non-existence, of such surface S with an A_{19} singularity (therefore unique), respectively D_{19}, due classification of sublattices of type A_{19} and D_{19} in the K3 lattice [KN, Propositions 1 & 2], [Ur3, Corollary 0.3 & Proposition 3.5]. Kato & Naruki

[KN] show that there exists a unique $S \subset \mathbb{P}^3$ having an A_{19} singularity, up to automorphisms of $\mathbb{P}^3$. In fact, taking $P = (1:0:0:0) \in \mathbb{P}^3_{(w:x:y:z)}$ to be the singular point of S, its equation in affine coordinates in $\{w \neq 0\} \simeq \mathbb{A}^3_{(x,y,z)}$ is given by

$$\begin{aligned}16(x^2+y^2)+32xz^2-16y^3+16z^4-32yz^3+8(2x^2-2xy+5y^2)z^2\\ +8(2x^3-5x^2y-6xy^2-7y^3)z+20x^4+44x^3y+65x^2y^2+40xy^3+41y^4=0.\end{aligned}$$

In particular, we must have $\rho(S)=1$. It is also possible to justify the uniqueness of S in such a case by analyzing the corresponding coarse moduli space of quartics.

Example 5.1.10. From the local analytic description of surface canonical singularities, we can produce some straightforward examples of quartic surfaces in $\mathbb{P}^3$ having a single canonical singularity. Allowing these local analytic coordinates to be global, we have the following examples, where we take $P = (1:0:0:0) \in \mathbb{P}^3_{(w:x:y:z)}$ to be the singular point and write the equation in affine coordinates $\{w \neq 0\} \simeq \mathbb{A}^3_{(x,y,z)}$:

A_3: $x^2+y^2+z^4$,

D_5: $x^2+y^2z+z^4$,

E_6: $x^2+y^3+z^4$,

E_7: $x^2+y^3+yz^3$.

One way to produce more explicit examples is to study the homogeneous equation of degree 4 by means of finding relations between the coefficients so that we have such singularities. This work is considerably hard, but it was done so that we can show Theorem 5.2.1. See criteria in 5.3.2.1.

Deformation classification. There also exists a deformation classification of quartic surfaces with canonical singularities. Such varieties are called *simple quartics* by Aktaş in [Ak1, Ak2]. Her work is also based on Lattice Theory and assures the existence of configurations of canonical singularities obtained by a perturbation of certain sets of singularities of type ADE. In particular, we have the existence of a quartic surface with a single singularity D_{18} [Ak1, Theorem 2]. The complete classification was initially done in [Ak1] for *non-special quartics*, whose definition is quite technical, and completed in [Ak2] contemplating the complementary cases.

5.1.4 Explicit resolution of Du Val singularities

An interesting property of singularities is that the way in which they are resolved (usually through a minimal resolution) is independent of which variety they live in. In the case of canonical surface singularities, as we have mentioned before, it is known that the intersection graph of the exceptional divisor of a minimal resolution is given by a simple-laced Dynkin diagram of type ADE.

In this subsection, we discuss the resolution of the Du Val singularities of type A_n. We determine how many blowups at singular points we need to do, and describe the configuration of the singularities in the intermediate steps. By the previous paragraph, the same behavior can be observed in the resolution process of a quartic surface in $\mathbb{P}^3$.

Since discrepancies are invariants of the corresponding singularities (centers of the respective divisors), we can use convenient models to compute them. In our case, we will use the normal forms of the Du Val singularities which arise after a suitable local biholomorphism in the analytic category.

Lemma 5.1.11. *The resolution of a Du Val singularity of type A_n can be reached after $\left\lceil \frac{n}{2} \right\rceil$ blowups at nonsingular points of the ambient space.*

Proof. Let D be an irreducible normal surface having a Du Val singularity of type A_n. We may assume that $D = V(x_1^2 + x_2^2 + x_3^{n+1}) \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$.

Identify $\mathbb{A}^3_{(x_1,x_2,x_3)} \simeq \{x_0 \neq 0\} \subset \mathbb{P}^3_{(x_0:x_1:x_2:x_3)}$ so that the singularity corresponds to $P = (1:0:0:0)$. The automorphism of $\mathbb{P}^3$ given by

$$(x_0 : x_1 : x_2 : x_3) \mapsto \left(x_0 : \frac{-x_1 + x_2}{2} : \frac{-i(x_1 + x_2)}{2} : x_3\right)$$

takes D to the surface $V(x_0^{n-1}x_1x_2 - x_3^{n+1}) \subset \mathbb{P}^3$ and P to itself. The latter has an equation easier to manipulate.

By abuse of notation, let us still denote such surface by D. In $\{x_0 \neq 0\}$ it has the equation $\{x_1x_2 - x_3^{n+1} = 0\}$ with the same type of singularity at P.

Let us blow up the point P. Set $f = x_1x_2 - x_3^{n+1}$ and $\pi \colon X \to \mathbb{P}^3$ the blowup of $\mathbb{P}^3$ at P. Since $P \in \{x_0 \neq 0\}$, we must only analyze what occurs in $\pi^{-1}(\{x_0 \neq 0\}) \simeq \mathrm{Bl}_0(\mathbb{A}^3)$, which by abuse of notation we will still denote by X.

We can describe X as

$$V(\{x_iy_j - x_jy_i; 1 \leq i,j \leq 3\}) \subset \mathbb{P}^3_{(x_0:x_1:x_2:x_3)} \times \mathbb{P}^2_{(y_1:y_2:y_3)}.$$

In this case $\mathrm{Exc}(\pi) =: E \simeq \mathbb{P}^2$.

One has X is covered by three affine charts $W_i := X \cap \{y_i \neq 0\} \simeq \mathbb{A}^3$ for $i \in \{1,2,3\}$. Abusing the notation letting $W_i \simeq \mathbb{A}^3_{(x_1,x_2,x_3)}$ for all i, one can check that the equation $f_1 = 0$ of $\pi_*^{-1}D =: \tilde{D}$, the strict transform of D, in the charts W_1, W_2 and W_3 is given respectively by

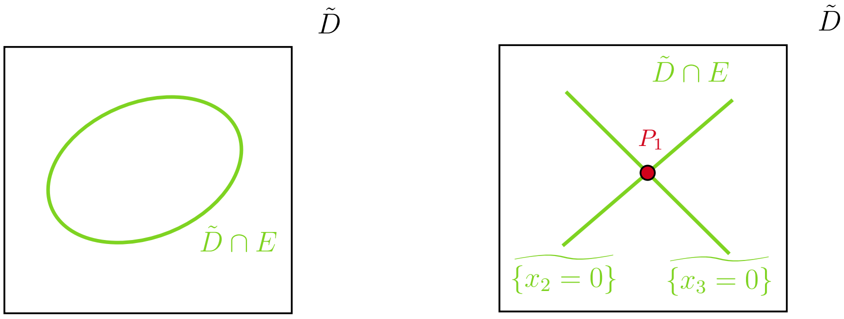
$$\begin{aligned} x_2 - x_1^{n-1}x_3^{n+1} &= 0, \\ x_1 - x_2^{n-1}x_3^{n+1} &= 0 \text{ and} \\ x_1x_2 - x_3^{n-1} &= 0. \end{aligned}$$

Furthermore, one has $E \cap W_i = \{x_i = 0\}$.

Since $\dfrac{\partial f_1}{\partial x_2} = 1 \neq 0$ and $\dfrac{\partial f_1}{\partial x_1} = 1 \neq 0$ in the affine charts W_1 and W_2, respectively, we have that $\tilde{D}$ is nonsingular in both charts by the Jacobian Criterion.

In the affine chart W_3, from the equation of $\tilde{D}$ it follows that $\tilde{D}$ has a Du Val singularity of type A_{n-2} at 0 if $n \geq 3$ and it is nonsingular if $n \in \{1, 2\}$. In fact, making use of the Jacobian Criterion for $n = 1$, we have that the mutual vanishing of all partial derivatives implies that the points which are candidates to be singularities are of the form $(0, 0, \lambda)$, where $\lambda \in \mathbb{C}$. But no point of this form belongs to $\tilde{D}$ in the affine chart W_3. For $n = 2$ and by the same criterion, this is a consequence of $\dfrac{\partial f_1}{\partial x_3} = 1 \neq 0$.

We have the following picture for the exceptional divisor on $\tilde{D}$.



conic if $n = 1$ two concurrent lines if $n \geq 2$

Figure 5.2: Exceptional divisor on $\tilde{D}$.

Now we are ready to show the statement of the lemma. Let us show it by induction on n. The basis of induction $n = 1$ is done, since from the previous paragraphs only $1 = \left\lceil \dfrac{1}{2} \right\rceil$ blow up is necessary to resolve the singularity of D.

Suppose that the statement holds for $n - 1 > 0$. Let us show that it also holds for n.

After performing the first blowup, we have that $\tilde{D}$ has a Du Val singularity of type A_{n-2} in one of the charts and it is nonsingular in the others. By the induction hypothesis, we can resolve such singularity after $\left\lceil \dfrac{n-2}{2} \right\rceil$ blowups at nonsingular points of the ambient space.

Thus, using this resolution for the singularity of $\tilde{D}$ in addition to the first blowup, we can resolve the singularity of D after $1 + \left\lceil \dfrac{n-2}{2} \right\rceil = \left\lceil \dfrac{n}{2} \right\rceil$ of such blowups.

$\square$

The previous lemma tells us that each point blowup weakens the singularity from A_n to A_{n-2}. From this, we can also infer the behavior of the corresponding exceptional divisor along this resolution process. The dynamic of the exceptional divisor of an A_n singularity is to grow from the middle to the “outermost” components. The following pictures illustrate this phenomenon:

The next lemma says that we can resolve a Du Val singularity of type A_n by blowing up rational curves $C \simeq \mathbb{P}^1$ passing through the singularity.

Lemma 5.1.12. *A general Du Val singularity of type A_n can be resolved by blowing up the singular point followed by $n - 2$ blowups along components of the exceptional divisor of the previous blowups.*

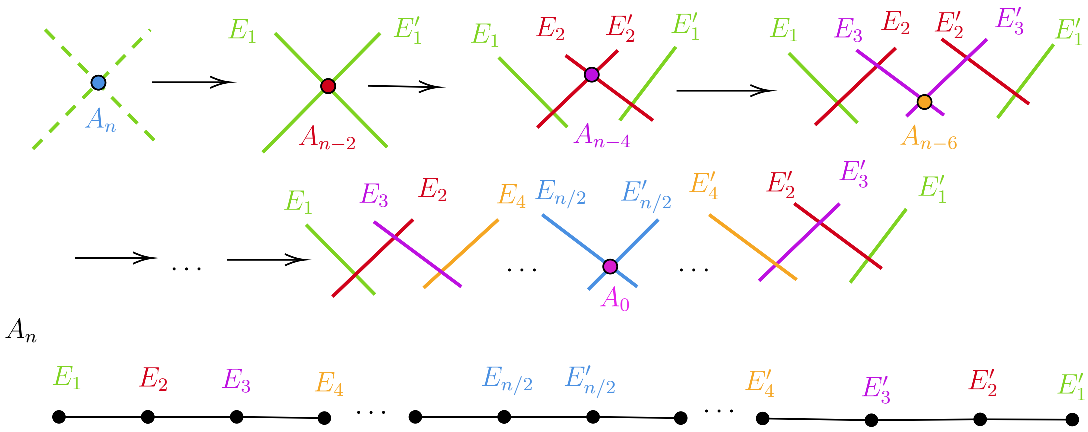


Figure 5.3: Resolution of the singularity A_n for n even.

Proof. We will keep the same notation and identifications as in Lemma 5.1.11 with the appropriate adjustments.

Without loss of generality, after the first blowup at the singular point, we may assume $D = V(x_1x_2 - x_3^{n+1}) \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$. In this case, by Lemma 5.1.11 we started with an A_{n+2} singularity.

Set $L \coloneqq \{x_1 = x_3 = 0\} \subset D$ and consider $\pi \colon X \to \mathbb{A}^3$ the blowup of $\mathbb{A}^3$ along L.

We can describe X as

$$V(x_1y_3 - x_3y_1) \subset \mathbb{A}^3_{(x_1,x_2,x_3)} \times \mathbb{P}^2_{(y_1:y_3)}.$$

One has X is covered by two affine charts $V_i \coloneqq X \cap \{y_i \neq 0\} \simeq \mathbb{A}^3$ for $i \in \{1,3\}$. Abusing the notation letting $V_i \simeq \mathbb{A}^3_{(x_1,x_2,x_3)}$ for all i, one can check that the equation $f_1 = 0$ of $\pi_*^{-1}D =: \tilde{D}$, the strict transform of D, in the charts V_1 and V_3 is given respectively by

$$x_2 - x_1(x_1x_3)^n = 0 \text{ and } x_1x_2 - x_3^n = 0.$$

Since $\dfrac{\partial f_1}{\partial x_2} = 1 \neq 0$ in the affine chart V_1, we have that $\tilde{D}$ is nonsingular on it by the Jacobian Criterion.

In the affine chart V_3, from the equation of $\tilde{D}$ it follows that $\tilde{D}$ has a Du Val singularity of type A_{n-1} at 0 if $n \geq 2$ and it is nonsingular if $n = 1$. In fact, making use of the Jacobian Criterion for the second case, we have that $\dfrac{\partial f_1}{\partial x_3} = 1 \neq 0$.

We have exactly the same picture as in Lemma 5.1.11 for the exceptional divisor on $\tilde{D}$. See Figure 5.2.

Now the statement of the lemma follows directly by induction on n because the respective blowup weakens the singularity from A_n to A_{n-1}.

□

The previous lemma tells us that it is possible to resolve a singularity by blowing up along a subvariety containing it.

The proof of the Lemma 5.1.12 also gives us an illustration of another fact: the blowup of a variety along a divisor that is not Cartier is not isomorphic to the variety itself. Indeed, for all $n \geq 1$ the prime divisor on D given by L as above is not Cartier. See [Har, Example

6.11.3] for $n = 1$ and adapt the same arguments for the case of $n > 1$. For $n = 1$ we have just checked that $\tilde{D}$ is nonsingular and hence $\mathrm{Bl}_L(D) = \tilde{D} \not\simeq D$.

Actually what is hidden behind the most known interpretation of the blowup as a tool to resolve singularities is the fact that blowups produce Cartier divisors. More precisely, the exceptional divisor produced is always an effective Cartier divisor. See [EH, Definition IV-16] and [Vak, Section 22.2] for more details.

5.2 Volume preserving Sarkisov links

According to Definition 3.1.17, volume preserving Sarkisov links are Sarkisov links endowed with additional data and property.

The aim of this section is to study volume preserving Sarkisov links whose domain is a Calabi-Yau pair $(\mathbb{P}^3, D)$ with $\mathrm{coreg}(\mathbb{P}^3, D) = 2$. Recall that in this case, by Lemma 5.1.6, D is an irreducible normal quartic surface with canonical singularities. The main result is the following:

Theorem 5.2.1. *Let $(\mathbb{P}^3, D)$ be a log Calabi-Yau pair of coregularity 2 and $\pi\colon (X, D_X) \to (\mathbb{P}^3, D)$ be a volume preserving toric $(1, a, b)$-weighted blowup of a torus invariant point. Then this point is necessarily a singularity of D and, up to permutation, the only possibilities for the weights, depending on the type of singularities, are listed in the following Table 5.4.*

type of singularity	volume preserving weights
A_1	(1,1,1)
A_2	(1,1,1), (1,1,2),
A_3	(1,1,1), (1,1,2), (1,1,3)
A_4	(1,1,1), (1,1,2), (1,1,3), (1,2,3)
A_5	(1,1,1), (1,1,2), (1,1,3), (1,2,3)
A_6	(1,1,1), (1,1,2), (1,1,3), (1,2,3), (1,2,5), (1,3,4)
A_7	(1,1,1), (1,1,2), (1,1,3), (1,2,3), (1,2,5), (1,3,4), (1,3,5)

Table 5.4: Table summarizing volume preserving weights, up to permutation.

Given a Mori fibered space $X/\operatorname{Spec}(\mathbb{C})$, where X is a Fano variety of Picard rank 1, the first step in the Sarkisov decomposition of any birational map with source $X/\operatorname{Spec}(\mathbb{C})$ is a divisorial extraction $\pi\colon Y \to X$ which will initiate the first link. This first link is necessarily of type I or II because $\rho(X) = 1$.

A divisorial contraction from a terminal variety to a nonsingular Fano threefold is either the blowup of a curve or the weighted blowup of a point in local analytic coordinates. The latter is a consequence of a result by Kawakita [Kaw1, Theorem 1.1] which says that for suitable analytic coordinates at the point, this divisorial extraction can be described as the weighted blowup with weights $(1, a, b)$, where $\mathrm{GCD}(a, b) = 1$. We call this map a *Kawakita blowup* of the point.

We point out that if one disregards the nonsingularity assumption on the threefold, additional classes of divisorial contractions beyond these last two may arise. See [Tzi, Zik].

Now consider a reduced Weil divisor D on X such that $(X, D) \to \operatorname{Spec}(\mathbb{C})$ has the structure of a Mori fibered Calabi-Yau pair. By the previous considerations and Proposition 3.1.25, the possible volume preserving divisorial extractions are

$$\pi = \begin{cases} \mathrm{Bl}_C, & \text{blowup of a nonsingular curve } C \subset D^{\mathrm{reg}} := D \setminus \operatorname{Sing}(D), \\ \mathrm{Bl}_{(1,a,b)}, & \text{weighted blowup of a point with } \operatorname{GCD}(a,b) = 1 \end{cases}.$$

We will be interested in the case of Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2. By Lemma 5.1.6 this is equivalent to requiring that D is irreducible and normal and it has at worst canonical singularities. We will see at a glance that such blown up point must be a singularity of D. The following lemma deduced from Proposition 3.1.25 can be readily verified for the case where the divisorial extraction is an ordinary blowup by computing some discrepancies and comparing divisors.

Lemma 5.2.2. *Let $\pi\colon (Y, D_Y) \to (X, D_X)$ be a volume preserving terminal divisorial extraction between threefold Calabi-Yau pairs contracting a divisor $E \subset Y$ to a closed point $P \in X$. Assume (X, D_X) has canonical singularities. Then P is a singularity of D_X, and D_Y is the strict transform of D_X in Y.*

Proof. By the first part of Proposition 3.1.25, $P \in D_X$. Suppose P is not a singularity of D_X. For surfaces, being terminal at a point is equivalent to being nonsingular at a point. So D_X is terminal at P, and hence by Proposition 3.1.25 we must have $\operatorname{codim}_X P = 2$. But this is absurd, since $\operatorname{codim}_X P = 3$. Therefore $P \in \operatorname{Sing}(D_X)$. By Proposition 3.1.20 we get that $D_Y = \pi_*^{-1} D_X$. □

Strict (or purely) canonical singularities for surfaces are the same as Du Val singularities as explained in Subsection 5.1.2. Our problem is the following:

Given the Du Val singularity A_n at a point $P \in D$, to determine for which weights $(1, a, b)$ the Kawakita blowup $\pi\colon (X, \tilde{D}) \to (\mathbb{P}^3, D)$ of P with weights $(1, a, b)$ is volume preserving.

The following result by Guerreiro will allow us to restrict our possibilities for the weights if we are interested in those ones inducing Sarkisov links.

Theorem 5.2.3 (cf. [Gue] Theorem 1.1)**.** *Let $\varphi\colon X \to \mathbb{P}^3$ be the toric $(1, a, b)$-weighted blowup of a point. Then φ initiates a (toric) Sarkisov link from $\mathbb{P}^3$ if and only if, up to permutation of a and b,*

$$(a, b) \in \{(1, 1), (1, 2), (2, 3), (2, 5)\}.$$

The following table extracted from [Gue, Table 1] describes the whole Sarkisov link initiated by the toric weighted blowup of a point whose weights are listed in Theorem 5.2.3. We refer the reader to [Gue] for more details.

(a,b)	τ	φ'	Mori fibered space
$(1,1)$		Fibration	$\mathbb{P}^1$-bundle over $\mathbb{P}^2$
$(1,2)$		Divisorial Contraction to $\mathbb{P}^1$	$\mathbb{P}(1,1,1,2)$
$(2,3)$	$(1,1,-1,-2)$	$(1,1,2)$-Weighted blowup of a smooth point	$\mathbb{P}(1,1,2,3)$
$(2,5)$	$(1,1,-1,-4)$	Kawamata blowup of $\frac{1}{3}(1,1,2)$	$\mathbb{P}(1,3,4,5)$

Table 5.5: Table summarising the results obtained by Guerreiro [Gue, Table 1].

The first column denotes the weights of the $(1,a,b)$-Kawakita blowup of a coordinate point. The second column is a GIT description of terminal small $\mathbb{Q}$-factorial modification (flips, flops, antiflips when that is the link includes them). The third column is either the fibration or the last birational morphism in the link, and the last column denotes the new Mori fibered space.

We remark that the ordinary blowup initiates a Sarkisov link of type I, and the remaining weights initiate Sarkisov links of type II.

The following result is a partial volume preserving version of Theorem 5.2.3 for the case where $\operatorname{coreg}(\mathbb{P}^3, D) = 2$.

Theorem 5.2.4. *Let $(\mathbb{P}^3, D)$ be a log Calabi-Yau pair of coregularity 2 and $\pi\colon (X, D_X) \to (\mathbb{P}^3, D)$ be a volume preserving toric $(1,a,b)$-weighted blowup of a torus invariant point. Then this point is necessarily a singularity of D and, up to permutation, the only possibilities for the weights initiating a volume preserving Sarkisov link, depending on the type of singularities, are listed in the following Table 5.6.*

type of singularity	volume preserving weights
A_1	(1,1,1)
A_2	(1,1,1), (1,1,2)
A_3	(1,1,1), (1,1,2)
A_4	(1,1,1), (1,1,2), (1,2,3)
A_5	(1,1,1), (1,1,2), (1,2,3)
$A_{\geq 6}$	(1,1,1), (1,1,2), (1,2,3), (1,2,5)

Table 5.6: Table summarizing volume preserving weights initiating Sarkisov links, up to permutation.

Since we will deal with (t,c) Calabi-Yau pairs, by Proposition 3.1.20 and Lemma 3.1.23, the boundary divisors with respect to which we want the Sarkisov link to be volume preserving are precisely the strict transforms of the initial one.

Before continuing to deal with this problem, we check that our hypotheses are nonempty, i.e., that there exists an irreducible normal quartic surface having a canonical singularity of type A_n. The answer is given by Kato & Naruki, Urabe and Yang [KN, Ur1, Ur2, Ur3, Ya] who analyzed all the possibilities of combinations of singularities in our context, as explained in Subsection 5.1.3. For the cases A_n and D_n, for instance, we know that n is at most 19 and 18, respectively. For the remaining case E_n, there also exists a quartic surface with this type of singularity.

The volume preserving property of the toric $(1, a, b)$-weighted blowup can be detected by the vanishing of $a(E, \mathbb{P}^3, D)$, the discrepancy of $E = \operatorname{Exc}(\pi)$ with respect to $(\mathbb{P}^3, D)$. Indeed, we have the following:

$$(1, a, b)\text{-weighted blowup is volume preserving} \Leftrightarrow a(E, \mathbb{P}^3, D) = 0.$$

The $(\Rightarrow)$ direction holds because π volume preserving implies that $a(E, X, \tilde{D}) = a(E, \mathbb{P}^3, D)$, and $a(E, X, \tilde{D}) = 0$ because $E \subset X$ and $E \not\subset \operatorname{Supp}(\tilde{D})$, that is, E is not a component of $\tilde{D}$. The $(\Leftarrow)$ direction holds due to Proposition 3.1.12 and Lemma 3.1.23, since in this case we have $K_X + \tilde{D} = \pi^*(K_{\mathbb{P}^3} + D)$.

Concerning the definition and main ideas involving weighted projective spaces and weighted blowups, we refer the reader to [CKS, CLS, Ful2]. They can be realized as toric varieties and geometric quotients. This realization will be crucial in our work.

We usually ask that the $(n+1)$-tuple of positive integers defining the weights consists of coprime elements.

First approach to the problem: Let $\sigma = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ and consider $\pi \colon X \to \mathbb{P}^n$ the σ-weighted blowup of $\mathbb{P}^n$ at the point $P = (1 : 0 : \ldots : 0)$. Set $E \coloneqq \operatorname{Exc}(\pi)$.

We define the σ-weight of a monomial $M = x_1^{p_1} \cdots x_n^{p_n}$ as $\operatorname{wt}_\sigma(M) = p_1 a_1 + \cdots + p_n a_n$. Given a nonzero homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, write its dehomogenization with respect to the variable x_0 as

$$f_* = \sum_I \alpha_I M_I \in \mathbb{C}[x_1, \ldots, x_n],$$

where M_I runs through the monomials in $x_1, \ldots, x_n$.

The σ-weight of f is defined by

$$\operatorname{wt}_\sigma(f) \coloneqq \min\{\operatorname{wt}_\sigma(M_I) | \alpha_I \neq 0\}.$$

Let $D = V(f) \subset \mathbb{P}^n$ be a hypersurface. Set $\operatorname{wt}_\sigma(D) \coloneqq \operatorname{wt}_\sigma(f)$. By [Hay, Lemma 3.5] we have

$$\begin{aligned} \tilde{D} &= \pi^* D - (\operatorname{wt}_\sigma(D))E, \text{ and} \\ K_X &= \pi^* K_{\mathbb{P}^n} + (a_1 + \cdots + a_n - 1)E \ . \end{aligned}$$

Assume $1 \leq a < b$ in the context of our problem. Thus one has $a(E, \mathbb{P}^3, D) = a + b - \operatorname{wt}_\sigma(D)$. The $(1, a, b)$-weighted blowup is volume preserving if and only $a(E, \mathbb{P}^3, D) = 0$, that is, we must have $a + b = \operatorname{wt}_\sigma(D)$.

Thus, one can proceed with a straightforward analysis of the homogeneous equation f that defines D. This can be done by verifying a lot of conditions on f which imply that certain monomials will not appear. However, this analysis will be very tedious if we make the singularity worse. Moreover, this strategy depends on the coordinates chosen.

Recalling that our problem is to determine "volume preserving weights", our task is to check when $a(E, \mathbb{P}^3, D) = 0$. Realizing this discrepancy by a divisorial extraction π is relatively feasible because weighted blowups can be described in terms of charts and they locally look like affine spaces up to an unramified cover. However, this would require a lot of discrepancy computations.

Using the property that discrepancies only depend on the valuations associated to divisors, we can have an alternative way to perform the task of finding the volume preserving weights. There exists a realization of the divisorial valuation ν_E associated to E based on a toric description of the weighted blowup. This was the same strategy adopted by Araujo, Corti & Massarenti [ACM], which has the advantage of being coordinate-free.

As a toric variety, X is determined by the fan Σ in $\mathbb{R}^3$ given by the union of all possible 3-dimensional cones and its subcones generated by the vectors in $\{v_0, v_i, v_j\}$ and in $\{v_i, v_j, v\}$ for $i, j \in \{1, 2, 3\}$, where $v_0 = (-1, -1, -1)$, $v_i = e_i$ for $i \in \{1, 2, 3\}$, and $v = (1, a, b)$.

In the following picture, we depict the fan Σ, where the colorful arrows indicate the 3-dimensional cones generated by the vectors $\{v_i, v_j, v\}$ for $i, j \in \{1, 2, 3\}$.

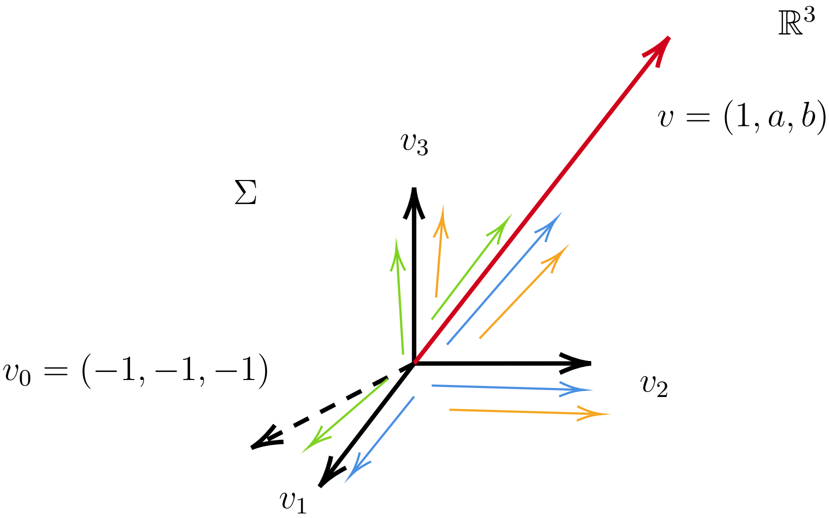


Figure 5.4: Fan of the $(1, a, b)$-weighted blowup of $\mathbb{P}^3$.

The "toric description of the weighted blowup" consists of a finite sequence of ordinary blowups with nonsingular centers such that the valuation on $\mathbb{P}^3$ corresponding to the last exceptional divisor coincides with the valuation associated to E on $\mathbb{P}^3$.

This corresponds to the toric realization of a resolution of singularities of the weighted blowup by adding rays in order to make its corresponding fan Σ smooth (or regular), that is, in such a way that the minimal generators of all its subcones form part of a $\mathbb{Z}$-basis of the corresponding lattice N. See [CLS, Theorem 11.1.9]. It is straightforward to check that the set of minimal generators $\{v_1, v_2, v\}$ is not a $\mathbb{Z}$-basis for the lattice $\mathbb{Z}^3$ unless $(a, b) = (1, 1)$. So the first blowup corresponds to adding the ray $(1, 1, 1)$, then $(1, 2, 2)$, and so on until adding the ray $(1, a, a)$. Then we add the rays $(1, a, a + 1), \ldots, (1, a, b)$. This process of adding rays ends with a regular fan Σ' that is a refinement of Σ, obtained by a star subdivision of Σ such that the toric morphism $X_{\Sigma'} \to X_\Sigma$ is a projective resolution of singularities. See [CLS, Definition 3.3.13 & Section 11.1].

Let Σ_i be the fan in $\mathbb{R}^3$ corresponding to the toric variety X_i at the i-th step of this process. According to the toric description, Σ_i is obtained by a star subdivision of Σ_{i-1} along a cone σ_{i-1}, that is,

$$\Sigma_i := \Sigma^*_{i-1}(\sigma_{i-1}).$$

The induced toric morphism $X_i \to X_{i-1}$ corresponds to blowing up the orbit of the torus action on X_{i-1} associated to the cone σ_{i-1} by means of the Orbit-Cone Correspondence Theorem. See [CLS, Theorem 3.2.6].

Inductively we have

$$X_b \xrightarrow{\pi_b} X_{b-1} \xrightarrow{\pi_{b-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 := \mathbb{P}^3,$$

where $X_i \xrightarrow{\pi_i} X_{i-1}$ is the blowup of the center $z_{i-1} = z_E X_{i-1}$ of the valuation E on X_{i-1}. Notice that $X_1 \to X_0$ is the blow-up of $z_0 = P$. For every i, we denote by $E_i \subset X_i$ the exceptional divisor of π_i, and for $j > i$ we denote by $E_i^j \subset X_j$ the strict transform of E_i in X_j. The following key properties follow directly from the toric description of the weighted blowup:

1. For all $0 \leq j < a$, the center z_j is a closed point of X_j. If $j \geq 1$, then $z_j \in E_j \subset X_j$, and if $j \geq 2$, then
$$z_j \in E_j \setminus E^j_{j-1}.$$
2. The center $z_a \in X_a$ is the generic point of a line $L_a \subset E_a \simeq \mathbb{P}^2$. If $a \geq 2$, then
$$L_a \not\subset E^a_{a-1}.$$
3. For all $a + 1 \leq j < b$, the center $z_j \in X_j$ is a section
$$L_j \subset E_j \setminus E^j_{j-1}$$
of the projection $E_j \to L_{j-1}$.
4. $E_b = E$ (by this we mean that the exceptional divisors E_b and E induce the same valuation on $X_0 = \mathbb{P}^3$).

Set $\pi' := \pi_1 \circ \cdots \circ \pi_b$. We have the following

$$\begin{tikzcd} X \arrow[d, "\pi"'] & & X_b \arrow[dll, "\pi_1 \circ \cdots \circ \pi_b"] \\ \mathbb{P}^3 & & \end{tikzcd}.$$

Thus E and $E_b = \mathrm{Exc}(\pi_b)$ induce the same valuation on $K(\mathbb{P}^3) \simeq K(X) \simeq K(X_b)$.

Roughly speaking, $a = \sharp\{\text{blown up points}\}$ and $b - a = \sharp\{\text{blown up curves}\}$ in the sequence.

We remark that the exceptional divisors $\mathrm{Exc}(\pi_i)$, for $i \geq a+1$ are Hirzebruch surfaces, since $\mathrm{Exc}(\pi_i) \simeq \mathbb{P}(\mathcal{N}^{\vee}_{L_{i-1}/X_{i-1}})$ and $\mathcal{N}^{\vee}_{L_{i-1}/X_{i-1}}$ is a rank 2 vector bundle over $L_{i-1} \simeq \mathbb{P}^1$. In fact, $z_a \simeq \mathbb{P}^1$ and so are all the remaining centers given by sections z_j. Up to isomorphism, a rank 2 vector bundle over $\mathbb{P}^1$ is of the form $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ for some $n \in \mathbb{Z}_{\geq 0}$. See [Bea, Proposition III.15 (i)] or face this as a particular case of the famous Grothendieck Theorem about finite rank vector bundles over $\mathbb{P}^1$.

The following Key Lemma will be essential because it will give us bounds for the possibilities for a and b, besides the relation $a + b = \mathrm{wt}_\sigma(D)$.

Lemma 5.2.5 (Key Lemma)**.** *Let $D \subset \mathbb{P}^3$ be an irreducible normal quartic surface with an isolated canonical singularity at $P \in \mathbb{P}^3$. The weighted blowup at P with coprime weights $(1, a, b)$ is volume preserving if and only if each blowup in its toric description is volume preserving.*

Proof. Let $\pi\colon X \to \mathbb{P}^3$ be the $(1, a, b)$-weighted blowup of P. Consider the chain of blowups which realizes the valuation associated to $E = \mathrm{Exc}(\pi)$, as explained above:

$$X_b \xrightarrow{\pi_b} X_{b-1} \xrightarrow{\pi_{b-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 =: \mathbb{P}^3.$$

Denote by D_i the strict transform of D on X_i.

($\Rightarrow$) Let us show by a contrapositive argument. Suppose some blowup in the toric description is not volume preserving, and define

$$i_0 := \max\{i \mid \pi_1, \dots, \pi_i \text{ are all volume preserving}\}.$$

Thus, π_{i_0+1} is the first blowup which is not volume preserving. Since the pair $(\mathbb{P}^3, D)$ is canonical, the corresponding discrepancies are always nonnegative.

By Proposition 3.1.12, such discrepancies are zero if and only if the corresponding blowup is volume preserving. This implies that $a(E_{i_0+1}, \mathbb{P}^3, D) > 0$. We have two possible cases:

Case 1: $z_{i_0} \not\subset D_{i_0}$.
It follows from the toric description that the next centers of the blowups will no longer belong to the strict transforms of D, that is, $z_j \not\subset D_j$ for all $j \geq i_0 + 1$. Consequently we will have $a(E_j, \mathbb{P}^3, D) > 0$ for all $j \geq i_0 + 1$.

Thus, $a(E_b, \mathbb{P}^3, D) > 0$ and therefore π is not volume preserving.

Case 2: $z_{i_0} \subset D_{i_0}$.
In this case, z_{i_0} is either a curve or a closed point. If z_{i_0} is a curve, then $\pi_{i_0+1}\colon X_{i_0+1} \to X_{i_0}$ is volume preserving, since one has $a(E_{i_0+1}, X_{i_0}, D_{i_0}) = 0$.

If z_{i_0} is a closed point, then either $z_{i_0} \in \mathrm{Sing}(D_{i_0})$ or $z_{i_0} \notin \mathrm{Sing}(D_{i_0})$.

In the first scenario, since canonicity is preserved under volume preserving maps by Lemma 3.1.23, it follows that z_{i_0} is a strict canonical singularity of D_{i_0}. Hence, one also has $a(E_{i_0+1}, X_{i_0}, D_{i_0}) = 0$.

So we are inevitably in the second scenario $z_{i_0} \notin \mathrm{Sing}(D_{i_0})$. By the toric description, we observe that necessarily $i_0 < a$. It follows that $z_i = P_i \in D_i$ is a nonsingular point for every i such that $i_0 \leq i \leq a-1$. Indeed, if $z_i \not\subset D_i$ for some i, arguing analogously as in Case 1, we deduce that π is not volume preserving.

Therefore, $a(E_a, X_{a-1}, D_{a-1}) > 0$.

Next, we analyze what happens for $i \geq a$. Recall that we have $z_a = D_a \cap E_a \subset D_a$ and $\pi_{a+1} \colon X_{a+1} \to X_a$ is volume preserving.

We have the following:

$$X_{a-1} \xleftarrow{\mathrm{Bl}_{P_{a-1}}} X_a \xleftarrow{\mathrm{Bl}_{L_a}} X_{a+1} \xleftarrow{\mathrm{Bl}_{L_{a+1}}} \cdots \xleftarrow{\mathrm{Bl}_{L_{b-1}}} X_b \ .$$

We have that

$$\begin{tikzcd}
D_{a-1} \arrow[r, "\pi_a^*"] & D_a + E_a \arrow[r, "\pi_{a+1}^*"] & D_{a+1} + E_a^{a+1} + 2E_{a+1} \arrow[d, "\pi_{a+2}^*"] \\
 & & D_{a+2} + E_a^{a+2} + 2E_{a+1}^{a+2} + 3E_{a+2} \arrow[d, "\pi_{a+3}^*"] \\
 & & \vdots \arrow[d, "\pi_b^*"] \\
 & & D_b + E_a^b + 2E_{a+1}^b + 3E_{a+2}^b + \cdots + (b-a-1)E_{b-1}^b + (b-a)E_b
\end{tikzcd},$$

which implies

$$\begin{aligned}
D_a &= \pi_a^* D_{a-1} - E_a, \\
D_{a+1} &= \pi_{a+1}^* D_a - E_{a+1} \\
&= \pi_{a+1}^*(\pi_a^* D_{a-1} - E_a) - E_{a+1} \\
&= (\pi_a \circ \pi_{a+1})^* D_{a-1} - E_a^{a+1} - 2E_{a+1}, \\
D_{a+2} &= (\pi_a \circ \pi_{a+1} \circ \pi_{a+2})^* D_{a-1} - E_a^{a+2} - 2E_{a+1}^{a+2} - 3E_{a+2}, \\
&\ \vdots \\
D_b &= (\pi_a \circ \cdots \circ \pi_b)^* D_{a-1} - E_a^b - 2E_{a+1}^b - 3E_{a+2}^b - \cdots - (b-a-1)E_{b-1}^b - (b-a)E_b.
\end{aligned}$$

By the Adjunction Formula, we have that

$$\begin{aligned}
K_{X_a} &= \pi_a^* K_{X_{a-1}} + 2E_a, \\
K_{X_{a+1}} &= \pi_{a+1}^*(K_{X_a}) + E_{a+1}
\end{aligned}$$

$$
\begin{aligned}
&= \pi_{a+1}^*(\pi_a^* K_{X_{a-1}} + 2E_a) + E_{a+1} \\
&= (\pi_a \circ \pi_{a+1})^* K_{X_{a-1}} + 2E_a^{a+1} + 3E_{a+1}, \\
K_{X_{a+2}} &= (\pi_a \circ \pi_{a+1} \circ \pi_{a+2})^* K_{X_{a-1}} + 2E_a^{a+2} + 3E_{a+1}^{a+2} + 4E_{a+2}^{a+2}, \\
&\vdots \\
K_{X_b} &= (\pi_a \circ \cdots \circ \pi_b)^* K_{X_{a-1}} + 2E_a^b + 3E_{a+1}^b + 4E_{a+2}^b + \cdots + (b-a)E_{b-1}^b + (b-a+1)E_b.
\end{aligned}
$$

Therefore,

$$
K_{X_b} + D_b = (\pi_a \circ \cdots \circ \pi_b)^*(K_{X_{a-1}} + D_{a-1}) + E_a^b + E_{a+1}^b + E_{a+2}^b + \cdots + E_{b-1}^b + E_b.
$$

It follows that $a(E_b, X_{a-1}, D_{a-1}) = 1$, whence we deduce $a(E_b, \mathbb{P}^3, D) = 1 > 0$. Consequently π is not volume preserving.

($\Rightarrow$) Since all the blowups are volume preserving, the corresponding discrepancies

$$
a(E_1, \mathbb{P}^3, D), \ldots, a(E_b, X_{b-1}, D_{b-1})
$$

will be zero as a consequence of the fact that composition of volume preserving maps is also volume preserving. See Remark 3.1.15.

Thus, $a(E, \mathbb{P}^3, D) = a(E_b, \mathbb{P}^3, D) = 0$ and therefore π is volume preserving.

$\square$

Key Lemma 5.2.5 indicates that after realizing the divisorial valuation ν_E associated to E by means of a sequence of ordinary blowups at points or curves through a toric description, we must verify that all these intermediate blowups are volume preserving.

Concerning the A_n case of the problem, Key Lemma 5.2.5 restricts the possibilities for the weights.

Corollary 5.2.6. *For $D \subset \mathbb{P}^3$ a quartic surface with an isolated canonical singularity of type A_n, a sequence of $1 \leq m \leq \left\lceil \frac{n}{2} \right\rceil$ ordinary blowups at the singular points of the strict transforms of D is volume preserving. On the other hand, the weights of the form $(1, a, b)$ with $a > \left\lceil \frac{n}{2} \right\rceil$ are not volume preserving.*

Proof. The first part is direct from the Lemma 5.1.11, which tells us that D is resolved after $\left\lceil \frac{n}{2} \right\rceil$ blowups. If $a > \left\lceil \frac{n}{2} \right\rceil$, by the toric description this means that we must blowup points outside the singular locus of the strict transform of D. These blowups are not volume preserving, and hence by the Key Lemma 5.2.5 the assertion follows. $\square$

5.3 Determination of the volume preserving weights

With the results of the previous sections, we are ready to determine the desired volume preserving weights. The content of this section is a classification of all possible toric weighted blowups $\pi\colon X \to \mathbb{P}^3$ that are volume preserving for a Calabi-Yau pair $(\mathbb{P}^3, D)$ with $\mathrm{coreg}(\mathbb{P}^3, D) = 2$, depending on the singularities on D. This classification will culminate in another one of which weights will initiate volume preserving Sarkisov links starting with such Calabi-Yau pair $(\mathbb{P}^3, D)$.

If we are interested in which ones will initiate a volume preserving Sarkisov link, we can restrict ourselves to the finite number of cases described in Theorem 5.2.3.

In what follows we will determine in a generic way which weights will be volume preserving and not necessarily the candidates to initiate a volume preserving Sarkisov link. The answer will depend on the type of canonical singularity of the quartic surface.

Even if some volume preserving weights do not yield a volume preserving link, they will be relevant to producing a model of the quartic surface embedded in a toric variety or links of a log version of the Sarkisov Program. See Remark 5.1.5.

Due to the difficulty level of this problem in terms of the number of computations needed, we will restrict ourselves to singularities A_n with $1 \leq n \leq 7$. This will be enough for the second classification to contemplate all types of strict canonical singularities of type A_n.

Remark 5.3.1. Throughout the proof of Theorem 5.2.1, we only consider global toric weighted blowups. In this context, we do not describe all divisorial extractions to $\mathbb{P}^3$ which are in local analytic coordinates a $(1, a, b)$-weighted blowup according to [Kaw1, Theorem 1], much less all volume preserving ones to Calabi-Yau pairs $(\mathbb{P}^3, D)$. Indeed, the weighted blowup does depend on the local coordinates chosen (algebraic x analytic).

Remark 5.3.2. By the toric description of the weighted blowup, if the weights $(1, a, b)$ are volume preserving for an A_n singularity, then $(1, a, b)$ are volume preserving for an A_m singularity with $m \geq n$.

5.3.1 Notation for $D \subset \mathbb{P}^3$ quartic surface and some generalities

Let $D \subset \mathbb{P}^3$ be an irreducible normal quartic surface having a strict canonical singularity at P so that we have $m_P(D) = 2$. Fix homogeneous coordinates such that $P = (1:0:0:0)$ and the equation of D has the form

$$x_0^2 A + x_0 B + C = 0,$$

where $A, B, C \in \mathbb{C}[x_1, x_2, x_3]$ are homogeneous polynomials of degrees 2, 3 and 4, respectively. Notice that the tangent cone of D at P is given by $TC_P D = \{A = 0\}$.

Given a positive integer n, we will say that P *is of type* $A_{\geq n}$ to express that P is a singularity of type A_m with $m \geq n$. Analogously for the D_n case. Sometimes, instead of saying "P is a singularity of type A_n", we will abbreviate it by "P is A_n".

From the local description of the Du Val singularities, we see that:

$$\left.\begin{array}{r} P \text{ is } A_1 \Leftrightarrow \operatorname{rank}(A) = 3 \\ P \text{ is } A_{\geq 2} \Leftrightarrow \operatorname{rank}(A) = 2 \end{array}\right\} \Rightarrow P \text{ is } A_n \Leftrightarrow \operatorname{rank}(A) > 1 \text{ and}$$

$$P \text{ is } D\text{-}E \Leftrightarrow \operatorname{rank}(A) = 1.$$

Notice that the projectivization of TC_PD is a plane conic, not necessarily irreducible. The ranks 3, 2 and 1 correspond to an irreducible conic, two concurrent lines and a double line (nonreduced), respectively.

Therefore, we have an immediate criterion to detect an A_1 singularity and necessary conditions for the other types.

Possibly after changing homogeneous coordinates, if $\operatorname{rank}(A) = 2$ or 1, we may assume that $A = x_2x_3$ or $A = x_3^2$, respectively.

We will adopt the following notation which will ease our computations. Write

$$B(x_1, x_2, x_3) = \sum_{i=0}^{3} b_i x_1^{3-i}, \text{ where } b_i \in \mathbb{C}[x_2, x_3]_i \text{ and}$$
$$C(x_1, x_2, x_3) = \sum_{i=0}^{4} c_i x_1^{4-i}, \text{ where } c_i \in \mathbb{C}[x_2, x_3]_i.$$

Also, write the following, where all the Greek letters indicating coefficients are complex numbers:

- $b_1 = \beta_2 x_2 + \beta_3 x_3$;
- $b_2 = \rho_2 x_2^2 + \rho_{23} x_2 x_3 + \rho_3 x_3^2$;
- $b_3 = \sum_{i=0}^{3} \sigma_i x_2^{3-i} x_3^i = \sigma_0 x_2^3 + \sigma_1 x_2^2 x_3 + \sigma_2 x_2 x_3^2 + \sigma_3 x_3^3$;
- $c_1 = \delta_2 x_2 + \delta_3 x_3$;
- $c_2 = \varepsilon_2 x_2^2 + \varepsilon_{23} x_2 x_3 + \varepsilon_3 x_3^2$;
- $c_3 = \sum_{i=0}^{3} \tau_i x_2^{3-i} x_3^i = \tau_0 x_2^3 + \tau_1 x_2^2 x_3 + \tau_2 x_2 x_3^2 + \tau_3 x_3^3$;
- $c_4 = \sum_{i=0}^{4} \lambda_i x_2^{4-i} x_3^i = \lambda_0 x_2^4 + \lambda_1 x_2^3 x_3 + \lambda_2 x_2^2 x_3^2 + \lambda_3 x_2 x_3^3 + \lambda_4 x_3^4$.

5.3.2 The A_n case

Let $D \subset \mathbb{P}^3$ be an irreducible normal quartic surface having a canonical singularity of type A_n at $P = (1 : 0 : 0 : 0)$. Abusing notation, extend it by writing $A_i, i \leq 0$ to mean that D is nonsingular at P, that is, D is terminal at P.

Let $\pi \colon X \to \mathbb{P}^3$ be the toric $(1, a, b)$-weighted blowup at P and $E \coloneqq \operatorname{Exc}(\pi)$.

According to the toric description of the $(1, a, b)$-weighted blowup, the valuation associated to E can be realized by an exceptional divisor on a sequence of a blowups at points followed by $b - a$ blowups along curves. We will follow the notation adopted so far.

By Lemma 5.1.11 the effect of a blowups at singular points is to weaken the singularity from A_n to A_{n-2a}, whereas by Lemma 5.1.12 the remaining $b - a$ blowups along curves will weaken it from A_{n-2a} to $A_{n-2a-(b-a)} = A_{n-a-b}$.

Lemma 5.3.3. *Let $X, D, P \in D$ and $\pi\colon X \to \mathbb{P}^3$ be as above and suppose that D has an A_n singular point at P. If π is volume preserving, then $a \leq \left\lceil \frac{n}{2} \right\rceil$ and*

$$b - a \leq n - 2a + 1 \Rightarrow a + b \leq n + 1.$$

Proof. The first inequality is simply Lemma 5.2.6. If $b - a > n - 2a + 1$, then the types of singularity of the strict transforms of D in the first $n - a + 1$ steps of the toric description of π are as follows:

$$\underbrace{A_n \xrightarrow{\mathrm{Bl}_{P_0}} A_{n-2} \xrightarrow{\mathrm{Bl}_{P_1}} \dots \xrightarrow{\mathrm{Bl}_{P_{a-1}}} A_{n-2a}}_{a \text{ blowups at singular points}} \underbrace{\xrightarrow{\mathrm{Bl}_{L_a}} A_{n-2a-1} \xrightarrow{\mathrm{Bl}_{L_{a+1}}} \dots \xrightarrow{\mathrm{Bl}_{L_{n-a-2}}} A_1 \xrightarrow{\mathrm{Bl}_{L_{n-a-1}}} A_0 \xrightarrow{\mathrm{Bl}_{L_{n-a}}} A_{-1}}_{n-2a+1 \text{ blowups along curves}}. \tag{5.3.1}$$

In terms of the ambient varieties we have:

$$\underbrace{\mathbb{P}^3 = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_a} X_a}_{a \text{ blowups at singular points}} \underbrace{\xleftarrow{\pi_{a+1}} X_{a+1} \xleftarrow{\pi_{a+2}} \dots \xleftarrow{\pi_{n-a-1}} X_{n-a-1} \xleftarrow{\pi_{n-a}} X_{n-a} \xleftarrow{\pi_{n-a+1}} X_{n-a+1}}_{n-2a+1 \text{ blowups along curves}}. \tag{5.3.2}$$

We observe that $n - 2a \geq -1$. Otherwise,

$$n - 2a < -1 \Rightarrow a > \frac{n+1}{2} \geq \left\lceil \frac{n}{2} \right\rceil,$$

contradicting Lemma 5.2.6.

By the Key Property 3 of the toric description, the next center $z_{n-a+1} \in X_{n-a+1}$ in 5.3.1 is the generic point of a section $L_{n-a+1} \subset E_{n-a+1}$ of the projection $E_{n-a+1} \to L_{n-a}$ disjoint from E_{n-a}^{n-a+1}.

On the other hand, since $D_{n-a+1} \subset X_{n-a+1}$ is a nonsingular surface, z_{n-a+1} is the generic point of a curve in $D_{n-a+1} \cap E_{n-a+1}$ by Proposition 3.1.25. Since

$$L_{n-a+1} \coloneqq D_{n-a+1} \cap E_{n-a+1} \simeq \mathbb{P}^1$$

is irreducible, we must have $z_{n-a+1} = L_{n-a+1}$.

Let us show that the curves L_{n-a+1} and $E_{n-a}^{n-a+1} \cap E_{n-a+1}$ intersect. This contradicts the Key Property 3 of the toric description, and therefore we must have $b - a \leq n - 2a + 1$.

This nonempty intersection follows from the subsequent discussion.

By description 5.3.1, D_j is singular at a point P_j if $j \leq n - a - 1$, and is nonsingular otherwise.

Suppose $a + 1 \leq j \leq n - a < b$. Then $E_j \simeq \mathbb{P}(\mathcal{N}^\vee_{L_{j-1}/X_{j-1}}) \simeq \mathbb{F}_m$ for some $m \geq 0$ and $E_j \cap D_j = L_j \cup L'_j$ consists of a fiber L'_j of $p_j\colon E_j \to L_{j-1}$ corresponding to all the normal directions to L_{j-1} at P_{j-1}, and a horizontal section L_j of the projection $p_j\colon E_j \to L_{j-1}$. The setting is depicted in Figure 5.5.

Furthermore, $L'_j \cap E^j_{j-1} = \{Q_j\}$ and Q_j corresponds to the normal direction L'_{j-1} to L_{j-1} determined by $T_{P_{j-1}} E_{j-1}$. One has $L_j \cap L'_j = \{P_j\}$ and P_j corresponds to the "limit" normal direction to L_{j-1} at P_{j-1} coming from the normal directions that define $T_Q D_{j-1}$ for $Q \neq P_{j-1}$.

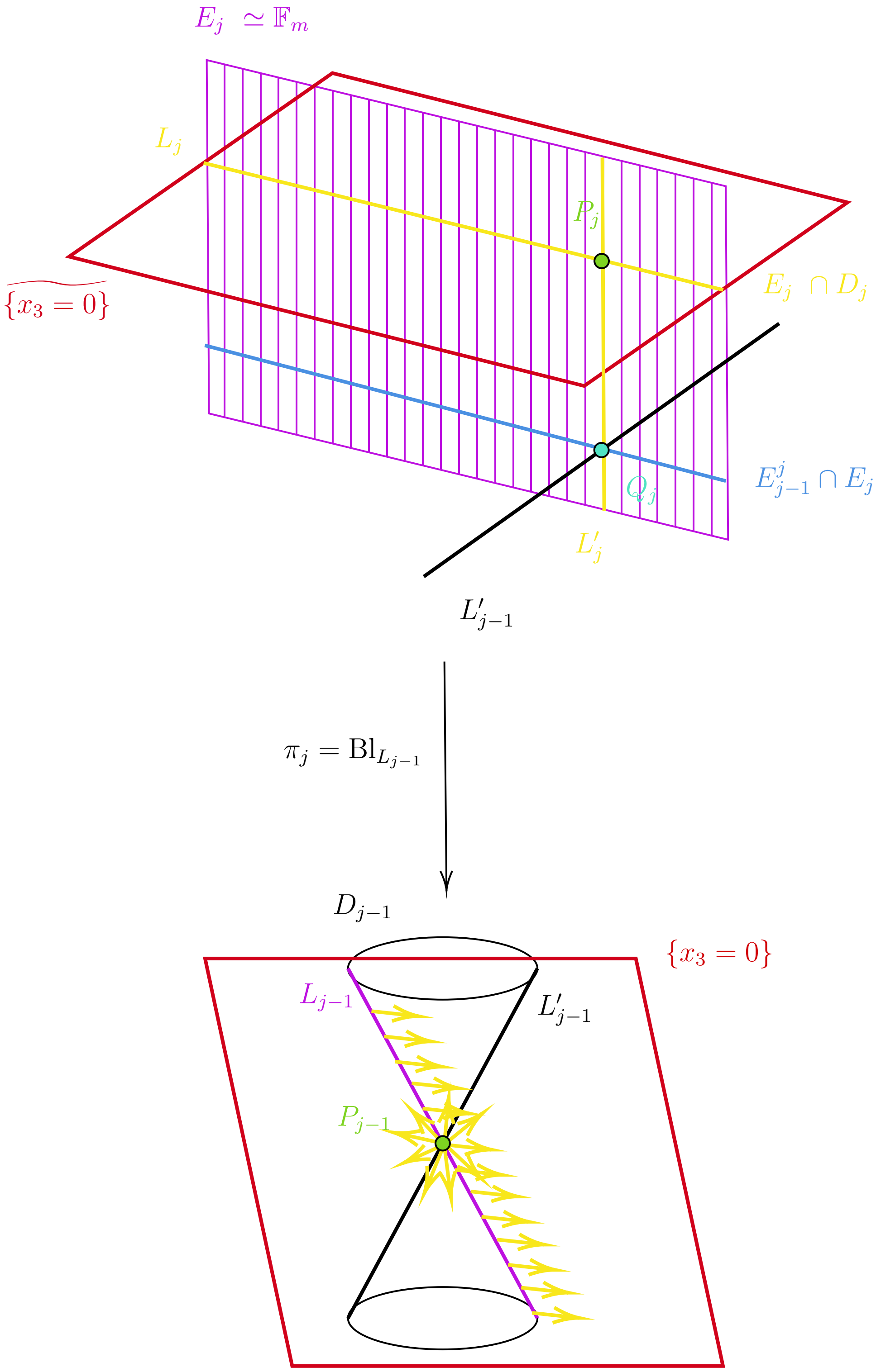


Figure 5.5: Case $a + 1 \leq j \leq n - a < b$.

Note that D_{n-a} is nonsingular and $\pi_{n-a+1}\colon X_{n-a+1} \to X_{n-a}$, the blowup of $L_{n-a} \subset D_{n-a}$, is such that $D_{n-a+1} = \widetilde{D_{n-a}} \simeq D_{n-a}$. So, for $j = n - a + 1$, we have that $E_{n-a+1} \cap D_{n-a+1}$ is a section of the projection $p_{n-a+1}\colon E_{n-a+1} \to L_{n-a}$.

Indeed, since D_{n-a} is nonsingular, for each $Q' \in L_{n-a}$, there exists only one normal direction to L_{n-a} at Q' that is tangent to D_{n-a}.

Furthermore, $E_{n-a+1} \cap D_{n-a+1} \cap E_{n-a}^{n-a+1} = \{Q_{n-a+1}\} \neq \emptyset$, where Q_{n-a+1} corresponds to the normal direction to L_{n-a} at P_{n-a} determined by $T_{P_{n-a}} E_{n-a} = T_{P_{n-a}} D_{n-a}$, and this concludes the proof. The setting is depicted in Figure 5.6.

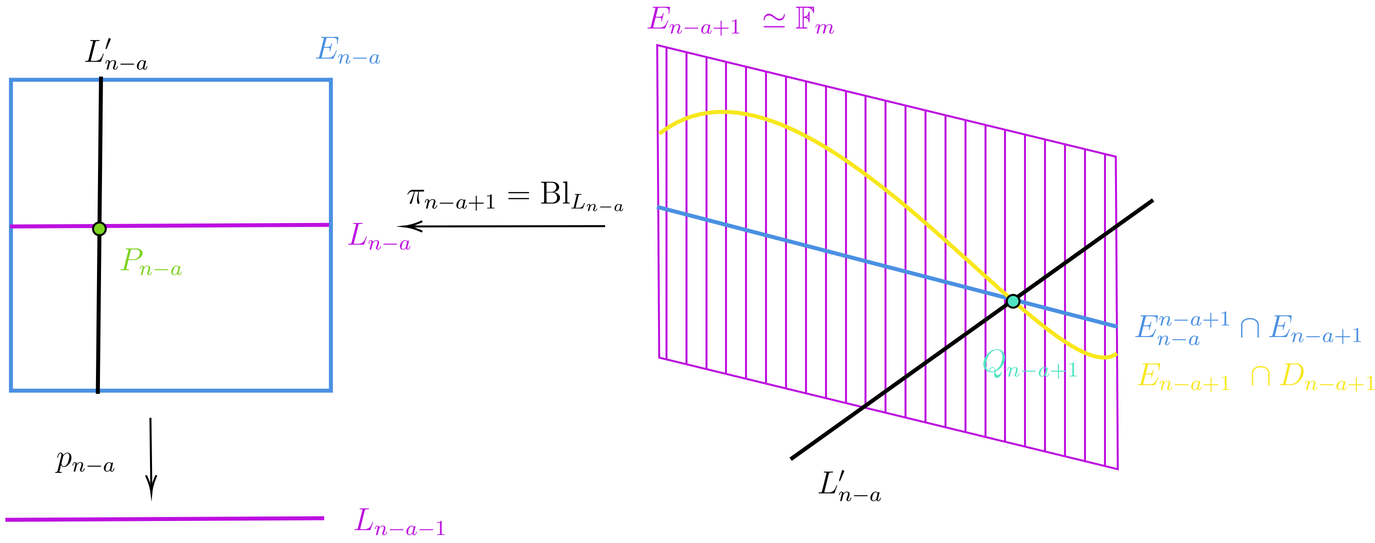


Figure 5.6: Case $j = n - a + 1$.

□

5.3.2.1 Criteria for A_n singularities on a quartic surface

Let $D = \{x_0^2 A + x_0 B + C = 0\} \subset \mathbb{P}^3$ be a quartic surface having a canonical singularity of type A_n at $P = (1:0:0:0)$, $n \in \{1, \ldots, 19\}$. Our task now is to establish conditions on A, B and C so that P is of type A_n.

This was done in a different setting in [KN], where Kato & Naruki also give a description of the coarse moduli space of some of these quartics.

However, we are working in different coordinates than [KN]. We will obtain our conditions by analyzing how the singularity is resolved by a sequence of blowups at singular points, and checking the geometry of the exceptional divisor along the process. The bigger n is, the more complicated these explicit criteria will be in terms of the coefficients of the quartic.

In view of Theorem 5.2.3 and Lemma 5.3.3, we will do it for $n \leq 7$.

In fact, $n \leq 6$ would be enough for the problem of determining which weighted blowups initiate a volume preserving Sarkisov link, but we will also deal with the case $n = 7$, since they were still manageable.

From what was explained in the Subsection 5.3.1 concerning the notation for the coefficients of the homogeneous equation that defines D, we have that

$$P \text{ is } A_1 \Leftrightarrow \operatorname{rank}(A) = 3.$$

Let us move to the $A_{\geq 2}$ case. Since a blowup at the singular point weakens the singularity from A_n to A_{n-2}, to obtain the criteria for $n = 7$, by Lemma 5.1.11 we will need to blow up 4 times.

We will denote by D_i the strict transform of D in the i-th blowup and E_i the exceptional divisor. All the following equations are with respect to the "first affine chart" $\{y_1 \neq 0\}$ in the corresponding blowup. We observe that $E_i = \{x_1 = 0\}$ in this affine chart.

At Step 0 we have our initial situation, and at Step i the situation after the i-th blowup. After performing many computations and applying the Jacobian Criterion, one obtains the following:

Step 0: $D_0 = \{x_0^2 A + x_0 B + C = 0\}$. In the affine open $\{x_0 \neq 0\}$ we have

$$D_0 = \{A + B + C = 0\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}.$$

Set $P_0 := (0,0,0) \in \mathrm{Sing}(D_0)$. One has

$$P_0 \text{ is } A_{\geq 2} \Leftrightarrow \mathrm{rank}(A) = 2 \Leftrightarrow A = x_2x_3, \text{ without loss of generality.}$$

Step 1: $D_1 = \{x_2x_3 + x_1B(1,x_2,x_3) + x_1^2C(1,x_2,x_3) = 0\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$. We have

$$E_1 \cap D_1 = \{x_1 = x_2x_3 = 0\} = L_1 \cup L_1',$$

where L_1 and L_1' are lines on $D_1 \subset \mathbb{A}^3$. Observe that $P_1 := (0,0,0) \in \mathrm{Sing}(E_1 \cap D_1)$. One has

$$P_0 \text{ is } A_2 \Leftrightarrow P_1 \text{ is } A_0 \Leftrightarrow b_0 \neq 0,$$
$$P_0 \text{ is } A_{\geq 3} \Leftrightarrow P_1 \text{ is } A_{\geq 1} \Leftrightarrow b_0 = 0 \ (*_1).$$

Step 2:

$$D_2 = \left\{ x_2x_3 + b_1(x_2,x_3) + x_1b_2(x_2,x_3) + x_1^2b_3(x_2,x_3) + \sum_{i=0}^{4} x_1^i c_i(x_2,x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}.$$

We have that

$$E_1 \cap D_1 = \{x_1 = x_2x_3 + \beta_2x_2 + \beta_3x_3 + c_0 = 0\},$$

is a conic (not necessarily irreducible) on $D_1 \subset \mathbb{A}^3$. One has

$$P_0 \text{ is } A_3 \Leftrightarrow P_1 \text{ is } A_1 \Leftrightarrow (*_1) \text{ and } c_0 \neq \beta_2\beta_3,$$
$$P_0 \text{ is } A_{\geq 4} \Leftrightarrow P_1 \text{ is } A_{\geq 2} \Leftrightarrow (*_1) \text{ and } c_0 = \beta_2\beta_3 \ (*_2).$$

In the latter case, we have

$$E_2 \cap D_2 = \{x_1 = (x_2 + \beta_3)(x_3 + \beta_2) = 0\} = L_2 \cup L_2',$$

where L_2 and L_2' are lines on D_2. Observe that $P_2 := (0, -\beta_3, -\beta_2) \in \mathrm{Sing}(E_2 \cap D_2)$. One has

$$P_0 \text{ is } A_4 \Leftrightarrow (*_2) \text{ and } b_2(\beta_3, \beta_2) - c_1(\beta_3, \beta_2) = \rho_2\beta_3^2 + \rho_{23}\beta_2\beta_3 + \rho_3\beta_2^2 - \delta_2\beta_3 - \delta_3\beta_2 =: \zeta \neq 0,$$
$$P_0 \text{ is } A_{\geq 5} \Leftrightarrow P_2 \text{ is } A_{\geq 1} \Leftrightarrow (*_2) \text{ and } \zeta = 0 \ (*_3).$$

Step 3:

$$D_3 = \Big\{ x_2x_3 + \rho_2x_1x_2 - 2\rho_2\beta_3x_2 + \rho_{23}(x_1x_2x_3 - \beta_2x_2 - \beta_3x_3) + \rho_3x_1x_3 - 2\rho_3\beta_2x_3$$
$$+ b_3(x_1x_2 - \beta_3, x_1x_3 - \beta_2) + \delta_2x_2 + \delta_3x_3 + \sum_{i=2}^{4} x_1^{i-2}c_i(x_1x_2 - \beta_3, x_1x_3 - \beta_2) = 0 \Big\}$$
$$\subset \mathbb{A}^3_{(x_1,x_2,x_3)}.$$

Set

$$\xi_2 := -2\rho_2\beta_3 - \rho_{23}\beta_2 + \delta_2,$$
$$\xi_3 := -2\rho_3\beta_2 - \rho_{23}\beta_3 + \delta_3 \text{ and}$$
$$\alpha := -b_3(\beta_3, \beta_2) + c_2(\beta_3, \beta_2).$$

We have that

$$E_3 \cap D_3 = \{x_1 = x_2x_3 + \xi_2x_2 + \xi_3x_3 + \alpha = 0\},$$

is a conic (not necessarily irreducible) on $D_3 \subset \mathbb{A}^3$. One has

$$P_0 \text{ is } A_5 \Leftrightarrow P_1 \text{ is } A_3 \Leftrightarrow (*_3) \text{ and } \xi_2\xi_3 \neq \alpha,$$
$$P_0 \text{ is } A_{\geq 6} \Leftrightarrow P_1 \text{ is } A_{\geq 4} \Leftrightarrow (*_3) \text{ and } \xi_2\xi_3 = \alpha \ (*_4).$$

In the latter case, we have

$$E_3 \cap D_3 = \{x_1 = (x_2 + \xi_3)(x_3 + \xi_2) = 0\} = L_3 \cup L_3',$$

where L_3 and L_3' are lines on $D_3 \subset \mathbb{A}^3$. Observe that $P_3 := (0, -\xi_3, -\xi_2) \in \mathrm{Sing}(E_3 \cap D_3)$.

Set

$$\omega := -3\sigma_0\xi_3\beta_3^2 + \sigma_1(-2\xi_3\beta_2\beta_3 - \xi_2\beta_3^2) + \sigma_2(-\xi_3\beta_2^2 - 2\xi_2\beta_2\beta_3) - 3\sigma_3\xi_2\beta_2^2 \text{ and}$$
$$\eta := 2\varepsilon_2\xi_3\beta_3 + \varepsilon_{23}\xi_3\beta_2 + \varepsilon_{23}\xi_2\beta_3 + 2\varepsilon_3\xi_2\beta_2.$$

One has

$$P_0 \text{ is } A_6 \Leftrightarrow (*_4) \text{ and } b_2(\xi_3, \xi_2) + \omega + \eta - c_3(\beta_3, \beta_2) =: \theta \neq 0,$$
$$P_0 \text{ is } A_{\geq 7} \Leftrightarrow P_1 \text{ is } A_{\geq 5} \Leftrightarrow (*_4) \text{ and } \theta = 0 \ (*_5).$$

Step 4: Set

$$\gamma_2 \coloneqq -2\rho_2\xi_3 - \rho_{23}\xi_2 + 3\sigma_0\beta_3^2 + 2\sigma_1\beta_2\beta_3 + \sigma_2\beta_2^2 - 2\varepsilon_2\beta_3 - \varepsilon_{23}\beta_2,$$
$$\gamma_3 \coloneqq -\rho_{23}\xi_3 - 2\rho_3\xi_2 + 3\sigma_3\beta_2^2 + 2\sigma_2\beta_2\beta_3 + \sigma_1\beta_3^2 - 2\varepsilon_3\beta_2 - \varepsilon_{23}\beta_3 \text{ and}$$

$$\begin{aligned}\mu \coloneqq & -3\sigma_0\beta_3\xi_3^2 - 3\sigma_3\beta_2\xi_2^2 - 2\sigma_1\beta_3\xi_2\xi_3 - 2\sigma_2\beta_2\xi_2\xi_3 - \sigma_1\xi_3^2\beta_2 - \sigma_2\xi_2^2\beta_3 + \varepsilon_2\xi_3^2 + \varepsilon_{23}\xi_2\xi_3 + \varepsilon_3\xi_2^2 \\ & - 3\tau_0\beta_3^2\xi_3 - 3\tau_3\beta_2^2\xi_2 + \tau_1(-2\beta_2\beta_3\xi_3 - \beta_3^2\xi_2) + \tau_2(-\xi_3\beta_2^2 - 2\beta_2\beta_3\xi_2) + c_4(\beta_3, \beta_2).\end{aligned}$$

In this step

$$D_4 = \{x_2x_3 + \gamma_2x_2 + \gamma_3x_3 + \mu + (\text{higher order terms in } x_1) = 0\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}.$$

We have that

$$E_4 \cap D_4 = \{x_1 = x_2x_3 + \gamma_2x_2 + \gamma_3x_3 + \mu = 0\},$$

is a conic (not necessarily irreducible) on $D_4 \subset \mathbb{A}^3$. One has

$$P_0 \text{ is } A_7 \Leftrightarrow P_1 \text{ is } A_5 \Leftrightarrow (*_5) \text{ and } \gamma_2\gamma_3 \neq \mu,$$
$$P_0 \text{ is } A_{\geq 8} \Leftrightarrow P_1 \text{ is } A_{\geq 6} \Leftrightarrow (*_5) \text{ and } \gamma_2\gamma_3 = \mu.$$

In the latter case, we have

$$E_4 \cap D_4 = \{x_1 = (x_2 + \gamma_3)(x_3 + \gamma_2) = 0\} = L_4 \cup L_4',$$

where L_4 and L_4' are lines on $D_4 \subset \mathbb{A}^3$. Observe that $P_4 \coloneqq (0, -\gamma_3, -\gamma_2) \in \operatorname{Sing}(E_4 \cap D_4)$.

We will stop at this step.

We observe that all the criteria obtained present symmetries. The degree of the conditions with respect to the coefficients of the quartic increases as n increases.

In [KN], Kato & Naruki found an equation of a quartic surface with a single A_{19} singularity. We may use it to double-check all our criteria.

In [KN], the equation of this quartic is in affine coordinates such that the tangent cone at P is given by $\{x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2) = 0\}$, namely,

$$\begin{aligned}16(x_1^2 + x_2^2) + 32x_1x_3^2 - 16x_2^3 + 16x_3^4 - 32x_2x_3^3 + 8(2x_1^2 - 2x_1x_2 + 5x_2^2)x_3^2 \\ + 8(2x_1^3 - 5x_1^2x_2 - 6x_1x_2^2 - 7x_2^3)x_3 + 20x_1^4 + 44x_1^3x_2 + 65x_1^2x_2^2 + 40x_1x_2^3 + 41x_2^4 = 0.\end{aligned}$$

Performing the affine change of the coordinates induced by the projective change of coordinates

$$(x_0 : x_1 : x_2 : x_3) \mapsto \left(x_0 : \frac{x_2 + x_3}{8} : \frac{i(x_2 - x_3)}{8} : x_1\right),$$

we can bring this quartic to have $TC_PD = \{x_2x_3 = 0\}$. The equation becomes

$$\begin{aligned}16x_1^4 - 4ix_1^3x_2 + 4ix_1^3x_3 - \left(\frac{3}{8} + \frac{i}{4}\right)x_1^2x_2^2 + \frac{7}{4}x_1^2x_2x_3 + 4x_1^2x_2 \\ - \left(\frac{3}{8} - \frac{i}{4}\right)x_1^2x_3^2 + 4x_1^2x_3 + \frac{ix_2^3}{32} - \frac{3}{32}ix_2^2x_3 + \frac{3}{32}ix_2x_3^2 + x_2x_3 - \frac{ix_3^3}{32} \\ + \left(\frac{1}{8} + \frac{i}{32}\right)x_1x_2^3 - \frac{13}{32}ix_1x_2^2x_3 + \frac{13}{32}ix_1x_2x_3^2 + \left(\frac{1}{8} - \frac{i}{32}\right)x_1x_3^3 \\ + \left(-\frac{1}{1024} + \frac{i}{1024}\right)x_2^4 - \left(\frac{21}{1024} - \frac{21i}{512}\right)x_2^3x_3 \\ + \frac{31x_2^2x_3^2}{256} - \left(\frac{21}{1024} + \frac{21i}{512}\right)x_2x_3^3 - \left(\frac{1}{1024} + \frac{i}{1024}\right)x_3^4 = 0. \quad (5.3.3)\end{aligned}$$

One can easily check that P is indeed $A_{\geq 8}$ according to all our criteria.

5.3.3 Toric description of the weights $(1, a, b)$

In this part, we will analyze the toric description of the weighted blowup with weights $(1, a, b)$ and conditions imposed on its center so that it is volume preserving. By Proposition 3.1.25 and Lemma 5.2.2, the centers are:

$$z_i = \begin{cases} \text{singular points in } D_i \text{ and its strict transforms,} & \text{for } 0 \leq i \leq a-1, \\ \text{curves on } D_i, & \text{for } a \leq i \leq b-1. \end{cases}$$

Recalling our notation, consider $\pi\colon X \to \mathbb{P}^3$ the $(1, a, b)$-weighted blowup at $P \in \operatorname{Sing}(D)$. Identify $\mathbb{P}^{34}$ with the space of quartics in $\mathbb{P}^3$ in the following way:

$$\begin{aligned}\mathbb{P}^{34} &\longrightarrow \{\text{quartics in } \mathbb{P}^3\} \\ (a_0 : \ldots : a_{34}) &\longmapsto \left\{\sum_{i=0}^{34} a_iM_i = 0\right\},\end{aligned}$$

where $\{M_0, \ldots, M_{34}\}$ are all the monomials of degree 4 in $\mathbb{C}[x_0, x_1, x_2, x_3]$ with a fixed order.

Let $\mathcal{A}_{\geq n} \subset \mathbb{P}^{34}$ be the coarse moduli space of irreducible quartic surfaces passing through $P = (1 : 0 : 0 : 0)$ and having an $A_{\geq n}$ singularity at this point.

5.3.3.1 Toric description of the weights $(1, 1, b)$

Weights $(1, 1, 1)$: We need to insert in Σ_0 the ray

$$(1, 1, 1) \in \operatorname{Cone}(v_1, v_2, v_3).$$

This insertion corresponds to blowing up the orbit of the torus action on $X_0 = \mathbb{P}^3$ associated to the 3-dimensional cone $\operatorname{Cone}(v_1, v_2, v_3)$. This orbit is precisely

$$\{x_1 = 0\} \cap \{x_2 = 0\} \cap \{x_3 = 0\} = P = (1 : 0 : 0 : 0),$$

that is, $z_0 = P =: P_0$.

By Proposition 3.1.12, π_1 is volume preserving.

Therefore the weights $(1,1,1)$ are volume preserving for any quartic with an A_n singularity at P, $n \geq 1$.

Weights $(1,1,2)$**:** We need to insert in Σ_1 the ray

$$(1,1,2) \in \mathrm{Cone}((1,1,1),(0,0,1)).$$

This insertion corresponds to blowing up the orbit of the torus action on X_1 associated to the 2-dimensional cone $\mathrm{Cone}((1,1,1),v_3)$. This orbit is precisely the line $L_1 = E_1 \cap \widetilde{\{x_3 = 0\}}$, that is,

$$z_1 = L_1 \subset E_1 \simeq \mathbb{P}^2.$$

In order for this blowup to be volume preserving, by Proposition 3.1.25 we need that $L_1 \subset E_1 \cap D_1$.

If P is a singularity of type A_1 or A_2, by Lemma 5.1.8 and Proposition 5.1.11, $D_1 \subset X_1$ is a K3 surface. For the A_1 case, we have that $E_1 \cap D_1$ is an irreducible conic. So the weights $(1,1,2)$ are not volume preserving in this situation because L_1 is a line and $E_1 \cap D_1$ is an irreducible conic. Notice that this also follows from Lemma 5.3.3.

Consider the $A_{\geq 2}$ case from now on. The following equations are with respect to the "first affine chart" $\{y_1 \neq 0\}$ in X_1. The computations are analogous in the other ones.

One has

$$\begin{aligned} D_1 &= \{x_2x_3 + x_1B(1,x_2,x_3) + x_1^2C(1,x_2,x_3) = 0\} \\ &= \left\{ x_2x_3 + x_1 \sum_{i=0}^{3} b_i(x_2,x_3) + x_1^2 \sum_{i=0}^{4} c_i(x_2,x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)} \end{aligned}$$

and $E_1 = \{x_1 = 0\}$.

Therefore

$$E_1 \cap D_1 = \{x_1 = x_2x_3 = 0\} = \{x_1 = x_2 = 0\} \cup \{x_1 = x_3 = 0\}.$$

Observe that $L_1 = \{x_1 = x_3 = 0\}$ is indeed contained in $E_1 \cap D_1$.

By Proposition 3.1.12, π_2 is volume preserving.

Therefore the weights $(1,1,2)$ are volume preserving for any quartic in $\mathcal{A}_{\geq 2}$.

Weights $(1,1,3)$**:** By Lemma 5.3.3 we are led to consider the $A_{\geq 3}$ case.

We need to insert in Σ_2 the ray

$$(1,1,3) \in \mathrm{Cone}((1,1,2),(0,0,1)).$$

This insertion corresponds to blowing up the orbit of the torus action on X_2 associated to the 2-dimensional cone $\mathrm{Cone}((1,1,2),v_3)$. This orbit is precisely the line $L_2 = E_2 \cap \widetilde{\{x_3 = 0\}}$, that is,

$$z_2 = L_2 \subset E_2 \simeq \mathbb{P}(\mathcal{N}^\vee_{L_1/X_1}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{F}_2.$$

In order for this blowup to be volume preserving, by Proposition 3.1.25 we need that $L_2 \subset E_2 \cap D_2$.

In the "first affine chart" $\{y_1 \neq 0\}$ in X_2, one has

$$D_2 = \left\{ x_2x_3 + \sum_{i=0}^{3} b_i(x_2, x_1x_3) + x_1 \sum_{i=0}^{4} c_i(x_2, x_1x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$$

and $E_2 = \{x_1 = 0\}$.

Therefore

$$E_2 \cap D_2 = \{x_1 = x_2x_3 + b_0 + \beta_2 x_2 + \rho_2 x_2^2 + \sigma_0 x_2^3 = 0\}$$

and it is a conic or cubic depending on whether $\sigma_0 = 0$.

One can check

$$\begin{aligned} L_2 = \{x_1 = x_3 = 0\} \subset E_2 \cap D_2 &\Leftrightarrow b_0 = \beta_2 = \rho_2 = \sigma_0 = 0 \\ &\Leftrightarrow x_3 \mid B \\ &\Leftrightarrow D_1 \text{ is tangent to } \{x_3 = 0\} \text{ along } L_1. \end{aligned}$$

The last condition is a consequence of the geometric properties of the blowup.

Note that $L_2 \subset E_2 \setminus E_1^2$. Indeed E_1^2 does not appear in the affine chart $\{y_1 \neq 0\}$, that is,

$$E_1^2 \cap (\text{affine chart } \{y_1 \neq 0\}) = \emptyset,$$

and $E_1^2 \subset (\text{affine chart } \{y_3 \neq 0\})$, while

$$L_2 \cap (\text{affine chart } \{y_3 \neq 0\}) = \emptyset.$$

If $L_2 \subset E_2 \cap D_2$, then Proposition 3.1.12 implies that π_3 is volume preserving.

Therefore the weights $(1,1,3)$ are volume preserving for any quartic in $\mathcal{A}_{\geq 3}$ such that $b_0 = \beta_2 = \rho_2 = \sigma_0 = 0$. We point out that the condition $b_0 = 0$ on an element of $\mathcal{A}_{\geq 2}$ implies that it belongs to $\mathcal{A}_{\geq 3}$. Conversely,

$$\mathcal{A}_{\geq 2} \cap \{b_0 = 0\} = \mathcal{A}_{\geq 3} \subset \mathbb{P}^{34}.$$

So the weights $(1,1,3)$ are not volume preserving for a generic D in the corresponding coarse moduli space.

Besides the relation $b_0 = 0$, we need the extra closed conditions $\beta_2 = \rho_2 = \sigma_0 = 0$ on an element of $\mathcal{A}_{\geq 3}$ so that the weights $(1,1,3)$ are volume preserving.

$$(1,1,3) \text{ are volume preserving weights for } D \in \mathcal{A}_{\geq 3} \Leftrightarrow x_3 \mid B.$$

Observe that if $x_3 \mid B$, then $x_3 \nmid C$. Otherwise, x_3 divides the equation that defines D, and therefore D would be reducible. Write

$$C = \sum_{i=0}^{4} c_i' x_3^{4-i},$$

where $c_i' \in \mathbb{C}[x_1, x_2]_i$. The condition $x_3 \nmid C$ implies that $c_4' \neq 0$. Since c_4' is a homogeneous polynomial in 2 variables over an algebraically closed field, we can factorize it into linear factors

$$\begin{aligned} c_4' &= \upsilon_0 x_1^4 + \upsilon_1 x_1^3 x_2 + \upsilon_2 x_1^2 x_2^2 + \upsilon_3 x_1 x_2^3 + \upsilon_4 x_2^4 \\ &= \prod_{i=1}^{4} (\alpha_i x_1 + \varrho_i x_2), \end{aligned}$$

where $\upsilon_i, \alpha_i, \varrho_i \in \mathbb{C}$ for all i. Take $\ell_i \coloneqq \{x_3 = \alpha_i x_1 + \varrho_i x_2 = 0\}$ and notice that $D \supset \ell_i$.

Thus, an element of $\mathcal{A}_{\geq 3}$ for which the weights $(1,1,3)$ are volume preserving necessarily contains lines through the singular point P. The union of these lines ℓ_i constitutes a hyperplane section of D.

Weights $(1,1,4)$**:** By Lemma 5.3.3 we are led to consider the $A_{\geq 4}$ case. From the previous case, we must have $b_0 = \beta_2 = \rho_2 = \sigma_0 = 0$. We will consider elements in

$$\mathcal{A}_{\geq 4} \cap \{\beta_2 = \rho_2 = \sigma_0 = 0\} \subset \mathbb{P}^{34}.$$

We suppressed the condition $\{b_0 = 0\}$ because it is already satisfied for elements in $\mathcal{A}_{\geq 4}$. We need to insert in Σ_3 the vector

$$(1,1,4) \in \mathrm{Cone}((1,1,3),(0,0,1)).$$

This insertion corresponds to blowing up the orbit of the torus action on X_3 associated to the 2-dimensional cone $\mathrm{Cone}((1,1,3), v_3)$. This orbit is precisely the line $L_3 = E_3 \cap \widetilde{\{x_3 = 0\}}$, that is,

$$z_3 = L_3 \subset E_3 \simeq \mathbb{P}(\mathcal{N}^\vee_{L_2/X_2}) \simeq \mathbb{F}_{m_2}.$$

For this blowup to be volume preserving, by Proposition 3.1.25 we need that $L_3 \subset E_3 \cap D_3$. In the "first affine chart" $\{y_1 \neq 0\}$ in X_3, one has

$$D_3 = \left\{ x_2 x_3 + x_1 \sum_{i=1}^{3} b_i'(x_2, x_3) + \sum_{i=0}^{4} c_i(x_2, x_1^2 x_3) = 0 \right\} \subset \mathbb{A}_{(x_1,x_2,x_3)},$$

where $b_i'(x_2, x_3) \in \mathbb{C}[x_2, x_3]_i$ is such that $x_1^2 b_i'(x_2, x_3) = b_i(x_2, x_1^2 x_3)$ for $i \in \{1,2,3\}$, and $E_3 = \{x_1 = 0\}$. Recall that here we have $b_0 = \beta_2 = \rho_2 = \sigma_0 = 0$.

Therefore

$$E_3 \cap D_3 = \{x_1 = x_2 x_3 + c_0 + \delta_2 x_2 + \varepsilon_2 x_2^2 + \tau_0 x_2^3 + \lambda_0 x_2^4 = 0\}$$

and

$$\begin{aligned} L_3 = \{x_1 = x_3 = 0\} \subset E_3 \cap D_3 &\Leftrightarrow c_0 = \delta_2 = \varepsilon_2 = \tau_0 = \lambda_0 = 0 \\ &\Leftrightarrow x_3 \mid C. \end{aligned}$$

We have the following:

$$\begin{aligned}
&(1,1,4) \text{ are volume preserving weights for } D \in \mathcal{A}_{\geq 4} \\
\Leftrightarrow\ &(1,1,3) \text{ are volume preserving weights for } D \in \mathcal{A}_{\geq 4} \text{ and } x_3 \mid C \\
\Leftrightarrow\ &x_3 \mid B, C.
\end{aligned}$$

Observe that if $x_3 \mid B, C$, so x_3 divides the equation that defines D, and therefore D would be reducible.

Therefore the weights $(1,1,4)$ are not volume preserving for any element in $\mathcal{A}_{\geq 4}$.

5.3.3.2 Toric description of the weights $(1,2,b)$

The toric description of the weighted blowup with weights $(1,2,b)$ says that we must start by inserting in Σ_0 the ray

$$(1,1,1) \in \mathrm{Cone}(v_1, v_2, v_3).$$

We already discussed this step, which corresponds to the blowup of $P_0 = P$.

Then we need to insert in Σ_1 the ray

$$(1,2,2) \in \mathrm{Cone}((1,1,1), v_2, v_3).$$

This insertion corresponds to blowing up the orbit of the torus action on X_1 associated to the 3-dimensional cone $\mathrm{Cone}((1,1,1), v_2, v_3)$. This orbit is precisely

$$E_1 \cap \widetilde{\{x_2 = 0\}} \cap \widetilde{\{x_3 = 0\}} = (0,0,0) := P_1$$

in the "first affine chart" $\{y_1 \neq 0\}$.

In order for this blowup to be volume preserving, by Lemma 5.2.2 we need that $P_1 \in \mathrm{Sing}(D_1)$.

In the "first affine chart" $\{y_1 \neq 0\}$ in X_1, one has

$$\begin{aligned}
D_1 &= \{x_2 x_3 + x_1 B(1, x_2, x_3) + x_1^2 C(1, x_2, x_3) = 0\} \\
&= \left\{ x_2 x_3 + x_1 \sum_{i=0}^{3} b_i(x_2, x_3) + x_1^2 \sum_{i=0}^{4} c_i(x_2, x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1, x_2, x_3)}
\end{aligned}$$

and $E_1 = \{x_1 = 0\}$.

Notice that $P_1 \in D_1$ and by the Jacobian Criterion,

$$P_1 \in \mathrm{Sing}(D_1) \Leftrightarrow b_0 = 0.$$

In this case, Proposition 3.1.12 implies that π_2 is volume preserving.

Weights $(1,2,3)$**:** By Lemma 5.3.3 we are led to consider the $A_{\geq 4}$ case.

After inserting the rays $(1,1,1)$ and $(1,1,2)$, we need to insert in Σ_2 the ray

$$(1,2,3) \in \mathrm{Cone}((1,2,2), (0,0,1)).$$

This insertion corresponds to blowing up the orbit of the torus action on X_2 associated to the 2-dimensional cone $\mathrm{Cone}((1,2,2),v_3)$. This orbit is precisely the line $L_2 = E_2 \cap \widetilde{\{x_3 = 0\}}$, that is,

$$z_2 = L_2 \subset E_2 \simeq \mathbb{P}^2.$$

In order for this blowup to be volume preserving, by Proposition 3.1.25 we need that $L_2 \subset E_2 \cap D_2$.

If P is a singularity of type A_3, by Lemma 5.1.8 and Proposition 5.1.11, $D_2 \subset X_2$ is nonsingular and $E_2 \cap D_2$ is an irreducible conic. So the weights $(1,2,3)$ are not volume preserving. Suppose now P is a singularity of type $A_{\geq 4}$.

In the "first affine chart" $\{y_1 \neq 0\}$ in X_2, one has

$$\begin{aligned} D_2 &= \left\{ x_2x_3 + \sum_{i=1}^{3} x_1^{i-1} b_i(x_2,x_3) + \sum_{i=0}^{4} x_1^i c_i(x_2, x_1x_3) = 0 \right\} \\ &= \left\{ x_2x_3 + b_1(x_2,x_3) + x_1 b_2(x_2,x_3) + x_1^2 b_1(x_2,x_3) + \sum_{i=0}^{4} x_1^i c_i(x_2,x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)} \end{aligned}$$

and $E_2 = \{x_1 = 0\}$.

Therefore

$$E_2 \cap D_2 = \{x_1 = x_2x_3 + \beta_2 x_2 + \beta_3 x_3 + c_0 = 0\}$$

and

$$L_2 = \{x_1 = x_3 = 0\} \subset E_2 \cap D_2 \Leftrightarrow \beta_2 = c_0 = 0.$$

If $L_2 \subset E_2 \cap D_2$, then Proposition 3.1.12 implies that π_3 is volume preserving.

Therefore the weights $(1,2,3)$ are volume preserving for any quartic in $\mathcal{A}_{\geq 4}$ such that $\beta_2 = c_0 = 0$. We point out that the condition $c_0 = \beta_2\beta_3$ on an element of $\mathcal{A}_{\geq 3}$ implies that it belongs to $\mathcal{A}_{\geq 4}$. Conversely,

$$\mathcal{A}_{\geq 3} \cap \{c_0 = \beta_2\beta_3\} = \mathcal{A}_{\geq 4} \subset \mathbb{P}^{34}.$$

So the weights $(1,2,3)$ are not volume preserving for a generic D in the corresponding coarse moduli space.

Besides the relation $c_0 = \beta_2\beta_3$, we need the extra closed conditions $\beta_2 = c_0 = 0$ on an element of $\mathcal{A}_{\geq 4}$ so that the weights $(1,2,3)$ are volume preserving.

$$(1,2,3) \text{ are volume preserving weights for } D \in \mathcal{A}_{\geq 4} \Leftrightarrow \beta_2 = c_0 = 0.$$

Weights $(1,2,5)$: By Lemma 5.3.3 we are led to consider the $A_{\geq 6}$ case.

According to the toric description, we need to insert in Σ_3 the ray

$$(1,2,4) \in \mathrm{Cone}((1,2,3),(0,0,1)).$$

From the previous steps, we must have $b_0 = 0$ and $\beta_2 = c_0 = 0$.

These conditions imply that the quartic is in

$$\mathcal{A}_{\geq 5} \cap \{\beta_2 = 0\} \subset \mathbb{P}^{34}.$$

We suppressed the closed conditions $\{b_0 = 0\}$ and $\{c_0 = 0\}$ because the former is already satisfied for elements in $\mathcal{A}_{\geq 4}$, and the latter follows from the fact in $\mathcal{A}_{\geq 4}$ we have that

$$\beta_2 = 0 \Rightarrow c_0 = 0.$$

Inserting the ray $(1, 2, 4)$ corresponds to blowing up the orbit of the torus action on X_3 associated to the 2-dimensional cone $\mathrm{Cone}((1, 2, 3), v_3)$. This orbit is precisely the line $L_3 = E_3 \cap \widetilde{\{x_3 = 0\}}$, that is,

$$z_3 = L_3 \subset E_3 \simeq \mathbb{P}(\mathcal{N}^{\vee}_{L_2/X_2}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{F}_2.$$

For this blowup to be volume preserving, by Proposition 3.1.25 we need that $L_3 \subset E_3 \cap D_3$.

In the "first affine chart" $\{y_1 \neq 0\}$ in X_3, one has

$$D_3 = \left\{ (x_2 + \beta_3)x_3 + b_2(x_2, x_1x_3) + x_1 b_3(x_2, x_1x_3) + \sum_{i=1}^{4} x_1^{i-1} c_i(x_2, x_1x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$$

and $E_3 = \{x_1 = 0\}$.

Therefore

$$E_3 \cap D_3 = \{x_1 = (x_2 + \beta_3)x_3 + \rho_2 x_2^2 + \delta_2 x_2 = 0\}$$

and

$$\begin{aligned} L_3 = \{x_1 = x_3 = 0\} \subset E_3 \cap D_3 &\Leftrightarrow \rho_2 = \delta_2 = 0 \\ &\Leftrightarrow D_2 \text{ is tangent to } \{x_3 = 0\} \text{ along } L_2. \end{aligned}$$

If $L_3 \subset E_3 \cap D_3$, then Proposition 3.1.12 implies that π_4 is volume preserving.

Therefore π_4 is volume preserving for any quartic in $\mathcal{A}_{\geq 5}$ such that $\rho_2 = \delta_2 = 0$. We point out that the condition $\zeta = 0$ on an element of $\mathcal{A}_{\geq 4}$ implies that it belongs to $\mathcal{A}_{\geq 5}$. Conversely,

$$\mathcal{A}_{\geq 4} \cap \{\zeta = b_2(\beta_3, \beta_2) - c_1(\beta_3, \beta_2) = \rho_2\beta_3^2 + \rho_{23}\beta_2\beta_3 + \rho_3\beta_2^2 - \delta_2\beta_3 - \delta_3\beta_2 = 0\} = \mathcal{A}_{\geq 5} \subset \mathbb{P}^{34}.$$

The blowup π_4 is not volume preserving for a generic D in $\mathcal{A}_{\geq 5}$.

Besides the relation $\zeta = 0$, we need the extra closed conditions $\rho_2 = \delta_2 = 0$ on an element of $\mathcal{A}_{\geq 5}$ so that π_4 is volume preserving.

$$\pi_4 \text{ is volume preserving for } D \in \mathcal{A}_{\geq 5} \cap \{\beta_2 = 0\} \Leftrightarrow \rho_2 = \delta_2 = 0.$$

Recall that by Lemma 5.3.3 we are led to consider the $A_{\geq 6}$ case so that the weights $(1, 2, 5)$ are volume preserving. From the previous steps $(1, 2, 2), (1, 2, 3)$ and $(1, 2, 4)$, we must have $b_0 = 0$, $\beta_2 = c_0 = 0$ and $\rho_2 = \delta_2 = 0$.

Removing the redundant conditions, we must consider elements in

$$\mathcal{A}_{\geq 6} \cap \{\beta_2 = \rho_2 = \delta_2 = 0\} \subset \mathbb{P}^{34}.$$

We need to insert in Σ_4 the ray

$$(1,2,5) \in \mathrm{Cone}((1,2,4),(0,0,1)).$$

This insertion corresponds to blowing up the orbit of the torus action on X_4 associated to the 2-dimensional cone $\mathrm{Cone}((1,2,4),v_3)$. This orbit is precisely the line $L_4 = E_4 \cap \widetilde{\{x_3 = 0\}}$, that is,

$$z_4 = L_4 \subset E_4 \simeq \mathbb{P}(\mathcal{N}^\vee_{L_3/X_3}) \simeq \mathbb{F}_{m_3}.$$

For this blowup to be volume preserving, by Proposition 3.1.25 we need that $L_4 \subset E_4 \cap D_4$.

In the "first affine chart" $\{y_1 \neq 0\}$ in X_4, one has

$$D_4 = \left\{ (x_2+\beta_3)x_3 + \rho_{23}x_1x_2x_3 + \rho_3 x_1^3 x_3^2 + b_3(x_2, x_1^2x_3) + \delta_3 x_1 x_3 + \sum_{i=2}^{4} x_1^{i-2} c_i(x_2, x_1x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$$

and $E_4 = \{x_1 = 0\}$.

Therefore

$$E_4 \cap D_4 = \{x_1 = (x_2+\beta_3)x_3 + \sigma_0 x_2^3 + \varepsilon_2 x_2^2 = 0\}$$

and

$$\begin{aligned} L_4 = \{x_1 = x_3 = 0\} \subset E_4 \cap D_4 &\Leftrightarrow \sigma_0 = \varepsilon_2 = 0 \\ &\Leftrightarrow D_3 \text{ is tangent to } \{x_3 = 0\} \text{ along } L_3. \end{aligned}$$

If $L_4 \subset E_4 \cap D_4$, then Proposition 3.1.12 implies that π_5 is volume preserving.

Therefore the weights $(1,2,5)$ are volume preserving for any quartic in $\mathcal{A}_{\geq 6}$ such that $\sigma_0 = \varepsilon_2 = 0$. We point out that the condition $\xi_2\xi_3 = \alpha$ on an element of $\mathcal{A}_{\geq 5}$ implies that it belongs to $\mathcal{A}_{\geq 6}$. Conversely,

$$\mathcal{A}_{\geq 5} \cap \{\xi_2\xi_3 = \alpha\} = \mathcal{A}_{\geq 6} \subset \mathbb{P}^{34}.$$

So the weights $(1,2,5)$ are not volume preserving for a generic D in $\mathcal{A}_{\geq 6}$.

Besides the relation $\xi_2\xi_3 = \alpha$, we need the extra closed conditions $\sigma_0 = \varepsilon_2 = 0$ over an element of $\mathcal{A}_{\geq 6}$ so that the weights $(1,2,5)$ are volume preserving.

$$(1,2,5) \text{ are volume preserving weights for } D \in \mathcal{A}_{\geq 6} \cap \{\beta_2 = \rho_2 = \delta_2 = 0\} \Leftrightarrow \sigma_0 = \varepsilon_2 = 0.$$

At this point, for quartics in

$$D \in \mathcal{A}_{\geq 6} \cap \{\beta_2 = \rho_2 = \delta_2 = \sigma_0 = \varepsilon_2 = 0\}$$

the criteria $(*_5)$; $(*_5)$ and $\gamma_2\gamma_3 \neq \mu$; and $(*_5)$ and $\gamma_2\gamma_3 = \mu$ to detect whether P is $A_{\geq 7}$, A_7 and $A_{\geq 8}$, respectively, become much simpler because for $\mathcal{A}_{\geq 6} \cap \{\beta_2 = \rho_2 = \delta_2 = \sigma_0 = \varepsilon_2 = 0\}$ we have

- $(*_5)$: $(*_4)$ and $\theta = -\tau_0\beta_3^3$;
- $\gamma_2 = 0$;
- $\gamma_3 = \rho_{23}\xi_3 + \sigma_1\beta_3^2 - \varepsilon_{23}\beta_3$;
- $\mu = \beta_3^2(-3\tau_0\xi_3 + \lambda_0)$.

Since in this setting $\gamma_2\gamma_3 = 0$, to detect whether P is $A_{\geq 8}$ it is enough to check if μ equals 0.

5.3.3.3 Toric description of the weights $(1, 3, b)$

The toric description of the weighted blowup with weights $(1, 3, b)$ says that we must start by inserting in Σ_0 the ray

$$(1, 1, 1) \in \operatorname{Cone}(v_1, v_2, v_3),$$

followed by the insertion in Σ_1 of the ray

$$(1, 2, 2) \in \operatorname{Cone}((1, 1, 2), v_2, v_3).$$

We already discussed these steps which correspond to the blowup of $P_0 = P$ and $P_1 = E_1 \cap \widetilde{\{x_2 = 0\}} \cap \widetilde{\{x_3 = 0\}}$, respectively.

Then we need to insert in Σ_2 the ray

$$(1, 3, 3) \in \operatorname{Cone}((1, 2, 2), v_2, v_3).$$

This insertion corresponds to blowing up the orbit of the torus action on X_2 associated to the 3-dimensional cone $\operatorname{Cone}((1, 2, 2), v_2, v_3)$. This orbit is

$$E_2 \cap \widetilde{\{x_2 = 0\}} \cap \widetilde{\{x_3 = 0\}} = (0, 0, 0) := P_2.$$

For this blowup to be volume preserving weights, by Lemma 5.2.2 we need that $P_2 \in \operatorname{Sing}(D_2)$.

In the "first affine chart" $\{y_1 \neq 0\}$ in X_2, one has

$$\begin{aligned} D_2 &= \left\{ x_2x_3 + \sum_{i=1}^{3} x_1^{i-1} b_i(x_2, x_3) + \sum_{i=0}^{4} x_1^i c_i(x_2, x_3) = 0 \right\} \\ &= \left\{ x_2x_3 + b_1(x_2, x_3) + x_1 b_2(x_2, x_3) + x_1^2 b_1(x_2, x_3) + \sum_{i=0}^{4} x_1^i c_i(x_2, x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1, x_2, x_2)} \end{aligned}$$

and $E_2 = \{x_1 = 0\}$.

Notice that $P_2 \in D_2 \Leftrightarrow c_0 = 0$. By the Jacobian Criterion,

$$P_2 \in \operatorname{Sing}(D_2) \Leftrightarrow \beta_2 = \beta_3 = 0.$$

In this case, Proposition 3.1.12 implies that π_3 is volume preserving.

Weights $(1,3,4)$**:** By Lemma 5.3.3 we are led to consider the $A_{\geq 6}$ case. From the previous steps, we must have $b_0 = 0$ and $c_0 = \beta_2 = \beta_3 = 0$, that is, we must consider elements in

$$\mathcal{A}_{\geq 6} \cap \{\beta_2 = \beta_3 = 0\} \subset \mathbb{P}^{34}.$$

We suppressed the condition $\{b_0 = 0\}$ because it is already satisfied for elements in $\mathcal{A}_{\geq 6}$, since we have that

$$\mathcal{A}_{\geq 6} \subset \mathcal{A}_{\geq 2};$$

as well as the condition $\{c_0 = 0\}$ because we have that

$$\mathcal{A}_{\geq 6} \subset \mathcal{A}_{\geq 4},$$

since for elements in $\mathcal{A}_{\geq 4}$ the relation $c_0 = \beta_2\beta_3$ holds.

We need to insert in Σ_4 the ray

$$(1,3,4) \in \operatorname{Cone}((1,3,3),(0,0,1)).$$

This insertion corresponds to blowing up the orbit of the torus action on X_3 associated to the 2-dimensional cone $\operatorname{Cone}((1,3,3),v_3)$. This orbit is precisely the line $L_3 = E_3 \cap \widetilde{\{x_3 = 0\}}$, that is,

$$z_3 = L_3 \subset E_3 \simeq \mathbb{P}^2.$$

In order for this blowup to be volume preserving, by [ACM, Proposition 3.9] we need that $L_3 \subset E_3 \cap D_3$.

If P is a singularity of type A_5, by Proposition 5.1.11 and [**?**, Lemma 5.18], $D_3 \subset X_3$ is nonsingular and $E_3 \cap D_3$ is an irreducible conic. So the weights $(1,3,4)$ are not volume preserving.

Suppose now P is a singularity of type $A_{\geq 6}$.

In the "first affine chart" $\{y_1 \neq 0\}$ in X_3, one has

$$D_3 = \left\{ x_2x_3 + \sum_{i=2}^{3} x_1^{2i-3} b_i(x_2,x_3) + \sum_{i=1}^{4} x_1^{2i-2} c_i(x_2,x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$$

and $E_3 = \{x_1 = 0\}$.

Therefore

$$E_3 \cap D_3 = \{x_1 = x_2x_3 + \delta_2 x_2 + \delta_3 x_3 = 0\}$$

and

$$L_3 = \{x_1 = x_3 = 0\} \subset E_3 \cap D_3 \Leftrightarrow \delta_2 = 0.$$

If $L_3 \subset E_3 \cap D_3$, then Proposition 3.1.12 implies that π_4 is volume preserving.

Therefore the weights $(1,3,4)$ are volume preserving for any quartic in $\mathcal{A}_{\geq 6}$ such that $\delta_2 = 0$. We point out that the condition $\theta = 0$ on an element of $\mathcal{A}_{\geq 5}$ implies that it belongs to $\mathcal{A}_{\geq 6}$. Conversely,

$$\mathcal{A}_{\geq 5} \cap \{\theta = 0\} = \mathcal{A}_{\geq 6} \subset \mathbb{P}^{34}.$$

So the weights $(1,3,4)$ are not volume preserving for a generic D in $\mathcal{A}_{\geq 6}$.

Besides the relation θ, we have the extra closed condition $\delta_2 = 0$ on an element of $\mathcal{A}_{\geq 6}$ so that the weights $(1,3,4)$ are volume preserving.

$$(1,3,4) \text{ are volume preserving weights for } D \in \mathcal{A}_{\geq 6} \cap \{\beta_2 = \beta_3 = 0\} \Leftrightarrow \delta_2 = 0.$$

Weights $(1,3,5)$: By Lemma 5.3.3 we are led to consider the $A_{\geq 7}$ case. From the previous steps, we need to take into account $\beta_2 = \beta_3 = 0$ and $\delta_2 = 0$, that is, we must consider elements in

$$\mathcal{A}_{\geq 6} \cap \{\beta_2 = \beta_3 = \delta_2 = 0\} \subset \mathbb{P}^{34}.$$

We need to insert in Σ_4 the ray

$$(1,3,5) \in \operatorname{Cone}((1,3,4),(0,0,1)).$$

This insertion corresponds to blowing up the orbit of the torus action on X_4 associated to the 2-dimensional cone $\operatorname{Cone}((1,3,4), v_3)$. This orbit is precisely the line $L_4 = E_4 \cap \widetilde{\{x_3 = 0\}}$, that is,

$$z_4 = L_4 \subset E_4 \simeq \mathbb{P}(\mathcal{N}^\vee_{L_3/X_3}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{F}_2.$$

In order for this blowup to be volume preserving, by Proposition 3.1.25 we need that $L_4 \subset E_4 \cap D_4$.

In the "first affine chart" $\{y_1 \neq 0\}$ in X_4, one has

$$D_4 = \left\{ (x_2 + \delta_3)x_3 + \sum_{i=2}^{3} x_1^{2i-4} b_i(x_2, x_1x_3) + \sum_{i=2}^{4} x_1^{2i-3} c_i(x_2, x_1x_3) = 0 \right\} \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$$

and $E_4 = \{x_1 = 0\}$.

Therefore

$$E_4 \cap D_4 = \{x_1 = (x_2 + \delta_3)x_3 + \rho_2 x_2^3 = 0\}$$

and

$$\begin{aligned} L_4 = \{x_1 = x_3 = 0\} \subset E_4 \cap D_4 &\Leftrightarrow \rho_2 = 0 \\ &\Leftrightarrow D_3 \text{ is tangent to } \{x_3 = 0\} \text{ along } L_3. \end{aligned}$$

If $L_4 \subset E_4 \cap D_4$, then Proposition 3.1.12 implies that π_5 is volume preserving.

Therefore the weights $(1,3,5)$ are volume preserving for any quartic in $\mathcal{A}_{\geq 7}$ such that $\rho_2 = 0$. We point out that the condition $\gamma_2\gamma_3 = \mu$ on an element of $\mathcal{A}_{\geq 6}$ implies that it belongs to $\mathcal{A}_{\geq 7}$. Conversely,

$$\mathcal{A}_{\geq 6} \cap \{\gamma_2\gamma_3 = \mu\} = \mathcal{A}_{\geq 7} \subset \mathbb{P}^{34}.$$

So the weights $(1,3,5)$ are not volume preserving for a generic D in $\mathcal{A}_{\geq 6}$.

Besides the relation $\gamma_2\gamma_3 = \mu$, we have the extra condition $\rho_2 = 0$ on an element of $\mathcal{A}_{\geq 7}$ so that the weights $(1,3,5)$ are volume preserving.

$$(1,3,5) \text{ are volume preserving weights for } D \in \mathcal{A}_{\geq 7} \cap \{\beta_2 = \beta_2 = \delta_2 = 0\} \Leftrightarrow \rho_2 = 0.$$

At this point, for quartics in

$$D \in \mathcal{A}_{\geq 7} \cap \{\beta_2 = \beta_2 = \delta_2 = \rho_2 = 0\}$$

the criteria $(*_5)$ and $\gamma_2\gamma_3 \neq \mu$ and $(*_5)$ and $\gamma_2\gamma_3 = \mu$ to detect whether P is A_7 and $A_{\geq 8}$, respectively, become much simpler because for $\mathcal{A}_{\geq 7} \cap \{\beta_2 = \beta_2 = \delta_2 = \rho_2 = 0\}$ we have

- $\gamma_2 = 0$;
- $\gamma_3 = -\rho_{23}\xi_3$;
- $\mu = \varepsilon_2\xi_3^2$.

Since in this setting $\gamma_2\gamma_3 = 0$, to detect whether P is $A_{\geq 8}$ it is enough to check if μ equals 0.

In the following Table 5.7, we summarize all the necessary and sufficient conditions so that the insertion of the ray $(1,c,d)$ corresponds to a volume preserving blowup. If there do not exist any conditions, we will write "generic" as the case of the rays $(1,1,1)$ and $(1,1,2)$.

ray inserted	conditions
(1,1,1)	generic on $\mathcal{A}_{\geq 1}$
(1,1,2)	generic on $\mathcal{A}_{\geq 2}$
(1,1,3)	$b_0 = \beta_2 = \rho_2 = \sigma_0 = 0 \Leftrightarrow x_3 \mid B$
(1,1,4)	$c_0 = \delta_2 = \varepsilon_2 = \tau_0 = \lambda_0 = 0 \Leftrightarrow x_3 \mid C$
(1,2,2)	$b_0 = 0$
(1,2,3)	$\beta_2 = c_0 = 0$
(1,2,4)	$\rho_2 = \delta_2 = 0$
(1,2,5)	$\sigma_0 = \varepsilon_2 = 0$
(1,3,3)	$c_0 = \beta_2 = \beta_3 = 0$
(1,3,4)	$\delta_2 = 0$
(1,3,5)	$\rho_2 = 0$

Table 5.7: Table summarizing necessary and sufficient conditions for the induced blowup is volume preserving.

5.3.4 Proof of Theorems 5.2.1 & 5.2.4

So far we have obtained explicit criteria to recognize certain canonical singularities and at the same time criteria so that the weights analyzed are volume preserving. By comparing both criteria, we can obtain the desired weights listed in Tables 5.4 & 5.6.

Remark 5.3.4. For the unique quartic surface in $\mathbb{P}^3$, up to automorphisms of ambient space, with a singularity of type A_{19}, the only volume preserving weights are $(1,1,1)$ and $(1,1,2)$ according to the criteria exposed in Table 5.7. Indeed, from its equation 5.3.3 we have that $\beta_2 = -4i \neq 0$ and therefore the weights $(1,1,3)$ and the weights of the form $(1,2,b)$ and $(1,3,b)$ are not volume preserving.

5.4 Volume preserving x Sarkisov factorization

In this section, we will exhibit in detail the example mentioned in Section 4.4 to show that Theorem 4.2.2 and Theorem 4.2.6 do not admit generalizations in higher dimension.

First, let us just recall this example given in Section 4.4.

Set $D \subset \mathbb{P}^3$ as a general irreducible normal quartic surface having a single singularity of type A_1 at $P = (1:0:0:0)$.

It follows that the equation of D can be written in the form $x_0^2 A + x_0 B + C$, where $A, B, C \in \mathbb{C}[x_1, x_2, x_3]$ are general homogeneous polynomials of degrees 2, 3, 4, respectively. Additionally, A is a quadratic form of rank 3.

In the proof of [ACM, Claim 5.8], Araujo, Corti & Massarenti show that the birational involution

$$\phi\colon (x_0 : x_1 : x_2 : x_3) \mapsto (-Ax_0 - B : Ax_1 : Ax_2 : Ax_3)$$

belongs to $\mathrm{Dec}(D)$.

One verifies that $\mathrm{Bs}(\phi) = V(A, B)$, and it is composed of the union of six pairwise distinct lines through P if we take B general enough. This implies that $\mathrm{Bs}(\phi) \not\subset D$, which differs from the assertion of Theorem 4.2.2 in dimension 2, stating that the base locus lies within the boundary divisor.

By exploiting this fact, we can construct a Sarkisov factorization that is not volume preserving, thus demonstrating that an extension of Theorem 4.2.6 does not hold in higher dimensions.

Let us analyze carefully the map $\phi \in \mathrm{Dec}(D)$. Its associated linear system $\Gamma \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$ is in particular contained in the linear system of space cubics passing through the reducible curve $V(A, B)$. Write $V(A, B) = L_1 \cup \ldots \cup L_6$, where each L_i stands for a line through P.

Consider $\mathcal{H}$ a general member of Γ. Notice that P is a singularity of $\mathrm{Bs}(\phi)$ as well as of $\mathcal{H}$ and D. In particular, we have that $P \in \mathrm{Sing}(\mathcal{H})$ is a canonical singularity of type A_1 and $m_P(\mathcal{H}) = 2$. This observation will be important later on on many occasions.

Indeed, $\mathcal{H}$ is of the form

$$\lambda_0(-Ax_0 - B) + \lambda_1 Ax_1 + \lambda_2 Ax_2 + \lambda_3 Ax_3,$$

for some $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$ identified with Γ.

Dehomogenizing $\mathcal{H}$ with respect to x_0, we get the equation of $\mathcal{H}$ in $\{x_0 \neq 0\}$ becomes

$$\lambda_0(-A - B) + \lambda_1 Ax_1 + \lambda_2 Ax_2 + \lambda_3 Ax_3.$$

Thus, $TC_P\mathcal{H} = \{-\lambda_0 A = 0\}$ whose projectivization is an irreducible conic, since $\operatorname{rank}(A) = 3$. We can assure that $P \in \operatorname{Sing}(\mathcal{H})$ is of type A_1 because a single blowup of the ambient space will be enough to resolve the singularity. Analogously to the many computations made in this chapter, one can check that the corresponding exceptional divisor intersected with the strict transform of $\mathcal{H}$ is an irreducible conic.

Another way to argue why $P \in \operatorname{Sing}(\mathcal{H})$ is of type A_1 is by looking at $TC_P\mathcal{H}$ and comparing it with the tangent cones of normal forms of surface canonical singularities. We would be using the fact that canonical singularities are equivalent to rational double points in dimension 2.

Let us run the Sarkisov Program for ϕ. From now on we will follow the notation and algorithm described in [Cor1]. Although it has a slightly different notation, we refer the reader to [Mat, Flowchart 13-1-9] for an explicit flowchart.

Notation. Henceforth, abusing notation, sometimes we will denote divisors on varieties and their strict transforms or pushforwards in others with the same symbol. Moreover, to avoid confusion in some instances, we will denote certain strict transforms or pushforwards with a right lower index indicating the ambient variety. We will do the same for general members of linear systems.

We will exhibit in detail a possible Sarkisov factorization for ϕ proceeding in steps.

The starting point or Step 0 in the Sarkisov Program is to compute the Sarkisov degree (μ, c, e) of the corresponding birational map ϕ. It will guide us along the factorization process. Performing a lot of computations, one can find that the Sarkisov degree of ϕ is $\left(\frac{3}{4}, 1, 9\right)$.

Extend the notion of infinitesimal neighborhood, see Definition 2.2.1, analogously in higher dimension and for subvarieties other than closed points. The 9 crepant exceptional divisors with respect to the pair $(\mathbb{P}^3, c\mathcal{H}) = (\mathbb{P}^3, \mathcal{H})$ are the exceptional divisors corresponding to the blowups of

- P,
- a curve e in the first infinitesimal neighborhood of P,
- a curve e' in the first infinitesimal neighborhood of e and
- of the six lines $L_1, \ldots, L_6$.

Step 1: Following the Sarkisov algorithm in [Cor1], since $c = 1 < \frac{4}{3} = \frac{1}{\mu}$, the first link in the Sarkisov factorization is of type I or II. This link is initiated by an *extremal blowup* [Cor1, Proposition-Definition 2.10], which always exists in this situation.

This choice of the extremal blowup is not determined by the algorithm. We are free to choose it. In our case, we have seven possibilities for such maps which are the blowup of $\mathbb{P}^3$ at P or the blowup of $\mathbb{P}^3$ along one of the lines L_i through P.

Only the first option gives us a volume preserving map, whereas the remaining ones do not. Indeed, let $\sigma_P \colon Z_1 \to \mathbb{P}^3$ be the blowup of P and set $E_P := \operatorname{Exc}(\sigma_P)$. By the Adjunction

Formula, we have $K_{Z_1} = \sigma_P^* K_{\mathbb{P}^3} + 2E_P$ and because $m_P(D) = 2$, we have $D_{Z_1} \sim \sigma_P^* D - 2E_P$. Hence,

$$K_{Z_1} + D_{Z_1} = \sigma_P^*(K_{\mathbb{P}^3} + D),$$

and since $(\sigma_P)_* D_{Z_1} = D$, Proposition 3.1.12 implies that σ_P is volume preserving.

Without less of generality, let $\sigma_1 \colon Z_1' \to \mathbb{P}^3$ be the blowup of L_1 and set $E_1 \coloneqq \mathrm{Exc}(\sigma_1)$. By the Adjunction Formula, we have $K_{Z_1'} = \sigma_1^* K_{\mathbb{P}^3} + E_1$ and because $L_1 \not\subset D$, we have $D_{Z_1'} \sim \sigma_1^* D$. Hence,

$$K_{Z_1'} + D_{Z_1'} = \sigma_1^*(K_{\mathbb{P}^3} + D) + E_1.$$

Notice that $(K_{Z_1'}, D_{Z_1'})$ is no longer a Calabi-Yau pair. For the sake of contradiction, suppose that σ_1 is volume preserving for some reduced Weil divisor D_1 on Z_1' making $(K_{Z_1'}, D_1)$ a Calabi-Yau pair. One has $a(E_1, \mathbb{P}^3, D) = 1$ by the previous formula, and hence we must have $a(E_1, Z_1', D_1) = 1$. Since $E_1 \subset Z_1'$, by definition of discrepancy this implies the $E_1 \subset \mathrm{Supp}(D_1)$ and it has coefficient -1. But this is absurd because we are assuming D_1 reduced. Therefore, σ_1 is not volume preserving.

Let us proceed with σ_P. Consider $\mathcal{H}$ a general member of the linear system associated to ϕ. Since $c = 1$, the next thing to do is to run the $(K_{Z_1} + \mathcal{H}_{Z_1})$-MMP over $\mathrm{Spec}(\mathbb{C})$. One can verify it results in the Mori fibered space structure $\pi_1 \colon Z_1 \to \mathbb{P}^2$.

Thus, the first (volume preserving) Sarkisov link in a factorization of ϕ is of type I, and it is given by the blowup of $\mathbb{P}^3$ at P. We have that $E_P \simeq \mathbb{P}^2$, $\tilde{A} \simeq \mathbb{F}_2$ and (by abuse of notation) $e = E_P \cap D_{Z_1} \simeq \mathbb{P}^1$. Moreover, the curve e is contained in $\mathrm{Bs}(\phi \circ \sigma_P)$. All the lines L_i are separated in Z_1 and are, in particular, rulings of $\mathbb{F}_2$. Observe that π_1 maps e isomorphically onto the conic $\{A = 0\} \subset \mathbb{P}^2$. We have the following picture:

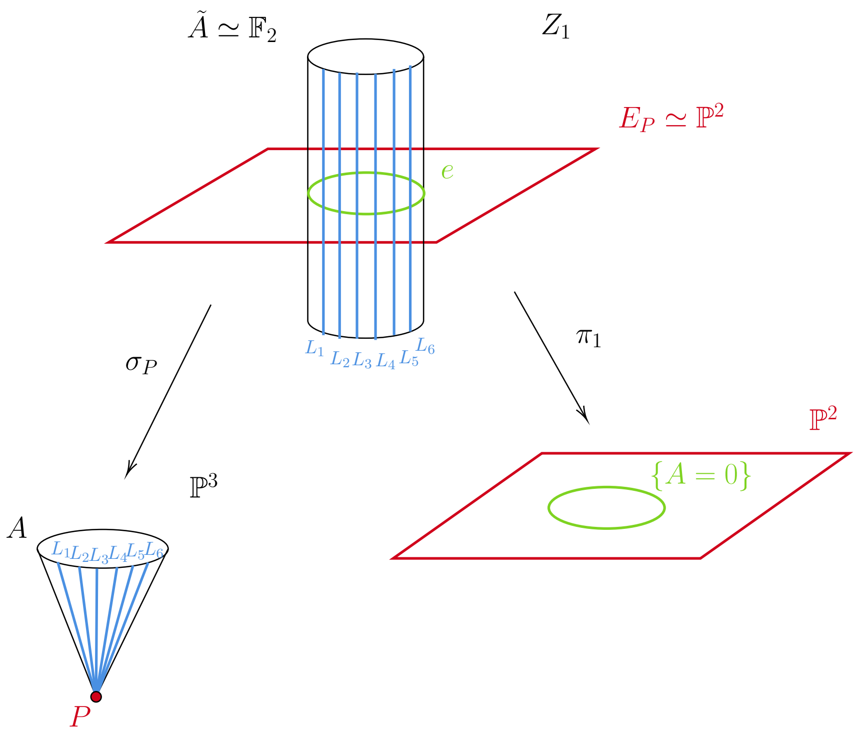


Figure 5.7: First link in a Sarkisov factorization of ϕ.

Step 2: We must compute the Sarkisov degree (μ_1, c_1, e_1) of the induced birational map $\phi \circ \sigma_P$. One can check that it equals $\left(\frac{1}{2}, 1, 8\right)$ and it is smaller than (μ, c, e) according to the partial ordering explained in Definition 2.3.1. So the birational map $\phi \circ \sigma_P$ is "simpler" than ϕ.

Since $c_1 = 1 < 2 = \dfrac{1}{\mu_1}$, the second link in the Sarkisov factorization is of type I or II. At this point, we also have seven extremal blowups to choose from. They are the blowup of Z_1 along e or the blowup of Z_1 along one of the lines L_i. Repeating exactly the same arguments as in Step 1, we can check that the first one yields a volume preserving map whereas the remaining ones do not. The reason behind this is that $e \subset D_{Z_1}$ and $L_i \not\subset D_{Z_1}$ for all $i \in \{1, \ldots, 6\}$.

Let us continue with $\sigma_e \colon Y_1 \to Z_1$ the blowup of Z_1 along e and set $E_e := \mathrm{Exc}(\sigma_e)$.

By [Har, Theorem 8.24, (b)], we have that $E_e \simeq \mathbb{P}(\mathcal{N}^\vee_{e/Z_1})$. One can compute $\mathcal{N}^\vee_{e/Z_1} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-4)$ and therefore $E_e \simeq \mathbb{F}_6$.

The curve $e' = E_e \cap D_{Y_1} \simeq \mathbb{P}^1$ is contained in $\mathrm{Bs}(\phi \circ \sigma_P \circ \sigma_e)$. Roughly speaking, we can say that e' is an *infinitely near curve* to e in analogy with the notion of infinitely near points in dimension 2. See Definition 2.2.1.

Since $c_1 = 1$, we need to run the $(K_{Y_1} + \mathcal{H}_{Y_1})$-MMP over $\mathbb{P}^2$.

One can verify that this log MMP results in the divisorial contraction

$$\alpha \colon Y_1 \to Z_2 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3)),$$

where $\pi_2 \colon Z_2 \to \mathbb{P}^2$ is the corresponding structure morphism making Z_2 a Mori fibered space. This birational morphism contracts exactly the rulings of $\tilde{A} \simeq \mathbb{F}_2$, that is, $\tilde{A} = \mathrm{Exc}(\alpha)$. The isomorphism $Z_2 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3))$ may be justified by analyzing the section of π_2 given by $\mathbb{P}^2 \to E_P \simeq \mathbb{P}^2$; and by $K_{Z_2} = \alpha_* K_{Y_1}$ written in terms of a basis for $\mathrm{Pic}(Z_2)$ and comparing it with the formula for the canonical class of a projective bundle.

Thus, the second (volume preserving) Sarkisov link in a factorization of ϕ is of type II, and it is given by the composition $\alpha \circ \sigma_e^{-1}$. Observe that α maps E_e isomorphically, via pushforward, onto the cylinder $E_e = \{A = 0\} \subset Z_2$. In particular, all the lines L_i are contracted by α. Moreover, we have that $\mathrm{Bs}(\phi \circ \sigma_P \circ \sigma_e \circ \alpha^{-1})$ consists of the curve $f = \alpha(e')$ which is mapped by π_2 isomorphically onto the conic $\{A = 0\} \subset \mathbb{P}^2$. We have the following picture in which we did not put the strict transforms of D to not pollute it:

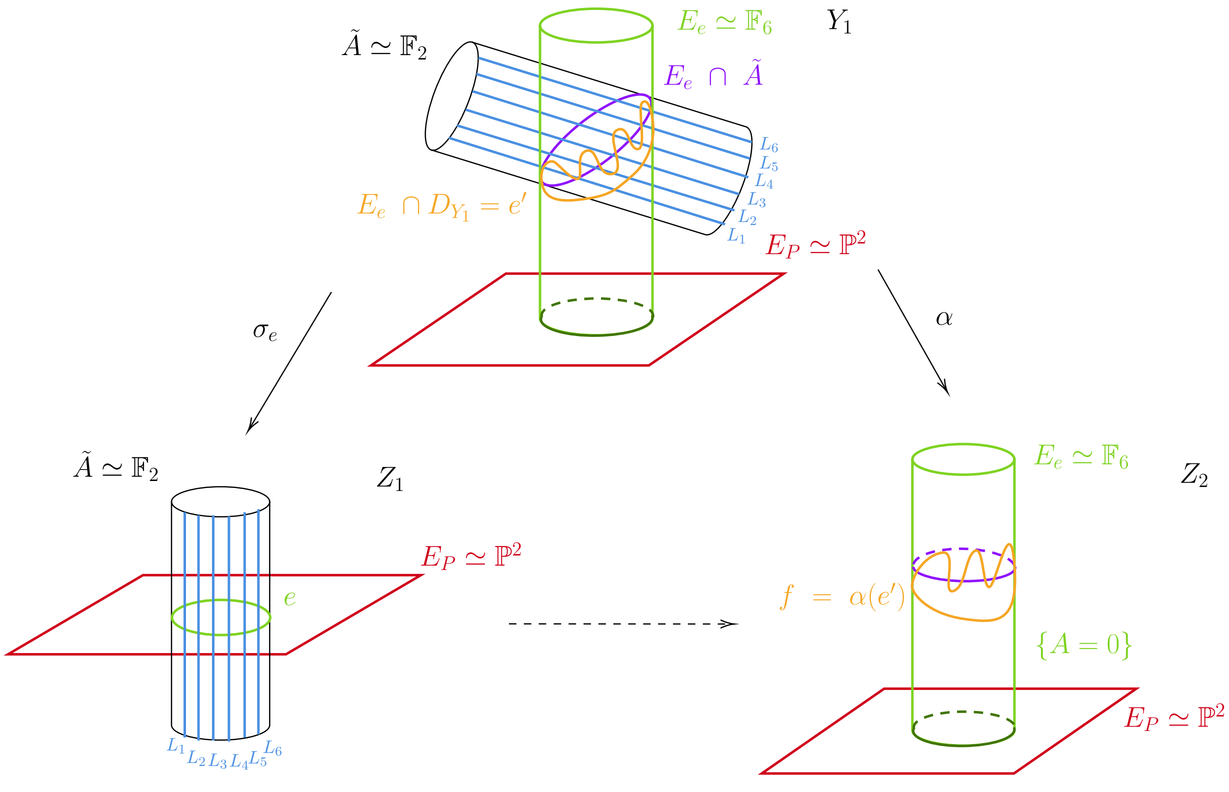


Figure 5.8: Second link in a Sarkisov factorization of ϕ.

Notice that we have similar behavior to the elementary transformations between the Hirzebruch surfaces $\mathbb{F}_n \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. Indeed, by [Har, Example 2.11.4] one has $Z_1 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. The composition $\alpha \circ \sigma_e^{-1}$ is the blowup along e followed by the contraction of the birational transforms of all fibers of π_1 through e, which correspond to the rulings of $\tilde{A} \simeq \mathbb{F}_2$. Geometrically, we have only interchanged a family of fibers of π_2 parameterized by $e \simeq \mathbb{P}^1$.

Step 3: Once again, we need to compute the Sarkisov degree (μ_2, c_2, e_2) of the induced birational map $\phi \circ \sigma_P \circ \sigma_e \circ \alpha^{-1}$. One can check that equals $\left(\frac{1}{2}, 1, 1\right)$ and it is smaller than (μ_1, c_1, e_1). Thus we have simplified the birational map $\phi \circ \sigma_P$.

Since $c_2 = 1 < 2 = \dfrac{1}{\mu_2}$, the third link in the Sarkisov factorization is of type I or II. The difference in this step is that we have only one possible extremal blowup given by the blowup of Z_2 along f. Consider $\sigma_f\colon Y_2 \to Z_2$ such map and denote $E_f \coloneqq \mathrm{Exc}(\sigma_f)$.

As in the previous steps, the map σ_f is volume preserving because $f \subset D_{Z_2}$. One can check that the map $\phi \circ \sigma_P \circ \sigma_e \circ \alpha^{-1} \circ \sigma_f$ is everywhere defined.

Since $c_1 = 1$, we need to run the $(K_{Y_2} + \mathcal{H}_{Y_2})$-MMP over $\mathbb{P}^2$. By [Har, Theorem 8.24, (b)], we have that $E_f \simeq \mathbb{P}(\mathcal{N}^\vee_{f/Z_2})$. One can compute $\mathcal{N}^\vee_{f/Z_2} \simeq \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$ and therefore $E_f \simeq \mathbb{F}_2$.

Repeating the same arguments as in Step 2, we obtain a divisorial contraction $\beta\colon Y_2 \to Z_3 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. The third (volume preserving) Sarkisov link in a factorization of ϕ is of type II, and it is given by the composition $\beta \circ \sigma_f^{-1}$ with analogous geometric properties to the previous one. We have the following picture:

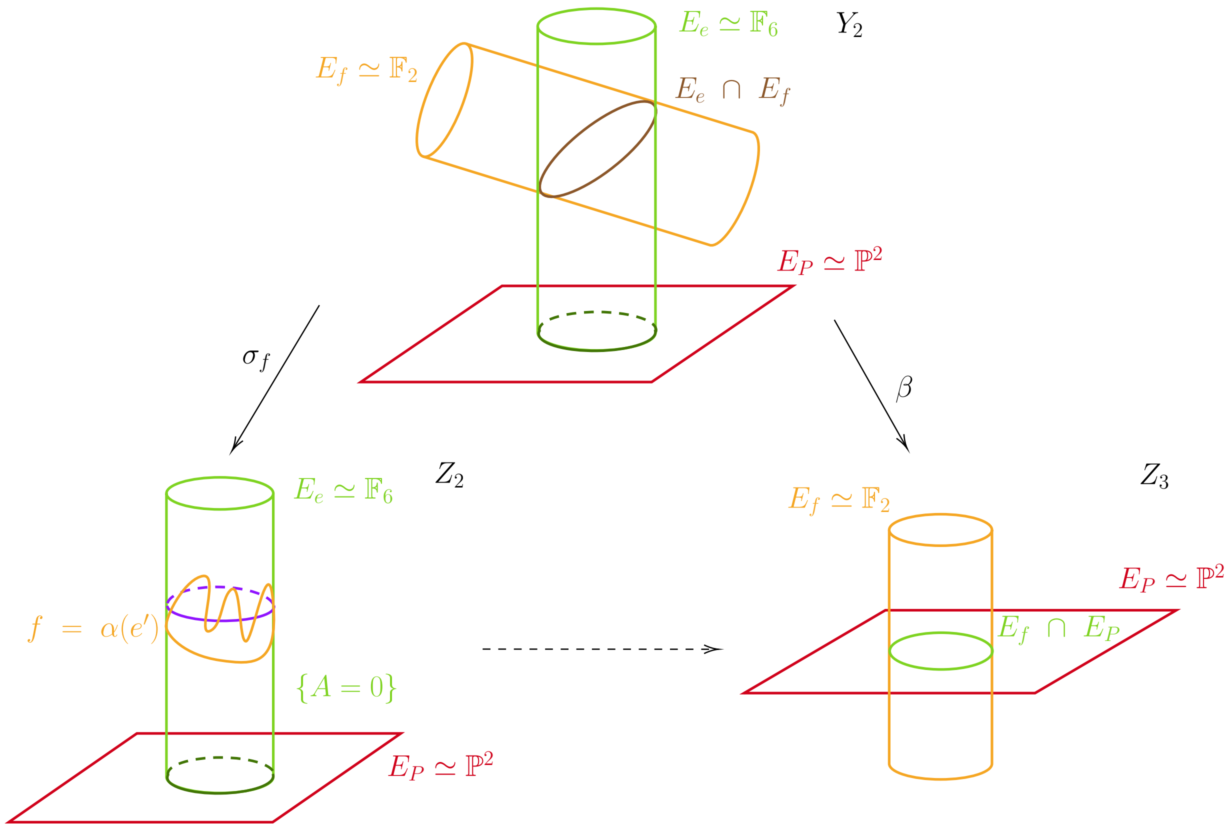


Figure 5.9: Third link in a Sarkisov factorization of ϕ.

Step 4: The computation of the Sarkisov degree of the induced birational map $\phi \circ \sigma_P \circ \sigma_e \circ \alpha^{-1} \circ \sigma_f \circ \beta^{-1}$ will be a little different. The issue here is that the pair $(Z_3, c\mathcal{H}_{Z_3})$ is canonical for any positive rational number c, since $\mathcal{H}_{Z_3}$ is base point free. So the notion of canonical threshold would lead us to c_3 "=" ∞. This is in accordance with Definition 2.3.1.

In this case, the Sarkisov degree (μ_3, c_3, e_3) becomes $\left(\frac{1}{2}, \infty, *\right)$ and it is smaller than (μ_2, c_2, e_2).

Since $c_3 = \infty$ "$\geq$" $2 = \dfrac{1}{\mu_3}$, the fourth link in the Sarkisov factorization is of type III or IV. We need to run the $(K_{Z_3} + 2\mathcal{H}_{Z_3})$-MMP over $\mathrm{Spec}(\mathbb{C})$. This log MMP results exactly in the divisorial contraction given by the blowup of $\mathbb{P}^3$ at P, that is, the map σ_P. We observe that E_f is precisely $\mathrm{Exc}(\sigma_P)$.

One can compute that the Sarkisov degree (μ_4, c_4, e_4) of the induced birational map $\phi \circ \sigma_P \circ \sigma_e \circ \alpha^{-1} \circ \sigma_f \circ \beta^{-1} \circ \sigma_P^{-1}$ is $\left(\frac{1}{4}, \infty, *\right)$, which is smaller than the previous one. Such Sarkisov degree implies that this map is an automorphism of $\mathbb{P}^3$.

The (volume preserving) factorization of ϕ is ended up by the Sarkisov link of type III given by σ_P. We have the following picture:

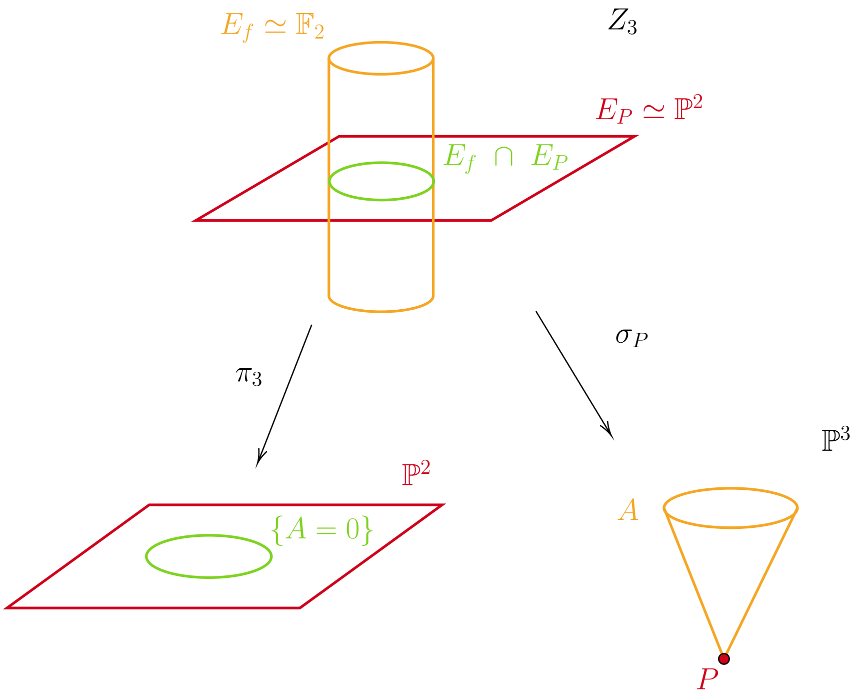


Figure 5.10: Fourth link in a Sarkisov factorization of ϕ.

We observe that the strict transforms of D remained nonsingular and isomorphic to D along the intermediate steps. In terms of a commutative diagram, this volume preserving factorization of ϕ is expressed in the following way:

$$\begin{tikzcd}
 & & Y_1 \arrow[dl, "\sigma_e"'] \arrow[dr, "\alpha"] & & Y_2 \arrow[dl, "\sigma_f"'] \arrow[dr, "\beta"] & & \\
 & Z_1 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \arrow[dl, "\sigma_P"'] \arrow[d, "\pi_1"'] \arrow[rr, dashed] & & Z_2 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3)) \arrow[d, "\pi_2"'] \arrow[rr, dashed] & & Z_3 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \arrow[d, "\pi_3"'] \arrow[dr, "\sigma_P"] & \\
\mathbb{P}^3 \arrow[d] \arrow[rrrrrr, dashed, bend right, "\phi"'] & \mathbb{P}^2 \arrow[rr, equal] & & \mathbb{P}^2 \arrow[rr, equal] & & \mathbb{P}^2 & \mathbb{P}^3 \arrow[d] \\
\mathrm{Spec}(\mathbb{C}) & & & & & & \mathrm{Spec}(\mathbb{C})
\end{tikzcd}$$

We have the following sequence of Sarkisov degrees of the induced birational maps:

$$\left(\frac{3}{4}, 1, 9\right) > \left(\frac{1}{2}, 1, 8\right) > \left(\frac{1}{2}, 1, 1\right) > \left(\frac{1}{2}, \infty, *\right) > \left(\frac{1}{4}, \infty, *\right).$$

Non-volume preserving factorization of ϕ: Let us make some comments if instead of proceeding with σ_e in Step 2, we had chosen σ_1. We invite the reader to fill out the details and make drawings in order to see what is happening geometrically.

For $i \in \{1,\ldots,6\}$, denote $P_i := L_i \cap E_P$. In this other Sarkisov factorization the $(i+1)$-th Sarkisov link is of type I, and it is given by $\sigma_i \colon \mathrm{Bl}_{L_1,\ldots,L_i}(Z_1) \to \mathrm{Bl}_{L_1,\ldots,L_{i-1}}(Z_1)$, where $\mathrm{Bl}_{L_1,\ldots,L_i}(Z_1) \to \mathrm{Bl}_{P_1,\ldots,P_i}(\mathbb{P}^2)$ is the corresponding structure of Mori fibered space. One can show that $E_i := \mathrm{Exc}(\sigma_i)$ is isomorphic to $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ for all $i \in \{1,\ldots,6\}$.

We have the following picture for σ_1, where $g = \tilde{E_P} \cap E_1 \simeq \mathbb{P}^1$. Geometrically, g represents all the normal directions to L_1 at P_1.

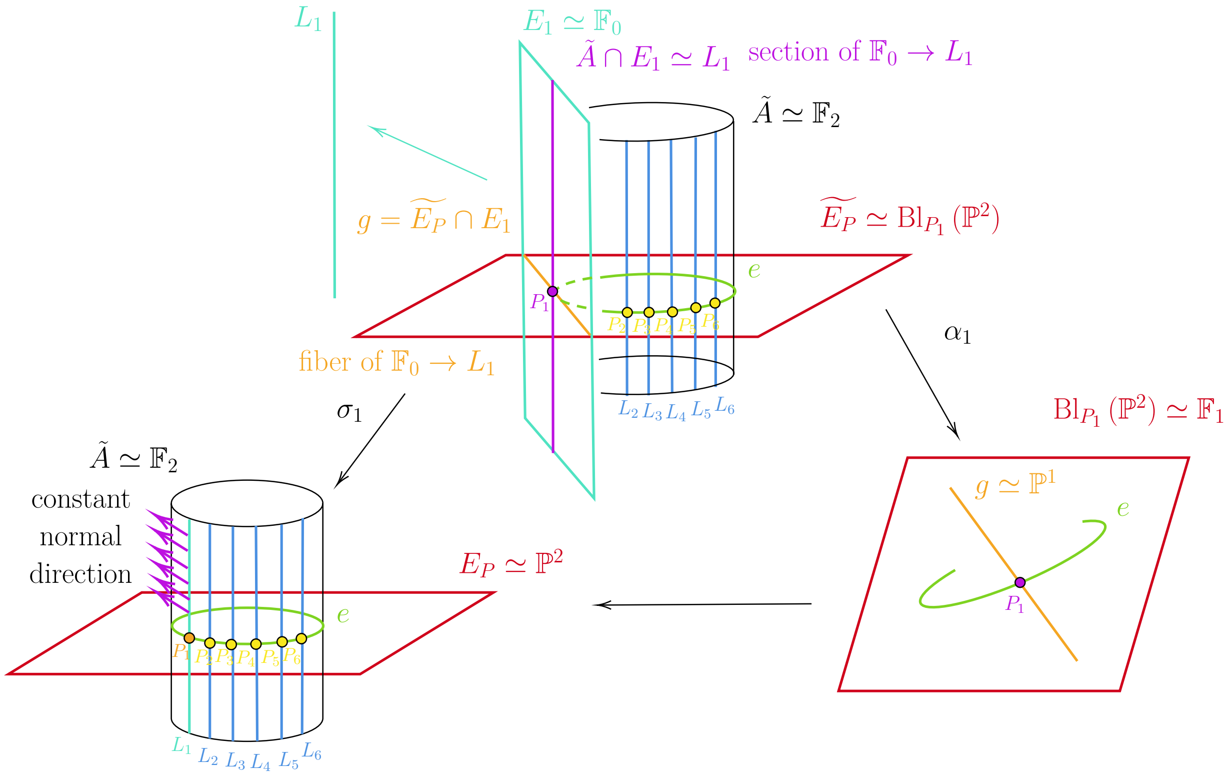


Figure 5.11: Sarkisov link σ_1.

After σ_6, the next two Sarkisov links are analogous to the intermediate ones in the volume preserving factorization described previously. In particular, they are volume preserving Sarkisov links of type II.

Finally, the remaining ones are given by the consecutive blowdowns of the corresponding strict transforms of $E_1,\ldots,E_6$ and E_P, respectively. All of them are Sarkisov links of type III and the last one is volume preserving. The setting is depicted in Figure 5.12.

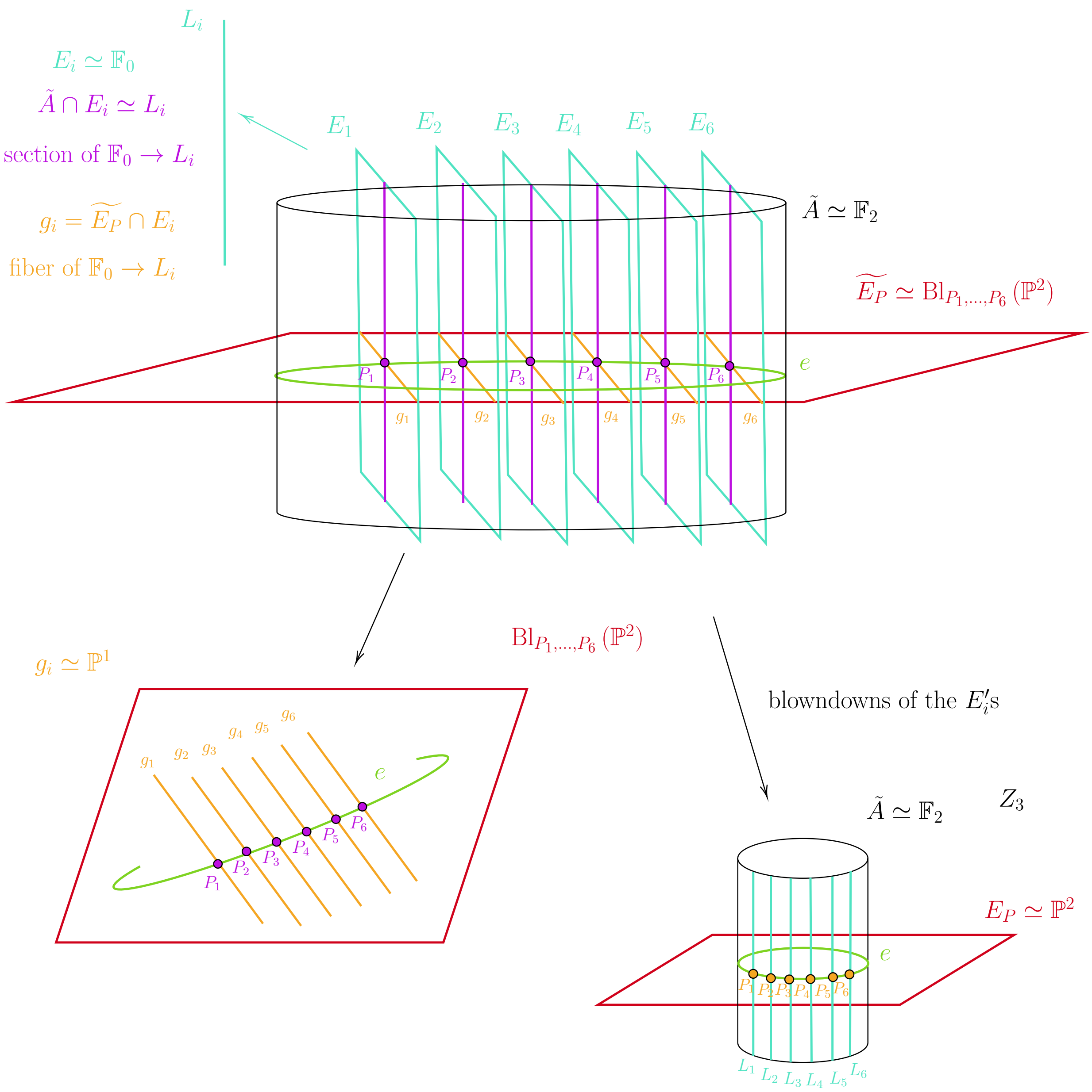


Figure 5.12: Blowdowns of the E_i's.

We have the following sequence of Sarkisov degrees of the induced birational maps:

$$\left(\frac{3}{4},1,9\right) > \underbrace{\left(\frac{1}{2},1,8\right) > \dots > \left(\frac{1}{2},1,2\right)}_{\text{relative to the } \sigma_i\text{'s}} > \left(\frac{1}{2},1,1\right) > \underbrace{\left(\frac{1}{2},2,6\right) > \dots > \left(\frac{1}{2},2,1\right)}_{\text{relative to the blowdowns of the } E_i\text{'s}}$$
$$> \left(\frac{1}{2},\infty,*\right) > \left(\frac{1}{4},\infty,*\right).$$

We remark that it is relevant that $P \in \mathrm{Sing}(\mathcal{H})$ is a singularity of type A_1 for the increasing of the canonical threshold in Step 8. The induced birational map before the Sarkisov links of type III has a base locus given by $\{P_1,\dots,P_6\}$.

Conclusion. By the previous detailed example, we can see the existence of a Sarkisov factorization for a volume preserving map that is not automatically volume preserving. This means that Theorem 4.2.6 is very particular for dimension 2.

The point is that the volume preserving factorization assured by [CK, Theorem 1.1] is induced by a standard Sarkisov factorization constructed in a special way [CK, Theorem 3.3]. But this special factorization may not be obtained by following algorithmically the Sarkisov Program, at least in dimension 3, if we make certain choices along the process.

In a careful analysis of the volume preserving decomposition of ϕ exhibited in [ACM], we show that it can be obtained by choosing conveniently the centers of the extremal blowups initiating Sarkisov links of type I and II. Furthermore, this factorization has the effect of resolving the map ϕ along the process as a consequence of the untwisting of the Sarkisov degree of the induced birational maps.

Appendix A

Resolution of the singularities D-E

In Chapter 5, we only treat the case where the canonical singularity is of type A_n.

The next step is to study the D-E case. We expect similar results to the A_n case.

Following the same strategy, explicit resolution of the singularities D-E is needed. This is the purpose of this Appendix, which follows the notation used in Chapter 5.

A.1 Resolution of the singularity D_n

We may assume $D = V(x_1^2 + x_2^2 x_3 + x_3^{n-1}) \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$, where $n \geq 4$.

Consider $\pi\colon X \to \mathbb{A}^3$ the blowup of $\mathbb{A}^3$ at P.

One has $E \cap W_i = \{x_i = 0\} \simeq \mathbb{A}^2$ for all $i \in \{1,2,3\}$. One can check that $E \cap \tilde{D} \cap W_1 = \emptyset$.

Abusing notation, in the affine chart W_2 the equation of $\tilde{D}$ is given by

$$f_1 = x_1^2 + x_2 x_3 + x_2^{n-3} x_3^{n-1} = 0.$$

We have $E \cap \tilde{D} = \{x_1^2 = x_2 = 0\}$.

If $\tilde{D}$ has singular points in W_2, they are necessarily of the form $(0,0,\lambda)$, where $\lambda \in \mathbb{C}$.

One computes

$$\begin{cases} \dfrac{\partial f_1}{\partial x_1} = 2x_1 \\ \dfrac{\partial f_1}{\partial x_2} = \begin{cases} x_3 + x_3^3, & \text{if } n = 4 \\ x_3 + (n-3)x_2^{n-4} x_3^{n-1}, & \text{if } n > 4 \end{cases} \\ \dfrac{\partial f_1}{\partial x_3} = x_2 + (n-1)x_2^{n-3} x_3^{n-2} \end{cases}.$$

Solving the system of equations given by the mutual vanishing of the partial derivatives and only regarding solutions of the form $(0,0,\lambda)$, we get the following by the Jacobian Criterion and analyzing the tangent cone:

- $P_2 \coloneqq (0,0,0), P_\pm \coloneqq (0,0,\pm i)$ are singularities of type A_1 if $n = 4$;
- $P_2 \coloneqq (0,0,0)$ is a singularity of type A_1 if $n > 4$.

Through an analogous analysis in the affine chart W_3, we get the following:

- $P_\pm \coloneqq (0,0,\mp i)$ are singularities of type A_1 also respectively appearing in W_2 if $n = 4$;
- $P_3 \coloneqq (0,0,0)$ is a singularity of type A_3 if $n = 5$, and of type D_{n-2} if $n > 5$.

We warn the reader that the points P_2 and P_3 are distinct. In fact, in $X = \mathrm{Bl}_P(\mathbb{A}^3)$ they have coordinates $((0,0,0);(0:1:0))$ and $((0,0,0);(0:0:1))$, respectively.

These facts imply the following result:

Lemma A.1.1. *The resolution of a Du Val singularity of type D_n can be done using $2 \cdot \left\lceil \dfrac{n-1}{2} \right\rceil$ blowups at nonsingular points of the ambient space.*

Proof. Let us show by induction over $n \geq 4$. The basis of induction $n = 4$ is done. After blowing up the singularity, from the previous computations in the affine chart W_2, we have that $\tilde{D}$ has 3 singularities of type A_1 and 2 of them also appear in the affine chart W_3. In the affine chart W_1 we have that $\tilde{D}$ is nonsingular.

Then, using Lemma 5.1.11, one has $1 + 3 \cdot 1 = 4 = 2 \cdot \left\lceil \dfrac{3}{2} \right\rceil$ blowups are necessary to resolve the singularity.

Suppose that the statement holds for $n - 1 > 3$. Let us show that it also holds for n. After performing the first blowup, we have that $\tilde{D}$ has no singularities in the affine chart W_1 and a single singularity of type A_1 in the affine chart W_2. In the affine chart W_3, $\tilde{D}$ admits a singularity of type A_3 if $n = 5$ and of type D_{n-2} if $n \geq 6$.

For $n = 5$, using Lemma 5.1.11, one has $1 + 1 + 2 = 4 = 2 \cdot \left\lceil \dfrac{4}{2} \right\rceil$ blowups are necessary to resolve the singularity.

For $n > 5$, by the induction hypothesis, we can resolve such singularity in W_3 after $2 \cdot \left\lceil \dfrac{n-3}{2} \right\rceil$ blowups at nonsingular points of the ambient space.

Thus, using this resolution for the singularity of $\tilde{D}$ in W_3, in addition to the first blowup and the blowup to resolve the singularity of type A_1 in W_2, we can resolve the singularity of D using $1 + 1 + 2 \cdot \left\lceil \dfrac{n-3}{2} \right\rceil = 2 \cdot \left\lceil \dfrac{n-1}{2} \right\rceil$ of such blowups. $\square$

The following pictures represent the resolution process of a singularity D_4, D_5, D_6 and $D_{>6}$, respectively.

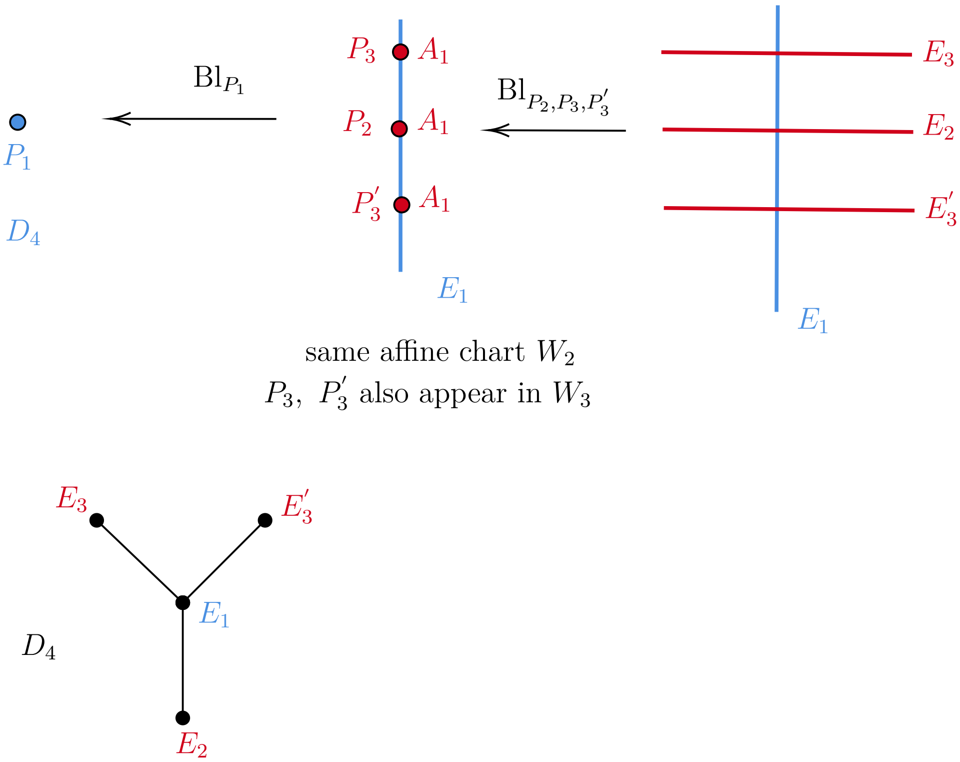


Figure A.1: Resolution of the singularity D_4

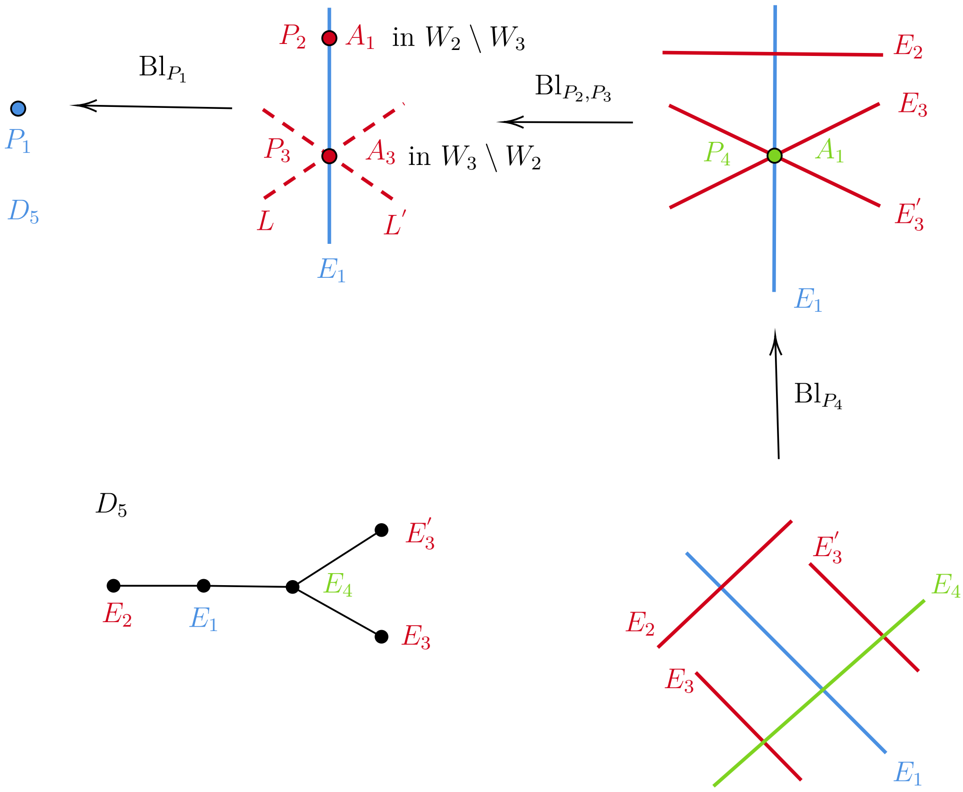


Figure A.2: Resolution of the singularity D_5.

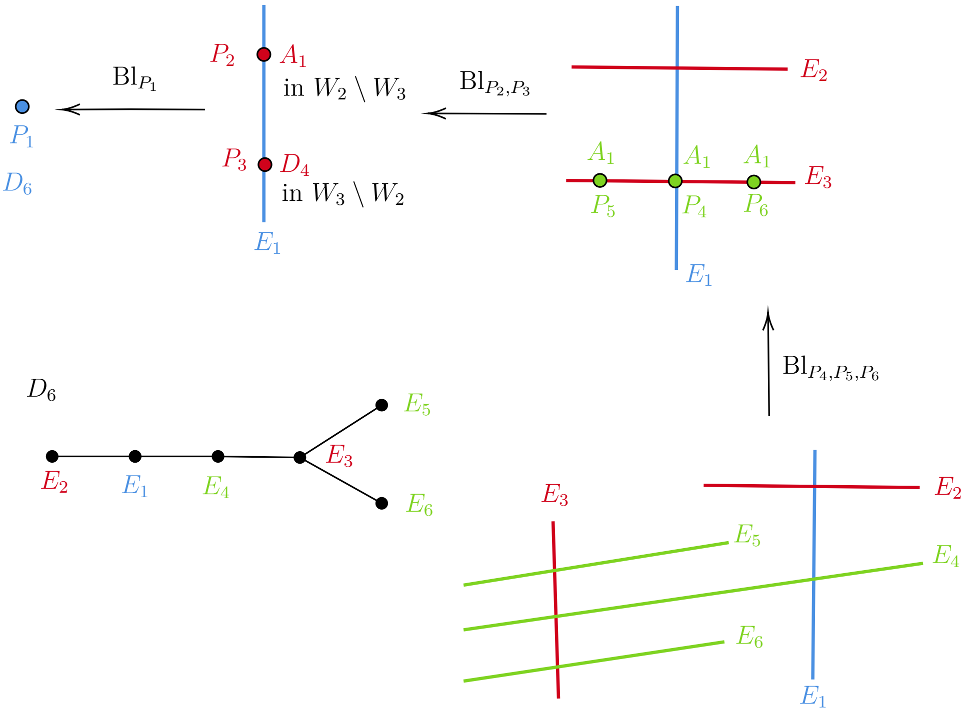


Figure A.3: Resolution of the singularity D_6.

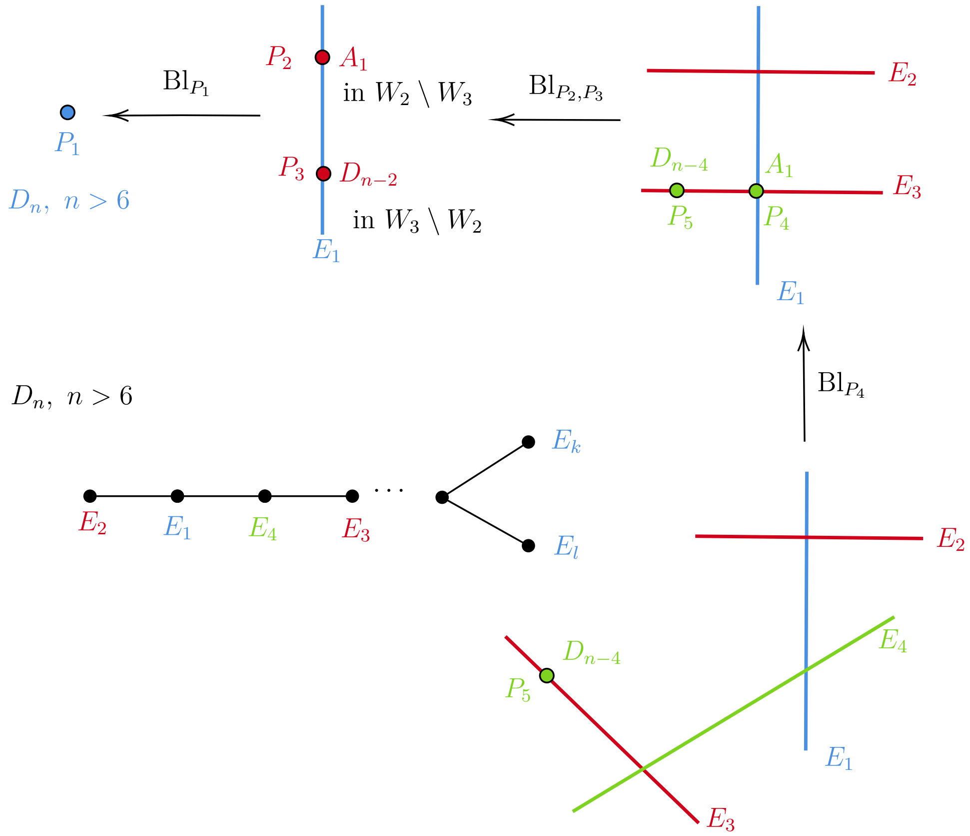


Figure A.4: Resolution of the singularity $D_{>6}$.

A.2 Resolution of the singularity E_7

We may assume $D = V(x_1^2 + x_2^3 + x_2x_3^3) \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$.

Consider $\pi\colon X \to \mathbb{A}^3$ the blowup of $\mathbb{A}^3$ at P. Following the previous notations, one can check $E \cap \tilde{D} \cap W_1 = \emptyset$.

Abusing notation, in the affine chart W_2 the equation of $\tilde{D}$ is given by

$$f_1 = x_2 + x_1^2 + x_2^2x_3^3 = 0.$$

We have $E \cap \tilde{D} = \{x_1^2 = x_2 = 0\}$.

If $\tilde{D}$ has singular points in W_2, they are necessarily of the form $(0,0,\lambda), \lambda \in \mathbb{C}$. Solving the system of equations given by the mutual vanishing of the partial derivatives and only regarding solutions of the form $(0,0,\lambda)$, we get that $\tilde{D}$ has no singularities in the affine chart W_2.

Through an analogous analysis in the affine chart W_3 and of the tangent cone, we get that $\tilde{D}$ admits a single singularity of type D_6.

Therefore we need $1 + 6 = 7$ blowups at nonsingular points of the ambient space to resolve a Du Val singularity of type E_7.

The following pictures represent the resolution process of a singularity E_7.

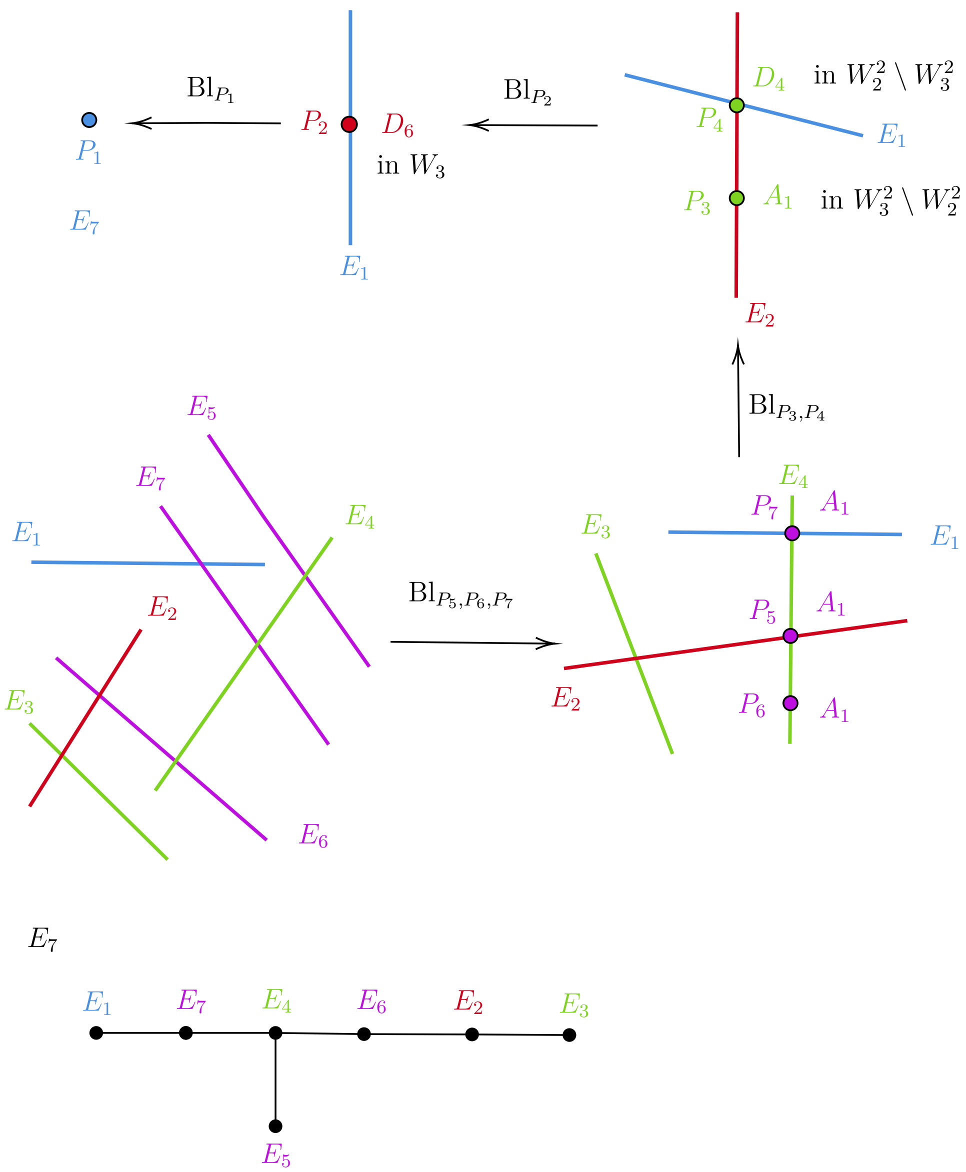


Figure A.5: Resolution of the singularity E_7.

A.3 Resolution of the singularities E_6 and E_8

We may assume $D = V(x_1^2 + x_2^3 + x_3^n) \subset \mathbb{A}^3_{(x_1,x_2,x_3)}$, where $n \in \{4, 5\}$ for a Du Val singularity of type E_6 and E_8, respectively.

Consider $\pi\colon X \to \mathbb{A}^3$ the blowup of $\mathbb{A}^3$ at P. Following the previous notations, one can check $E \cap \tilde{D} \cap W_1 = \emptyset$.

Abusing notation, in the affine chart W_2 the equation of $\tilde{D}$ is given by

$$f_1 = x_2 + x_1^2 + x_2^{n-2} x_3^n = 0,$$

for $n \in \{4, 5\}$. We have $E \cap \tilde{D} = \{x_1^2 = x_2 = 0\}$.

If $\tilde{D}$ has singular points in W_2, they are necessarily of the form $(0, 0, \lambda), \lambda \in \mathbb{C}$. Solving the system of equations given by the mutual vanishing of the partial derivatives and only regarding solutions of the form $(0,0,\lambda)$, we get that $\tilde{D}$ has no singularities in the affine chart W_2 for $n \in \{4,5\}$.

Through an analogous analysis in the affine chart W_3 together with tangent cones, we get that $\tilde{D}$ admits a single singularity of type A_5 if $n=4$, and of type E_7 if $n=5$.

Therefore we need $1+3=4$ and $1+7=8$ blowups at nonsingular points of the ambient space to resolve a Du Val singularity of type E_6 and E_8, respectively.

The following pictures represent the resolution process of a singularity E_6 and E_8, respectively.

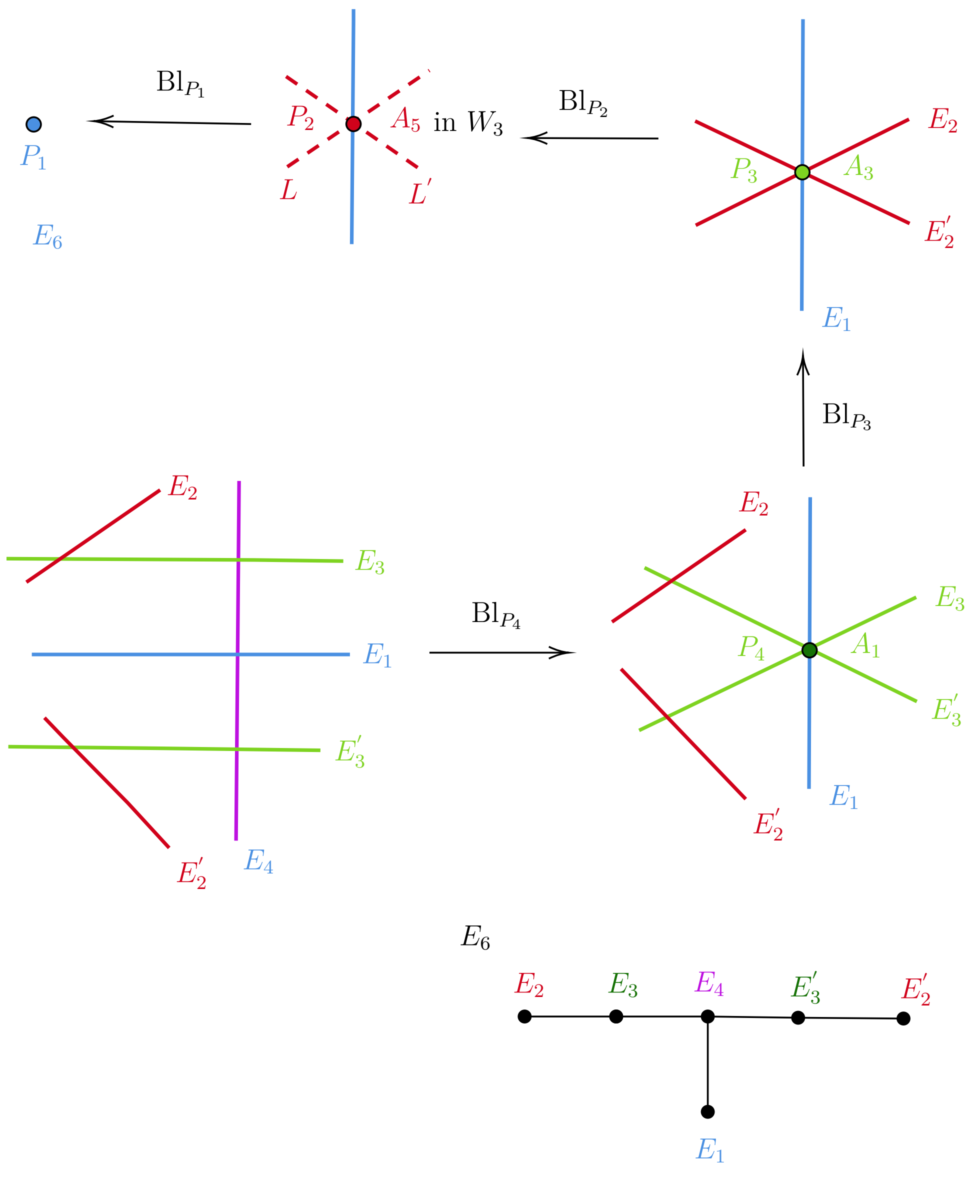


Figure A.6: Resolution of the singularity E_6.

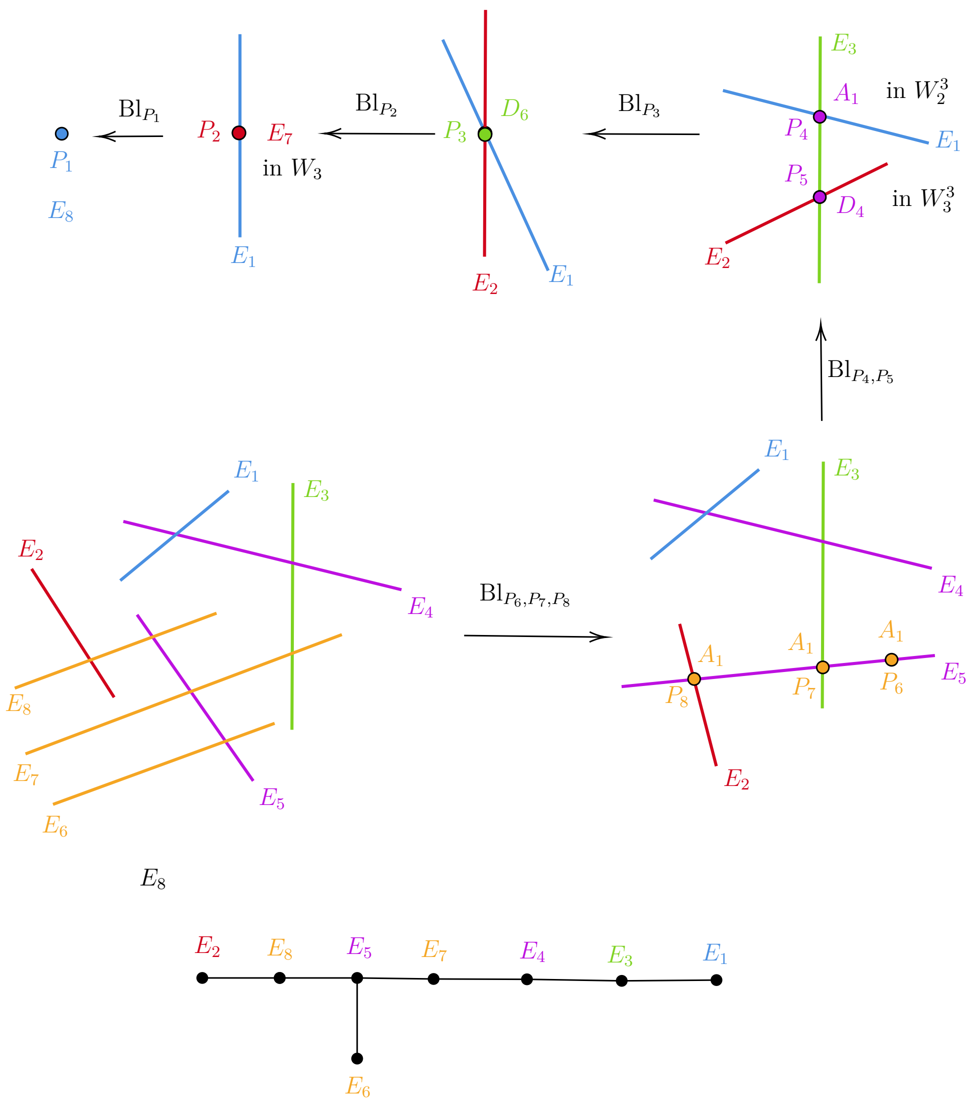


Figure A.7: Resolution of the singularity E_8.

We can summarize the previous results in the following table together with the A_n case:

type of singularity	number of point blowups to resolve
A_n	$\left\lceil \frac{n}{2} \right\rceil$
D_n	$2 \cdot \left\lceil \frac{n-2}{2} \right\rceil$
E_6	4
E_7	7
E_8	8

Table A.1: Number of point blowups to resolve the Du Val singularities.

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