

PBW BASES OF IRREDUCIBLE ISING MODULES

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ABSTRACT. To every $h + \mathbb{N}$ -graded module M over an \mathbb{N} -graded conformal vertex algebra V , we associate an increasing filtration $(G^p M)_{p \in \mathbb{Z}}$, which is compatible with the filtrations introduced by Haisheng Li. The associated graded vector space $\text{gr}^G(M)$ is naturally a module over the vertex Poisson algebra $\text{gr}^G(V)$. We study $\text{gr}^G(M)$ for the three irreducible modules over the Ising model $\text{Vir}_{3,4}$, namely $\text{Vir}_{3,4} = L(1/2, 0)$, $L(1/2, 1/2)$ and $L(1/2, 1/16)$. We obtain an explicit PBW basis of each of these modules and a formula for their refined characters, which are related to Nahm sums for the matrix $\begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}$.

Keywords. Vertex algebras, Ising model, Nahm sums, quantum algebra, combinatorics.

CONTENTS

0. Introduction	2
1. Vertex superalgebras and their modules	6
1.1. Formal calculus	6
1.2. Lie conformal superalgebras	8
1.3. Fields over vector spaces	15
1.4. Vertex superalgebras	21
1.5. Graded and conformal vertex superalgebras	28
1.6. Modules over vertex superalgebras	32
1.7. Lie algebras associated to vertex algebras	38
2. Highest weight representations of the Virasoro Lie algebra	41
2.1. Unitary and contravariant representations of Lie algebras	41
2.2. Verma modules	42
2.3. Kac determinant formula and singular vectors	44
3. Modules over the simple Virasoro vertex algebras	47
3.1. The Zhu algebra	47
3.2. The inverse of the lowest weight subspace functor	49
3.3. Modules over the Virasoro vertex algebras	50
4. Vertex Poisson algebras and filtrations of vertex algebras	52
4.1. Vertex Lie superalgebras and their modules	53
4.2. Vertex Poisson algebras and their modules	54
4.3. Filtrations of vertex algebras	56
4.4. Filtrations of modules over vertex algebras	60
4.5. The Zhu C_2 -algebra and the C_2 -cofiniteness condition	65
5. PBW basis of $L(1/2, 1/2)$ and its refined character	71
5.1. Conventions and notation	71
5.2. A combinatorial argument	73
5.3. Computing leading monomials	77
5.4. Proofs of the main theorems	81
Appendix A. Almost commutative algebras	82
Appendix B. Poisson algebras and their modules	83
Appendix C. Jet algebras and jet schemes	84
Appendix D. An explicit description of K	86
Appendix E. The case $L(1/2, 1/16)$	86
References	91

0. INTRODUCTION

Modules over the Virasoro Lie algebra, Kac-Moody Lie algebras, vertex algebras, and other related noncommutative algebraic objects are often equipped with natural filtrations, whose associated graded spaces bear interesting structures, such as differential structures, Poisson structures, etc. The prototype of this situation is, of course, the PBW theorem, one formulation of which is that for a Lie algebra \mathfrak{g} , the associated graded space $\text{gr}(U(\mathfrak{g}))$ of the universal enveloping algebra $U(\mathfrak{g})$ with respect to the usual PBW filtration is naturally isomorphic to the symmetric algebra $S(\mathfrak{g})$, i.e., it is a polynomial algebra.

A similar notion in the context of vertex operator algebras has garnered increasing interest in recent years. For a vertex operator algebra V , there is a natural epimorphism $JR_V \rightarrow \text{gr}_F(V)$ between the jet algebra of the Zhu C_2 -algebra R_V of V and the associated graded space $\text{gr}_F(V)$ of V with respect to the Li filtration. If this is an isomorphism, we say V is classically free. The property of classical freedom, and its failure, are related to problems in representation theory such as monomial bases of irreducible representations and combinatorial interpretations of classical number theoretic identities like the Rogers-Ramanujan identity.

The Ising model $\text{Vir}_{3,4}$ is a vertex operator algebra that turns out not to be classically free, as illustrated in [AVEH22]. However, it is still possible to find a monomial basis of $\text{Vir}_{3,4}$. In this thesis, we continue the work of Andrews, Van Ekeren and Heluani by studying the irreducible modules over $\text{Vir}_{3,4}$, their associated graded spaces and their PBW bases. As it turns out, we have to use a modified version of the PBW filtration when considering modules over $\text{Vir}_{3,4}$.

A vertex superalgebra can be thought of as a vector superspace V together with a distinguished element $|0\rangle \in V_0$, called vacuum vector, and \mathbb{Z} -many bilinear products

$$\begin{aligned} \bullet_{(n)}\bullet &: V \times V \rightarrow V, \\ (a, b) &\mapsto a_{(n)}b, \end{aligned}$$

called n -products, such that the generating function

$$\begin{aligned} [\bullet_\lambda\bullet] &: V \times V \rightarrow V[\lambda], \\ [a_\lambda b] &= \sum_{n \in \mathbb{N}} (a_{(n)}b) \frac{\lambda^n}{n!}, \end{aligned}$$

called λ -bracket, satisfies the axioms of a Lie conformal superalgebra, which is very similar to a Lie superalgebra, together with quasicommutativity, quasiassociativity and the noncommutative Wick formula. However, a more compact way to define vertex superalgebras is to gather all n -products into a single generating function

$$\begin{aligned} Y(\bullet, z) &: V \rightarrow \text{End}(V)[[z^{\pm 1}]], \\ a &\mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}z^{-n-1} \end{aligned}$$

satisfying the vacuum axiom, the translation covariance axiom and the locality axiom. Another, even shorter, way to describe vertex algebras is to require the vacuum axiom and the Borcherds identity. The equivalence of these definitions is explained in [DSK06, §1].

A common way to create vertex superalgebras is to start from regular formal distribution Lie superalgebras $(\mathfrak{g}, \mathfrak{F}, T)$, which are just Lie superalgebras \mathfrak{g} together with a family of \mathfrak{g} -valued formal distributions \mathfrak{F} generating \mathfrak{g} and a derivation T . Then, we can construct the associated vertex algebra $V(\mathfrak{g}, \mathfrak{F}, T)$. As we can see, vertex superalgebras are closely related to traditional Lie superalgebras.

We now introduce the universal Virasoro vertex algebra Vir^c , where $c \in \mathbb{C}$. The Virasoro Lie algebra, denoted by Vir , is the Lie algebra given by

$$\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C.$$

These elements satisfy the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} C \quad \text{for } m, n \in \mathbb{Z},$$

$$[\text{Vir}, C] = 0.$$

We have an important Vir-valued formal distribution

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

and we can verify that $(\text{Vir}, \{L(z), C\}, \text{ad}(L_{-1}))$ is a regular formal distribution Lie algebra. A certain quotient of the associated vertex algebra is the universal Virasoro vertex algebra of central charge c , which we describe now. First, we make the subalgebra $\text{Vir}_{\geq -1} \oplus \mathbb{C}C$ act on \mathbb{C} as follows:

$$L_n 1 = 0 \text{ for } n \geq -1 \text{ and } C1 = c.$$

It turns out that Vir^c is an induced Vir-module given by

$$\text{Vir}^c = \text{Ind}_{\text{Vir}_{\geq -1} \oplus \mathbb{C}C}^{\text{Vir}}(\mathbb{C}) = U(\text{Vir}) \otimes_{U(\text{Vir}_{\geq -1} \oplus \mathbb{C}C)} \mathbb{C}.$$

By the PBW theorem, a basis of Vir^c , also known as PBW basis, is given by

$$\{L_{-i_k} \dots L_{-i_1} | 0\rangle \mid i_k \geq \dots \geq i_1 \geq 2\}.$$

The vertex algebra Vir^c has a unique maximal proper ideal, and the quotient Vir_c is the simple Virasoro vertex algebra of central charge c . For a generic value of $c \in \mathbb{C}$, $\text{Vir}^c = \text{Vir}_c$, i.e., Vir^c is already simple. The smallest case in which this does not happen (other than the one dimensional vertex algebra $\mathbb{C}|0\rangle$) is the Ising model $\text{Vir}_{3,4} = \text{Vir}_{1/2}$. A natural question to ask is: Can an explicit PBW basis of $\text{Vir}_{3,4}$ be obtained? This is achieved in [AVEH22].

A module over a vertex superalgebra V can be defined as a vector superspace M together with a linear and parity preserving map $Y^M : V \rightarrow \text{End}(M)[[z^{\pm 1}]]$ satisfying the vacuum axiom and the Borcherds identity. When V is given by a formal distribution Lie superalgebra, modules over V are just smooth modules over the underlying Lie superalgebra. We often assume that a V -module M has a grading $\bigoplus_{n \in \mathbb{N}} M(n)$, which is compatible with the n -products in V . Such modules are called admissible.

Some of the most common examples of vertex superalgebras encountered in the literature are:

- (i) The commutative vertex algebras, which are just differential commutative associative algebras with unit;
- (ii) For $c \in \mathbb{C}$, the universal Virasoro vertex algebra Vir^c and the simple Virasoro vertex algebra Vir_c ;
- (iii) For a Lie superalgebra \mathfrak{g} with a supersymmetric bilinear form, the universal affine vertex superalgebra $V^k(\mathfrak{g})$ of level $k \in \mathbb{C}$;
- (iv) For a superspace V with an antisupersymmetric bilinear form, the fermionic vertex superalgebra $F(V)$;
- (v) For a lattice L of finite rank equipped with a positive-definite symmetric bilinear form, the lattice vertex algebra V_L ;
- (vi) The Moonshine module vertex algebra V^{\natural} , whose automorphism group is the Monster sporadic group and started the study of vertex algebras.

In this thesis, we will consider examples (i)–(iv). The Moonshine module is perhaps the most amazing example of a vertex algebra. However, the Virasoro vertex algebras are foundational in the sense that most vertex algebras V are required to be conformal, which means there is a distinguished vector $\omega \in V$ satisfying the Virasoro commutation relations. Even more, we usually require vertex algebras to be \mathbb{Z} -graded lower truncated or \mathbb{N} -graded. Thus, we define vertex operator algebras as \mathbb{Z} -graded lower truncated conformal vertex algebras, and this is often the object of study, not general vertex algebras.

Vertex algebras are often assumed to be C_2 -cofinite, which means R_V is finite dimensional, and rational, which means any admissible module is completely reducible. Rational vertex operator

algebras have finitely many isomorphism classes of irreducible admissible modules, while C_2 -cofinite vertex operator algebras have well-behaved modules.

Given a \mathbb{Z} -graded vertex algebra V , the Zhu algebra $\text{Zhu}(V)$ is an associative algebra with unit that describes the irreducible admissible V -modules by doing the following: given an irreducible admissible V -module M , the space of lowest weight vectors $M(0)$ is a $\text{Zhu}(V)$ -module. It turns out that it is possible to reverse this process, i.e., start with an irreducible $\text{Zhu}(V)$ -module and end up with an irreducible admissible V -module.

In this thesis, we focus on the Ising model $\text{Vir}_{3,4}$, which is conformal, \mathbb{N} -graded, C_2 -cofinite and rational. The irreducible modules over $\text{Vir}_{3,4}$, called Ising modules, are $\text{Vir}_{3,4} = L(1/2, 0)$, $L(1/2, 1/2)$ and $L(1/2, 1/16)$, which are irreducible highest weight representations of the Virasoro Lie algebra. To describe a PBW basis of Ising modules, we study filtrations of vertex algebras and their modules. These filtrations will yield a vertex Poisson algebra, in the case of vertex algebras, and a module over a vertex Poisson algebra, in the case of modules over vertex algebras. Thus, the situation is similar to that of Lie algebras: we start with a Lie algebra \mathfrak{g} , we make its universal enveloping algebra $U(\mathfrak{g})$, we consider the PBW filtration of $U(\mathfrak{g})$, and the resulting associated algebra $\text{gr}(U(\mathfrak{g}))$ is canonically isomorphic to the polynomial algebra $S(\mathfrak{g})$ (see Appendix A).

In [Li04], Li introduced an increasing filtration $(G^p V)_{p \in \mathbb{Z}}$ on an arbitrary \mathbb{N} -graded vertex algebra V . The associated graded space $\text{gr}^G(V)$ with respect to this increasing filtration then carries the structure of an \mathbb{N} -graded vertex Poisson algebra.

Then in [Li05], Li introduced a decreasing filtration $(F_p V)_{p \in \mathbb{Z}}$ on an arbitrary vertex algebra V , not necessarily \mathbb{N} -graded. The associated graded space $\text{gr}_F(V)$ with respect to this decreasing filtration again carries the structure of a vertex Poisson algebra. Li also introduced a decreasing filtration $(F_p M)_{p \in \mathbb{Z}}$ for modules M over a vertex algebra V and showed that the associated graded space $\text{gr}_F(M)$ is a module over the vertex Poisson algebra $\text{gr}_F(V)$.

In summary, Li constructed three functors:

$$\begin{aligned} \text{gr}^G &: \{\mathbb{N}\text{-graded vertex algebras}\} \rightarrow \{\mathbb{N}\text{-graded vertex Poisson algebras}\}, \\ \text{gr}_F &: \{\text{vertex algebras}\} \rightarrow \{\text{vertex Poisson algebras}\}, \\ \text{gr}_F &: \{V\text{-modules}\} \rightarrow \{\text{gr}_F(V)\text{-modules}\}. \end{aligned}$$

Then Arakawa showed in [Ara12, Proposition 2.6.1] that when V is \mathbb{N} -graded, $\text{gr}_F(V)$ and $\text{gr}^G(V)$ are isomorphic as vertex Poisson algebras.

In this thesis, we define an increasing filtration $(G^p M)_{p \in \mathbb{Z}}$ for $h + \mathbb{N}$ -graded modules M over an \mathbb{N} -graded conformal vertex algebra (V, ω) . We construct a functor

$$\text{gr}^G : \{h + \mathbb{N}\text{-graded } (V, \omega)\text{-modules}\} \rightarrow \{h + \mathbb{N}\text{-graded } \text{gr}^G(V)\text{-modules}\}.$$

Parallel to what was done in [Ara12], we show that $\text{gr}_F(M)$ and $\text{gr}^G(M)$ are isomorphic as modules. However, for our purposes, the filtration $(G^p M)_{p \in \mathbb{Z}}$ is better suited.

In [AVEH22], two theorems about the Virasoro minimal model $\text{Vir}_{3,4}$ are proved.

Theorem 0.1. *The refined character of $\text{gr}^G(\text{Vir}_{3,4})$ is given by*

$$\text{ch}_{\text{gr}^G(\text{Vir}_{3,4})}(t, q) = \sum_{k_1, k_2 \in \mathbb{N}} t^{4k_1 + 2k_2} \frac{q^{4k_1^2 + 3k_1 k_2 + k_2^2}}{(q)_{k_1} (q)_{k_2}} (1 - q^{k_1} + q^{k_1 + k_2}).$$

Let R^0 be the following set of partitions

$$\begin{aligned} [r, r, r], [r + 1, r, r], [r + 1, r + 1, r], [r + 2, r + 1, r], [r + 2, r + 2, r], & \quad (r \geq 2) \\ [r + 2, r, r], & \quad (r \geq 3) \\ [r + 3, r + 3, r, r], [r + 4, r + 3, r, r], [r + 4, r + 3, r + 1, r], [r + 4, r + 4, r + 1, r], & \quad (r \geq 2) \\ [r + 6, r + 5, r + 3, r + 1, r], & \quad (r \geq 2) \\ [5, 4, 2, 2], [7, 6, 4, 2, 2], [7, 7, 4, 2, 2], [9, 8, 6, 4, 2, 2]. & \end{aligned}$$

Let P^0 be the set of partitions $\lambda = [\lambda_1, \dots, \lambda_m]$ with $\lambda_m \geq 2$ that do not contain any partition in R^0 .

Theorem 0.2. *The set*

$$\{L_{-\lambda_1}L_{-\lambda_2}\dots L_{-\lambda_m}|0\rangle \mid \lambda = [\lambda_1, \dots, \lambda_m] \in P^0\}$$

is a vector space basis of $\text{Vir}_{3,4}$.

In this thesis, we prove two similar results about $L(1/2, 1/2)$ and $L(1/2, 1/16)$.

Theorem 0.3. *The refined character of $\text{gr}^G(L(1/2, 1/2))$ is given by*

$$\text{ch}_{\text{gr}^G(L(1/2, 1/2))}(t, q) = q^{1/2} \left(\sum_{k_1, k_2 \in \mathbb{N}} t^{4k_1+2k_2} \frac{q^{4k_1^2+3k_1k_2+k_2^2}}{(q)_{k_1}(q)_{k_2}} (q^{3k_1+2k_2} + q^{5k_1+2k_2+1}t + q^{6k_1+3k_2+2}t^2) \right).$$

Let $R^{1/2}$ be the following set of partitions

$$\begin{aligned} &[r, r, r], [r+1, r, r], [r+1, r+1, r], [r+2, r+1, r], [r+2, r+2, r], & (r \geq 3) \\ &[r+2, r, r], & (r \geq 3) \\ &[r+3, r+3, r, r], [r+4, r+3, r, r], [r+4, r+3, r+1, r], [r+4, r+4, r+1, r], & (r \geq 3) \\ &[r+6, r+5, r+3, r+1, r], & (r \geq 3) \\ &[2], [1, 1, 1], [3, 1, 1], [3, 3], [4, 3, 1], [4, 4, 1], [5, 4, 1, 1], [6, 5, 3, 1]. \end{aligned}$$

Let $P^{1/2}$ be the set of partitions that do not contain any partition in $R^{1/2}$.

Theorem 0.4. *The set*

$$\{L_{-\lambda_1}L_{-\lambda_2}\dots L_{-\lambda_m}|1/2\rangle \mid \lambda = [\lambda_1, \dots, \lambda_m] \in P^{1/2}\}$$

is a vector space basis of $L(1/2, 1/2)$.

Specializing the character formula in Theorem 0.3 to $t = 1$, we obtain that the character of $L(1/2, 1/2)$ is the sum of three Nahm sums for the same matrix $\begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}$ (cf. [Nah07] and [AVEH22]). The partitions in P^0 also have a combinatorial interpretation, as was noted in [AVEH22] and [Tsu23]. For example, for $n \in \mathbb{N}$, the number of partitions of n in P^0 is the number of partitions of n with parts congruent to $\pm 2, \pm 3, \pm 4$ and ± 5 modulo 16.

Theorem 0.5. *The refined character of $\text{gr}^G(L(1/2, 1/16))$ is given by*

$$\text{ch}_{\text{gr}^G(L(1/2, 1/16))}(t, q) = q^{1/16} \left(\sum_{k_1, k_2 \in \mathbb{N}} t^{4k_1+2k_2} \frac{q^{4k_1^2+3k_1k_2+k_2^2}}{(q)_{k_1}(q)_{k_2}} (q^{k_1+k_2} + q^{4k_1+2k_2+1}t + q^{7k_1+3k_2+3}t^3) \right).$$

Let $R^{1/16}$ be the following set of partitions

$$\begin{aligned} &[r, r, r], [r+1, r, r], [r+1, r+1, r], [r+2, r+1, r], [r+2, r+2, r], & (r \geq 3) \\ &[r+2, r, r], & (r \geq 3) \\ &[r+3, r+3, r, r], [r+4, r+3, r, r], [r+4, r+3, r+1, r], [r+4, r+4, r+1, r], & (r \geq 3) \\ &[r+6, r+5, r+3, r+1, r], & (r \geq 3) \\ &[2], [1, 1, 1, 1], [3, 1, 1, 1], [3, 3, 1], [4, 3, 1], [4, 4, 1, 1], [5, 4, 1, 1, 1], [5, 5, 1, 1, 1], \\ &[6, 5, 3, 1, 1], [6, 6, 3, 1, 1], [7, 6, 4, 1, 1, 1], [8, 7, 5, 3, 1, 1]. \end{aligned}$$

Let $P^{1/16}$ be the set of partitions that do not contain any partition in $R^{1/16}$. The set

$$\{L_{-\lambda_1}L_{-\lambda_2}\dots L_{-\lambda_m}|1/16\rangle \mid \lambda = [\lambda_1, \dots, \lambda_m] \in P^{1/16}\}$$

is a vector space basis of $L(1/2, 1/16)$.

In this thesis, we develop the theory needed to prove the theorems above. We assume only basic knowledge about algebra, commutative algebra, Lie algebras and algebraic geometry. Mathematical terms are typeset with *italics* when they are officially defined.

This thesis is organized as follows. In §1, we introduce the notion of vertex superalgebras and their modules while trying to compile several results. In §2, we review the theory of representations of the Virasoro Lie algebra with the objective of describing explicitly the maximal proper subrepresentations of Verma modules. In §3, we quickly review the theory of the Zhu algebra. In §4, the theory of filtrations of vertex algebras and their modules started by Haisheng Li is presented, we introduce the standard filtration for modules, and we prove it is compatible with the definitions already given by Li. Then, we introduce the Zhu C_2 -algebra and derive consequences of the C_2 -cofiniteness condition. In §5, we carry out the computations needed to obtain the PBW basis of $L(1/2, 1/2)$ and its refined character. We also prove Theorem 0.3 and Theorem 0.4. In the appendices, we recall some results about almost commutative algebras, Poisson algebras and Jet algebras, we show the SageMath [Sag22] program used to compute the exceptional partitions appearing in $R^{1/2}$, and we briefly consider the case $L(1/2, 1/16)$. The software systems Mathematica [Inc22] and Singular [DGPS22] were also very useful to verify the series identities and compute Gröbner bases.

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1. VERTEX SUPERALGEBRAS AND THEIR MODULES

In this section, we first review formal calculus and Lie conformal superalgebras in a purely algebraic way, meaning convergence of series is never considered. Next, we study the Virasoro Lie conformal algebra, the Current Lie conformal superalgebra and the fermionic Lie conformal superalgebra. Afterward, we specialize the theory of formal calculus to $\text{End}(V)$ -valued series, where V is a vector space. This leads to the concept of fields over vector spaces. Then, the concept of vertex superalgebra is introduced. It is shown how to construct a vertex superalgebra from a Lie conformal superalgebra. Later, we introduce Hamiltonians and conformal vectors of vertex superalgebras, which lead to graded and conformal vertex superalgebras, respectively. Next, modules over vertex superalgebras are introduced with emphasis on admissible modules. Finally, we construct a couple of functors from the category of vertex algebras to the category of Lie algebras.

1.1. Formal calculus. All vector spaces and all algebras are over \mathbb{C} , the field of complex numbers, unless otherwise stated. All tensor products are over \mathbb{C} , unless otherwise stated. The set of natural numbers $\{0, 1, \dots\}$ is denoted by \mathbb{N} , the set of integers is denoted by \mathbb{Z} , the set of positive integers $\{1, 2, \dots\}$ is denoted by \mathbb{Z}_+ , and the set of negative integers $\{-1, -2, \dots\}$ is denoted by \mathbb{Z}_- .

The vector space of *formal distributions in $n \in \mathbb{N}$ variables*, denoted by $\mathbb{C}[[x_1^{\pm 1}, \dots, x_n^{\pm 1}]]$, is the set of functions $f : \mathbb{Z}^n \rightarrow \mathbb{C}$, written as $f(x_1, \dots, x_n) = \sum_{m_1, \dots, m_n \in \mathbb{Z}} f_{m_1, \dots, m_n} x_1^{m_1} \dots x_n^{m_n}$, with the natural operations of addition and multiplication by a scalar. The field of *rational functions in n variables*, denoted by $\mathbb{C}(x_1, \dots, x_n)$, is the field of fractions $\text{Frac}(\mathbb{C}[x_1, \dots, x_n])$. The field of *formal Laurent series*, denoted by $\mathbb{C}((x))$, is the subspace of elements $f(x) \in \mathbb{C}[[x^{\pm 1}]]$ such that there is $N \in \mathbb{Z}$ with $f_n = 0$ for $n \leq N$. We also have $\mathbb{C}((x)) = \text{Frac}(\mathbb{C}[[x]])$, so $\mathbb{C}((x))$ is actually a field. The field of *joint Laurent series in n variables*, denoted by $\mathbb{C}((x_1, \dots, x_n))$, is $\text{Frac}(\mathbb{C}[[x_1, \dots, x_n]])$. If V is a vector space, we similarly define $V[[x_1^{\pm 1}, \dots, x_n^{\pm 1}]]$ and $V((x))$, but in this case, $V((x))$ is only a vector space. We can consider $V[[x_1^{\pm 1}, \dots, x_n^{\pm 1}]]$ a module over the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$.

Let V be a vector space. The *Fourier expansion of a formal distribution* $a(z) \in V[[z^{\pm 1}]]$, written as $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$, is conventionally written in the theory of vertex superalgebras as

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

where

$$a_{(n)} = a_{-n-1}.$$

The *residue of a formal distribution* $a(z) \in V[[z^{\pm 1}]]$ is defined as

$$\text{res}_z(a(z)) = a_{(0)} = a_{-1}.$$

If $P \in \mathbb{C}[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$ and $Q \in \mathbb{C}[[w_1^{\pm 1}, \dots, w_m^{\pm 1}]]$, then $PQ \in \mathbb{C}[[z_1^{\pm 1}, \dots, z_n^{\pm 1}, w_1^{\pm 1}, \dots, w_m^{\pm 1}]]$ is defined in the natural way. However, if both P and Q belong to $\mathbb{C}[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$, we may encounter difficulties because infinite sums may appear.

An important formal distribution in two variables z and w is the *formal delta distribution*, which is defined by

$$\delta(z, w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1} \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]].$$

The *expansion in the domain* $|z| > |w|$ is the field homomorphism $i_{z,w} : \mathbb{C}((z, w)) \rightarrow \mathbb{C}((z))((w))$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}[[z, w]] & \xrightarrow{\text{inc}} & \mathbb{C}((z, w)) \\ & \searrow \text{inc} & \downarrow i_{z,w} \\ & & \mathbb{C}((z))((w)) \end{array}$$

where inc denotes a natural inclusion. Similarly, the expansion in the domain $|w| > |z|$ is the field homomorphism $i_{w,z} : \mathbb{C}((z, w)) \rightarrow \mathbb{C}((w))((z))$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}[[z, w]] & \xrightarrow{\text{inc}} & \mathbb{C}((z, w)) \\ & \searrow \text{inc} & \downarrow i_{w,z} \\ & & \mathbb{C}((w))((z)) \end{array}$$

The diagram

$$\begin{array}{ccc} & \mathbb{C}((z, w)) & \\ i_{z,w} \swarrow & & \searrow i_{w,z} \\ \mathbb{C}((z))((w)) & & \mathbb{C}((w))((z)) \\ \text{inc} \searrow & & \swarrow \text{inc} \\ & \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] & \end{array}$$

does not commute. In fact, the formal delta distribution can be expressed as

$$\delta(z, w) = i_{z,w} \left(\frac{1}{z-w} \right) - i_{w,z} \left(\frac{1}{z-w} \right),$$

where we consider $i_{z,w}(\frac{1}{z-w})$ and $i_{w,z}(\frac{1}{z-w})$ as elements of $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$. From now on, we will consider $i_{z,w}$ and $i_{w,z}$ as mapped into $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$.

Let V be a vector space. A formal distribution $a(z, w) \in V[[z^{\pm 1}, w^{\pm 1}]]$ is *local* if there is $N \in \mathbb{N}$ such that

$$(z-w)^N a(z, w) = 0.$$

For example, the formal delta distribution $\delta(z, w)$ is local with $(z-w)\delta(z, w) = 0$.

Theorem 1.1.1 ([Kac98, Proposition 2.2]). *Let $a(z, w) \in V[[z^{\pm 1}, w^{\pm 1}]]$ be a local formal distribution. Then $a(z, w)$ can be written uniquely as a sum*

$$a(z, w) = \sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} c^j(w),$$

where $c^j(w) \in V[[w^{\pm 1}]]$ are formal distributions given by

$$c^j(w) = \text{res}_z((z-w)^j a(z, w)) \quad \text{for } j \in \mathbb{N}.$$

In addition, the converse is true.

Let V be a vector space, and let $a(z) \in V[[z^{\pm 1}]]$ be a formal distribution. We define

$$i_{z,w}a(z+w) = \sum_{n \in \mathbb{Z}} a_n i_{z,w}((z+w)^n).$$

Proposition 1.1.2 ([Noz08, Proposition 3.4.1]). *For a formal distribution $a(z) \in V[[z^{\pm 1}]]$, the usual Taylor series expansion holds, i.e.,*

$$i_{z,w}a(z+w) = \sum_{j \in \mathbb{N}} \frac{\partial^j a(z)}{j!} w^j.$$

We now define the notion of Fourier transform in two cases: in one and two variables. Let V be a vector space, and let $a(z) \in V[[z^{\pm 1}]]$. We define the *Fourier transform in one variable* of $a(z)$ by

$$F_z^\lambda a(z) = \text{res}_z(e^{\lambda z} a(z)) \in V[[\lambda]].$$

Proposition 1.1.3 ([Noz08, Proposition 1.5.2]). *The Fourier transform in one variable satisfies the following properties for $a(z) \in V[[z^{\pm 1}]]$:*

- (i) $F_z^\lambda \partial_z a(z) = -\lambda F_z^\lambda a(z)$;
- (ii) $F_z^\lambda (e^{zT} a(z)) = F_z^{\lambda+T} (a(z))$, where $T \in \text{End}(V)$ and $a(z) \in V((z))$;
- (iii) $F_z^\lambda (a(-z)) = -F_z^{-\lambda} a(z)$;
- (iv) $F_z^\lambda \partial_w^n \delta(z, w) = e^{\lambda w} \lambda^n$.

Now let $a(z, w) \in V[[z^{\pm 1}, w^{\pm 1}]]$. We define the *Fourier transform in two variables* of $a(z, w)$ by

$$F_{z,w}^\lambda a(z, w) = \text{res}_z(e^{\lambda(z-w)} a(z, w)) \in V[[w^{\pm 1}]][[\lambda]].$$

Expanding the definition of $F_{z,w}^\lambda$, we obtain another expression

$$F_{z,w}^\lambda a(z, w) = \sum_{j \in \mathbb{N}} \frac{\lambda^j}{j!} c^j(w),$$

where

$$c^j(w) = \text{res}_z((z-w)^j a(z, w)) \quad \text{for } j \in \mathbb{N}.$$

Proposition 1.1.4 ([Noz08, Proposition 1.5.4]). *The Fourier transform in two variables satisfies the following properties for $a(z, w) \in V[[z^{\pm 1}, w^{\pm 1}]]$:*

- (i) If $a(z, w)$ is local, then $F_{z,w}^\lambda a(z, w) \in V[[w^{\pm 1}]][[\lambda]]$;
- (ii) $F_{z,w}^\lambda \partial_z a(z, w) = -\lambda F_{z,w}^\lambda a(z, w) = [\partial_w, F_{z,w}^\lambda] a(z, w)$;
- (iii) If $a(z, w)$ is local, then $F_{z,w}^\lambda a(w, z) = F_{z,w}^{-\lambda - \partial_w} a(z, w)$, where we set $F_{z,w}^{-\lambda - \partial_w} a(z, w) = F_{z,w}^\mu a(z, w)|_{\mu = -\lambda - \partial_w}$.

Remark 1.1.5. In the theory of vertex superalgebras, one usually has to interpret equations like in Proposition 1.1.4(iii).

1.2. Lie conformal superalgebras. A *vector superspace* is a \mathbb{Z}_2 -graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$, $\bar{0} = 0 + 2\mathbb{Z}$ and $\bar{1} = 1 + 2\mathbb{Z}$. We call $V_{\bar{0}}$ the *even subspace* of V and $V_{\bar{1}}$ the *odd subspace* of V . Elements of $V_{\bar{0}} \cup V_{\bar{1}}$ are called *homogeneous*. If V is finite dimensional, we define its superdimension by setting $\text{sdim}(V) = \dim(V_{\bar{0}}) - \dim(V_{\bar{1}})$. A *superalgebra* is a \mathbb{Z}_2 -graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$. This means $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$.

We set $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. If $a \in V_\alpha$, $a \neq 0$ is homogeneous, we set $p(a) = \alpha$ and call it the *parity* of a . If a and b are homogeneous, we set $p(a, b) = (-1)^{p(a)p(b)}$. A *Lie superalgebra* is a superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear product $[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ called *Lie superbracket* satisfying the following properties for $a, b, c \in \mathfrak{g}$ homogeneous:

- (i) (Graded antisymmetry) $[a, b] = -p(a, b)[b, a]$;
- (ii) (Graded Jacobi identity) $p(a, c)[a, [b, c]] + p(b, a)[b, [c, a]] + p(c, b)[c, [a, b]] = 0$.

In an associative superalgebra A , we can define the superbracket of homogeneous elements $a, b \in A$ by

$$[a, b] = ab - p(a, b)ba.$$

It can then be extended by linearity to nonhomogeneous elements. With this superbracket, A becomes a Lie superalgebra called the *underlying Lie superalgebra of A* , and it is denoted by $[A]$.

Remark 1.2.1. The even part $\mathfrak{g}_{\bar{0}}$ of a Lie superalgebra \mathfrak{g} is just a standard Lie algebra. However, unlike superspaces and superalgebras, a Lie superalgebra is not always a Lie algebra. This is why we prefer the term Lie superalgebra to \mathbb{Z}_2 -graded Lie algebra.

Remark 1.2.2. When a formula involves $p(a)$, it is assumed that a is a homogeneous with parity $p(a)$, and the formula is extended to arbitrary a by linearity.

Probably the most important example of an associative superalgebra is the *endomorphism superalgebra of a superspace V* , denoted by $\text{End}(V)$, with the \mathbb{Z}_2 -grading given by

$$\text{End}(V)_\alpha = \{T \in \text{End}(V) \mid \text{for } \beta \in \mathbb{Z}_2, T(V_\beta) \subseteq V_{\alpha+\beta}\} \quad \text{for } \alpha \in \mathbb{Z}_2.$$

We denote $\mathfrak{gl}(V) = [\text{End}(V)]$.

Let A be a not necessarily associative superalgebra. A *superderivation of A* is a homogeneous endomorphism $\partial \in \text{End}(A)$ such that

$$\partial(ab) = \partial(a)b + (-1)^{p(\partial)p(a)}a\partial(b) \quad \text{for } a, b \in A.$$

The *subspace of superderivations of A* is denoted by $\text{Der}(A)$. A *differential superalgebra* is a superalgebra A together with a superderivation ∂ of A . A differential algebra is a differential superalgebra with odd subspace equal to 0. A homomorphism $f : (A, \partial_1) \rightarrow (A_2, \partial_2)$ of differential superalgebras is a linear and parity preserving map such that $f \circ \partial_1 = \partial_2 \circ f$.

We assume \mathfrak{g} is a Lie superalgebra and ∂ is a superderivation of \mathfrak{g} . We can form the universal enveloping superalgebra $U(\mathfrak{g})$, which is now an associative superalgebra. The derivation $\partial : \mathfrak{g} \rightarrow \mathfrak{g}$ can be extended uniquely to a derivation $DU(\partial) : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$. We have constructed a functor

$$DU : \{\text{differential Lie superalgebras}\} \rightarrow \{\text{associative differential superalgebras}\}, \\ DU(\mathfrak{g}, \partial) = (U(\mathfrak{g}), DU(\partial)).$$

Let \mathfrak{g} be a Lie superalgebra. We first extend the Lie superbracket on \mathfrak{g} to the Lie superbracket between two \mathfrak{g} -valued formal distributions in one variable. Starting from $a(z) = \sum_{m \in \mathbb{Z}} a_{(m)} z^{-m-1} \in \mathfrak{g}[[z^{\pm 1}]]$ and $b(w) = \sum_{n \in \mathbb{Z}} b_{(n)} w^{-n-1} \in \mathfrak{g}[[w^{\pm 1}]]$, we define a new formal distribution in two variables by defining the superbracket

$$[a(z), b(w)] = \sum_{m, n \in \mathbb{Z}} [a_{(m)}, b_{(n)}] z^{-m-1} w^{-n-1} \in \mathfrak{g}[[z^{\pm 1}, w^{\pm 1}]].$$

Let \mathfrak{g} be a Lie superalgebra. A pair $(a(z), b(z))$ of \mathfrak{g} -valued formal distributions is said *local* if $[a(z), b(w)]$ is local. By Theorem 1.1.1, this means that

$$[a(z), b(w)] = \sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} c^j(w),$$

where $c^j(w) = \text{res}_z((z-w)^j [a(z), b(w)]) \in \mathfrak{g}[[w^{\pm 1}]]$ for $j \in \mathbb{N}$. Equivalently, we can write this equation as

$$[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{N}} c^j(w)_{(m+n-j)} \quad \text{for } m, n \in \mathbb{Z}. \quad (1.2.1)$$

Remark 1.2.3. If $(a(z), b(z))$ is a local pair, then $(\partial_z a(z), b(z))$ is also a local pair.

Let \mathfrak{g} be a Lie superalgebra. A subset $\mathfrak{F} \subseteq \mathfrak{g}[[z^{\pm 1}]]$ of formal distributions is called a *local family* if all pairs of its elements are local. For $j \in \mathbb{N}$, the *j-product* is the \mathbb{C} -bilinear map defined by

$$\begin{aligned} \bullet_{(j)} \bullet : \mathfrak{g}[[w^{\pm 1}]] \times \mathfrak{g}[[w^{\pm 1}]] &\rightarrow \mathfrak{g}[[w^{\pm 1}]], \\ a(w)_{(j)} b(w) &= \operatorname{res}_z((z-w)^j [a(z), b(w)]). \end{aligned} \quad (1.2.2)$$

Expanding the right-hand side, we get

$$(a(w)_{(j)} b(w))_{(m)} = \sum_{k=0}^j \binom{j}{k} (-1)^k [a_{(j-k)}, b_{(m+k)}] \quad \text{for } j \in \mathbb{N} \text{ and } m \in \mathbb{Z}. \quad (1.2.3)$$

We define $a(w)_{(j)} \in \operatorname{End}(\mathfrak{g}[[w^{\pm 1}]])$ in the natural way. If $(a(z), b(z))$ is a local pair, then (1.2.1) becomes

$$[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{N}} \binom{m}{j} (a(w)_{(j)} b(w))_{(m+n-j)} \quad \text{for } m, n \in \mathbb{Z}. \quad (1.2.4)$$

By Theorem 1.1.1, we also have

$$[a(z), b(w)] = \sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} (a(w)_{(j)} b(w)).$$

All these identities led us to define the following new algebraic structure that encodes the relevant information compactly.

Let \mathfrak{g} be a Lie superalgebra. The λ -*bracket* is the \mathbb{C} -bilinear map given by

$$\begin{aligned} [\bullet_\lambda \bullet] : \mathfrak{g}[[w^{\pm 1}]] \times \mathfrak{g}[[w^{\pm 1}]] &\rightarrow \mathfrak{g}[[w^{\pm 1}]][[\lambda]], \\ [a(w)_\lambda b(w)] &= F_{z,w}^\lambda [a(z), b(w)]. \end{aligned}$$

It can easily be shown that the λ -bracket is related to the j -products by

$$[a(w)_\lambda b(w)] = \sum_{j \in \mathbb{N}} a(w)_{(j)} b(w) \frac{\lambda^j}{j!}.$$

This suggests seeing the λ -bracket as the generating function of the j -products. It allows us to gather all the j -products in one product alone, the price to pay being the additional formal variable λ . We note that for a local pair, the sum in the expansion of $[a(w)_\lambda b(w)]$ in terms of the j -products is finite, i.e., $[a(w)_\lambda b(w)] \in \mathfrak{g}[[w^{\pm 1}]][[\lambda]]$.

Theorem 1.2.4 ([Noz08, §2.3]). *The j -products and the λ -bracket satisfy the following properties for $a(w), b(w) \in \mathfrak{g}[[w^{\pm 1}]]$ and $j \in \mathbb{N}$:*

- (i) $(\partial a(w))_{(j)} b = -j a(w)_{(j-1)} b(w)$;
- (ii) $a(w)_{(j)} \partial b(w) = \partial(a(w)_{(j)} b(w)) + j a(w)_{(j-1)} b(w)$;
- (iii) $\partial(a(w)_{(j)} b(w)) = (\partial a(w))_{(j)} b(w) + a(w)_{(j)} \partial b(w)$;
- (iv) $[\partial a(w)_\lambda b(w)] = -\lambda [a(w)_\lambda b(w)]$;
- (v) $[a(w)_\lambda \partial b(w)] = (\partial + \lambda) [a(w)_\lambda b(w)]$;
- (vi) $\partial [a(w)_\lambda b(w)] = [\partial a(w)_\lambda b(w)] + [a(w)_\lambda \partial b(w)]$.

Remark 1.2.5. Properties (iii) and (vi) of Theorem 1.2.4 tell us that $\partial : \mathfrak{g}[[z^{\pm 1}]] \rightarrow \mathfrak{g}[[z^{\pm 1}]]$ acts as a derivation on the j -products and the λ -bracket.

Let V be a vector superspace. From now on, all coefficients of a formal distribution are assumed to have the same parity. Therefore, we can define the *parity of a formal distribution* $a(z) \in V[[z^{\pm 1}]]$ as $p(a(z)) = p(a_{(n)})$ for any $n \in \mathbb{Z}$.

Theorem 1.2.6 ([Noz08, §2.3]). *The j -products and the λ -bracket satisfy the following properties for $a(w), b(w) \in \mathfrak{g}[[w^{\pm 1}]]$, $p, m \in \mathbb{Z}$ and $j \in \mathbb{N}$:*

- (i) $b(w)_{(j)} a(w) = -p(a(w), b(w)) \sum_{l=0}^{\infty} (-1)^{j+l} \frac{\partial^l (a(w)_{(j+l)} b(w))}{l!}$ if $(a(w), b(w))$ is a local pair;
- (ii) $[a(w)_{(p)}, b(w)_{(m)}] = \sum_{k=0}^p \binom{p}{k} (a(w)_{(k)} b(w))_{(p+m-k)}$;

- (iii) $[b(w)_\lambda a(w)] = -p(a(w), b(w))[a(w)_{-\lambda-\partial} b(w)]$ if $(a(w), b(w))$ is a local pair;
- (iv) $[a(w)_\lambda [b(w)_\mu c(w)]] = [[a(w)_\lambda b(w)]_{\lambda+\mu} c(w)] + p(a(w), b(w))[b(w)_\mu [a(w)_\lambda c(w)]]$;
- (v) $F_z^{\lambda+\mu}[a(z)_\lambda b(z)] = [F_z^\lambda a(z), F_z^\mu b(z)]$.

Let \mathfrak{g} be a Lie superalgebra. A *formal distribution Lie superalgebra* is a pair $(\mathfrak{g}, \mathfrak{F})$, where \mathfrak{F} is a local family of \mathfrak{g} -valued formal distributions, denoted by $\{a^j(z) = \sum_{n \in \mathbb{Z}} a_{(n)}^j z^{-n-1}\}_{j \in J}$, such that the coefficients $\{a_{(n)}^j \mid j \in J, n \in \mathbb{Z}\}$ span the whole \mathfrak{g} . A *regular* formal distribution Lie superalgebra is a triple $(\mathfrak{g}, \mathfrak{F}, T)$ such that:

- (i) $(\mathfrak{g}, \mathfrak{F})$ is a formal distribution Lie superalgebra;
- (ii) $\mathbb{C}[\partial_z]\mathfrak{F}$ is closed under all n -th products for $n \in \mathbb{N}$;
- (iii) $T \in \text{Der}(\mathfrak{g})$ satisfies

$$T(a^j(z)) = \partial_z a^j(z) \quad \text{for } j \in J,$$

which is equivalent to

$$T(a_{(n)}^j) = -n a_{(n-1)}^j \quad \text{for } j \in J \text{ and } n \in \mathbb{Z}. \quad (1.2.5)$$

Remark 1.2.7. We note that (1.2.5) and the fact that $\{a_{(n)}^j \mid j \in J, n \in \mathbb{Z}\}$ spans \mathfrak{g} imply that if such T exists, it is even and unique. Thus, we could remove T from the notation, but we will not.

Let $(\mathfrak{g}, \mathfrak{F})$ be a formal distribution Lie superalgebra. The *annihilation subalgebra* of $(\mathfrak{g}, \mathfrak{F})$ is

$$\mathfrak{g}_- = \text{span}\{a_{(n)}^j \mid j \in J, n \in \mathbb{N}\},$$

and the *creation subalgebra* of $(\mathfrak{g}, \mathfrak{F})$ is

$$\mathfrak{g}_+ = \text{span}\{a_{(-n-1)}^j \mid j \in J, n \in \mathbb{N}\}.$$

By Remark 1.2.3, if $(\mathfrak{g}, \mathfrak{F})$ is a formal distribution Lie superalgebra, then $\mathbb{C}[\partial_z]\mathfrak{F}$ is a local family. The notions of j -products and λ -bracket were previously defined from \mathfrak{g} -valued formal distributions, with \mathfrak{g} being a given Lie superalgebra. Those products were shown to satisfy several properties, coming either from their definition or from the fact that \mathfrak{g} is a Lie superalgebra. We now take those properties as axioms of a new algebraic structure, defined intrinsically, without any reference either to \mathfrak{g} , nor to formal distributions. For this reason, we write ∂ instead of ∂_z in the following definition.

A $\mathbb{C}[\partial]$ -module \mathcal{R} is called a *Lie conformal superalgebra* if it is endowed with a \mathbb{C} -bilinear map called λ -bracket

$$[\bullet_\lambda \bullet] : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}[\lambda]$$

satisfying the following properties for $a, b, c \in \mathcal{R}$:

- (i) (Sesquilinearity) $[\partial a_\lambda b] = -\lambda [a_\lambda b]$;
- (ii) (Skewsymmetry) $[b_\lambda a] = -p(a, b)[a_{-\lambda-\partial} b]$;
- (iii) (Jacobi identity) $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + p(a, b)[b_\mu [a_\lambda c]]$.

If we write

$$[a_\lambda b] = \sum_{j \in \mathbb{N}} (a_{(j)} b) \frac{\lambda^j}{j!},$$

where $a_{(j)} \in \text{End}(\mathcal{R})$, these properties translate in terms of j -products as follows:

- (i) $(\partial a)_{(j)} = -j a_{(j-1)}$;
- (ii) $b_{(j)} a = -p(a, b) \sum_{l=0}^{\infty} (-1)^{j+l} \frac{\partial^l (a_{(j+l)} b)}{l!}$;
- (iii) $[a_{(p)}, b_{(m)}] = \sum_{k=0}^p \binom{p}{k} (a_{(k)} b)_{(p+m-k)}$.

Proposition 1.2.8 ([Noz08, Remark 2.5.3]). *Let \mathcal{R} be a Lie conformal superalgebra, and let $a, b \in \mathcal{R}$. Then*

$$[a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$$

or, equivalently,

$$a_{(j)} \partial b = \partial(a_{(j)} b) + j a_{(j-1)} b \quad \text{for } j \in \mathbb{N}.$$

In particular, ∂ is a derivation of the λ -bracket.

We have previously shown that any regular formal distribution Lie superalgebra $(\mathfrak{g}, \mathfrak{F}, T)$ was given the structure of Lie Conformal superalgebra \mathcal{R} , with $\mathcal{R} = \mathbb{C}[\partial_z]\mathfrak{F}$, $\partial = \partial_z$ and $[a(w)_\lambda b(w)] = F_{z,w}^\lambda([a(z), b(w)])$. It turns out that the process can be reverted: to any Lie conformal superalgebra, we can associate a regular formal distribution Lie superalgebra. According to the definition of a formal distribution Lie superalgebra, we first have to define a Lie superalgebra, denoted by $\text{Lie}(\mathcal{R})$, and then associate to it a conformal family \mathcal{R} of $\text{Lie}(\mathcal{R})$ -valued formal distributions, whose coefficients span $\text{Lie}(\mathcal{R})$, so that $(\text{Lie}(\mathcal{R}), \mathcal{R})$ is then the expected formal distribution Lie superalgebra. We proceed in two steps.

We first consider the space $\tilde{\mathcal{R}} = \mathcal{R}[t, t^{-1}] = \mathcal{R} \otimes \mathbb{C}[t, t^{-1}]$ with $\tilde{\partial} = \partial \otimes \text{Id}_{\mathbb{C}[t, t^{-1}]} + \text{Id}_{\mathcal{R}} \otimes \partial_t$. This space is called the *affinization* of \mathcal{R} . Its generating elements can be written $a \otimes t^m$, where $a \in \mathcal{R}$ and $m \in \mathbb{Z}$. For clarity, we will use the notation at^m for its elements, and we write $\tilde{\partial} = \partial + \partial_t$. We define the commutation relation on $\tilde{\mathcal{R}}$ as

$$[at^m, bt^n] = \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(j)}b) t^{m+n-j} \quad \text{for } a, b \in \mathcal{R} \text{ and } m, n \in \mathbb{Z},$$

which gives $\tilde{\mathcal{R}}$ the structure of algebra, denoted by $(\tilde{\mathcal{R}}, [\bullet, \bullet])$.

Now the second step. We have to check that the commutator verifies the antisymmetry and Jacobi identities, considering that the terms $a_{(j)}b$ of the definition of $[\bullet, \bullet]$ satisfy the axioms of a Lie conformal superalgebra. The latter ones are not sufficient. Another constraint has to be imposed on elements of $\tilde{\mathcal{R}}$, namely $\tilde{\partial}(at^m) = 0$. The algebraic formulation of the latter condition is as follows: the space $\tilde{\mathcal{R}}$ has to be quotiented by the subspace I spanned by the elements of the form $\{(\partial a)t^n + nat^{n-1} \mid n \in \mathbb{Z}\}$. Using $\tilde{\partial}$, we can write $I = \tilde{\partial}\tilde{\mathcal{R}}$. This process has two goals: first transferring on $\tilde{\mathcal{R}}/\tilde{\partial}\tilde{\mathcal{R}}$ the structure of algebra of $(\tilde{\mathcal{R}}, [\bullet, \bullet])$, and then endowing $(\tilde{\mathcal{R}}/\tilde{\partial}\tilde{\mathcal{R}}, [\bullet, \bullet])$ with the structure of Lie superalgebra. The first goal is not direct because $\tilde{\partial}\tilde{\mathcal{R}}$ has to be a two-sided ideal of the algebra $(\tilde{\mathcal{R}}, [\bullet, \bullet])$, which is the case.

Lemma 1.2.9 ([Noz08, Proposition 2.6.1]). *$\tilde{\partial}\tilde{\mathcal{R}}$ is a two-sided ideal of the algebra $(\tilde{\mathcal{R}}, [\bullet, \bullet])$.*

We define the homomorphism $\phi : \tilde{\mathcal{R}} \rightarrow \tilde{\mathcal{R}}/\tilde{\partial}\tilde{\mathcal{R}}$ as the natural quotient map. The commutator between two elements of $\tilde{\mathcal{R}}$ is defined by

$$[\phi(at^m), \phi(bt^n)] = \sum_{j \in \mathbb{N}} \binom{m}{j} \phi((a_{(j)}b)t^{m+n-j}) \quad \text{for } a, b \in \mathcal{R} \text{ and } m, n \in \mathbb{Z}.$$

Proposition 1.2.10 ([Noz08, Proposition 2.6.3]). *$(\tilde{\mathcal{R}}/\tilde{\partial}\tilde{\mathcal{R}}, [\bullet, \bullet])$ is a Lie superalgebra.*

We set

$$\text{Lie}(\mathcal{R}) = \tilde{\mathcal{R}}/\tilde{\partial}\tilde{\mathcal{R}}.$$

Abusing notation, we define the family \mathcal{R} of $\text{Lie}(\mathcal{R})$ -valued formal distributions, whose coefficients span $\text{Lie}(\mathcal{R})$, by

$$\mathcal{R} = \left\{ \sum_{n \in \mathbb{Z}} \phi(at^n) z^{-n-1} \mid a \in \mathcal{R} \right\}.$$

Theorem 1.2.11 ([Noz08, Proposition 2.6.4]). *Let \mathcal{R} be a Lie conformal superalgebra. Then $(\text{Lie}(\mathcal{R}), \mathcal{R}, -\partial_t)$ is a regular formal distribution Lie superalgebra.*

Remark 1.2.12. We have not defined the category of regular formal distribution Lie superalgebras nor the category of Lie conformal superalgebras. But it is clear how they should be, and they are equivalent categories

$$\begin{aligned} \{\text{regular formal distribution Lie superalgebra}\} &\leftrightarrow \{\text{Lie conformal superalgebra}\}, \\ (\mathfrak{g}, \mathfrak{F}, T) &\mapsto (\mathbb{C}[\partial_z]\mathfrak{F}, F_{z,w}^\lambda([\bullet, \bullet])), \\ (\text{Lie}(\mathcal{R}), \mathcal{R}, -\partial_t) &\leftarrow \mathcal{R}. \end{aligned}$$

Theorem 1.2.13. *Let \mathcal{R} be a Lie conformal superalgebra, let $a, b \in \mathcal{R}$, and let $j, m \in \mathbb{N}$. Then*

$$(a_{(j)}b)_{(m)} = \sum_{k=0}^j \binom{j}{k} (-1)^k [a_{(j-k)}, b_{(m+k)}].$$

Proof. This is just (1.2.3) in the language of Lie conformal superalgebras. \square

We now show three examples of regular formal distribution Lie superalgebras and their respective Lie conformal superalgebras.

Example 1.2.14 (Virasoro Lie conformal algebra). The *Virasoro Lie algebra*, denoted by Vir , is the Lie algebra given by

$$\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C.$$

These elements satisfy the following commutation relations:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} C \quad \text{for } m, n \in \mathbb{Z}, \\ [\text{Vir}, C] &= 0. \end{aligned} \tag{1.2.6}$$

We construct a Vir -valued formal distribution by setting

$$L(z) = \sum_{n \in \mathbb{Z}} L_{(n)} z^{-n-1} \quad \text{with } L_{(n)} = L_{n-1} \text{ for } n \in \mathbb{Z}.$$

We usually write $L(z)$ as

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

In terms of formal distributions, the commutation relations become:

$$\begin{aligned} [L(z), L(w)] &= \delta(z, w) \partial_w L(w) + \partial_w \delta(z, w) 2L(w) + \partial_w^3 \delta(z, w) \frac{C}{12}, \\ [L(z), C] &= 0, \end{aligned} \tag{1.2.7}$$

where C denotes the constant formal distribution equal to $C \in \text{Vir}$. In terms of the j -products, the commutation relations become:

$$\begin{aligned} L(z)_{(0)} L(z) &= \partial_z L(z), \\ L(z)_{(1)} L(z) &= 2L(z), \\ L(z)_{(3)} L(z) &= \frac{C}{2}, \\ L(z)_{(j)} L(z) &= 0 \quad \text{for } j \neq 0, 1, 3, \\ L(z)_{(j)} C &= 0 \quad \text{for } j \in \mathbb{N}. \end{aligned} \tag{1.2.8}$$

In terms of the λ -bracket, the commutation relations become:

$$\begin{aligned} [L(z)_\lambda L(z)] &= (\partial + 2\lambda)L(z) + \frac{\lambda^3}{12} C, \\ [L(z)_\lambda C] &= 0. \end{aligned} \tag{1.2.9}$$

By Theorem 1.1.1, $\{L(z), C\}$ is a local family. Therefore, $(\text{Vir}, \{L(z), C\})$ is a formal distribution Lie algebra. Moreover, we can verify directly that $(\text{Vir}, \{L(z), C\}, \text{ad}(L_{-1}))$ is regular. We obtain a Lie conformal algebra $\mathcal{R} = \mathbb{C}[\partial]L + \mathbb{C}C$, with $L = L(z)$, $\partial C = 0$ and $\partial = \partial_z$. This is actually a direct sum, and we get the Virasoro Lie conformal algebra

$$\text{Vir} = \mathbb{C}[\partial]L \oplus \mathbb{C}C.$$

Remark 1.2.15. The notation $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ is contradictory with the notation we wrote in §1.1. However, this notation will acquire a meaning when we treat the notion of weight of an eigendistribution. In fact, this notation usually simplifies calculations, as we will see later.

Let \mathfrak{g} be a Lie superalgebra, and let $C \in \mathfrak{g}$. A *Virasoro formal distribution of central charge* C is a \mathfrak{g} -valued formal distribution $L(z) \in \mathfrak{g}[[z^{\pm 1}]]$ satisfying (1.2.6) or, equivalently, (1.2.7), (1.2.8) or (1.2.9).

Example 1.2.16 (Current Lie conformal superalgebra). Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. A *supersymmetric* bilinear form is a bilinear map $(\bullet|\bullet) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ such that

$$(a|b) = (-1)^{p(a)}(b|a) \quad \text{for } a, b \in \mathfrak{g}, a \text{ homogeneous.}$$

Alternatively, we can define a supersymmetric bilinear form as a bilinear form that vanishes on $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and $\mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{0}}$, symmetric on $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{0}}$ and antisymmetric on $\mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{1}}$. A bilinear form $(\bullet|\bullet) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ is said *invariant* if

$$([a, b]|c) = (a|[b, c]) \quad \text{for } a, b, c \in \mathfrak{g}.$$

Let \mathfrak{g} be a Lie superalgebra endowed with a supersymmetric invariant bilinear form $(\bullet|\bullet)$. The associated *loop algebra of \mathfrak{g}* is the superalgebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \mathfrak{g}[t, t^{-1}]$, endowed with the superbracket defined by

$$[a \otimes f(t), b \otimes g(t)] = [a, b] \otimes f(t)g(t) \quad \text{for } a, b \in \mathfrak{g} \text{ and } f(t), g(t) \in \mathbb{C}[t, t^{-1}],$$

and with parity given by $p(a \otimes f(t)) = p(a)$. This makes $\tilde{\mathfrak{g}}$ into a Lie superalgebra. Replacing $a \otimes t^n$ by at^n for brevity, the commutation relations become

$$[at^m, bt^n] = [a, b]t^{m+n} \quad \text{for } a, b \in \mathfrak{g} \text{ and } m, n \in \mathbb{Z}.$$

The central extension of the loop algebra is the algebra $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}K$ with the superbracket defined by:

$$\begin{aligned} [at^m, bt^n] &= [a, b]t^{m+n} + m\delta_{m, -n}(a|b)K \quad \text{for } a, b \in \mathfrak{g} \text{ and } m, n \in \mathbb{Z}, \\ [\hat{\mathfrak{g}}, K] &= 0, \end{aligned}$$

and with parity given by $p(K) = \bar{0}$. This makes $\hat{\mathfrak{g}}$ into a Lie superalgebra called the *affinization of \mathfrak{g}* . If \mathfrak{g} is a finite-dimensional simple Lie superalgebra, then the affinization of \mathfrak{g} leads to a *Kac-Moody affinization*. We now construct $\hat{\mathfrak{g}}$ -valued formal distributions by setting

$$a(z) = \sum_{n \in \mathbb{Z}} at^n z^{-n-1} \quad \text{for } a \in \mathfrak{g}.$$

These formal distributions are called *currents*. In terms of currents, the commutation relations become:

$$\begin{aligned} [a(z), b(w)] &= \delta(z, w)[a, b](w) + \partial_w \delta(z, w)K(a|b), \\ [a(z), K] &= 0. \end{aligned}$$

In terms of the λ -bracket, the commutation relations become:

$$\begin{aligned} [a(z)_\lambda b(z)] &= [a(z), b(z)] + (a|b)K\lambda, \\ [a(z)_\lambda K] &= 0. \end{aligned}$$

By Theorem 1.1.1, $\{a(z) \mid a \in \mathfrak{g}\} \cup \{K\}$ is a local family. Therefore, $(\hat{\mathfrak{g}}, \{a(z) \mid a \in \mathfrak{g}\} \cup \{K\})$ is a formal distribution Lie superalgebra. Moreover, we can verify directly that $(\hat{\mathfrak{g}}, \{a(z) \mid a \in \mathfrak{g}\} \cup \{K\}, -\partial_t)$ is regular. Similarly to the Virasoro Lie conformal algebra, we obtain the current Lie conformal superalgebra

$$\text{Cur}(\mathfrak{g}) = \mathbb{C}[\partial]\mathfrak{g} \oplus \mathbb{C}K.$$

Let \mathfrak{g} be an abelian Lie superalgebra. In that case, $\text{Cur}(\mathfrak{g})$ is known as the *conformal algebra of free bosons* associated with the free bosons algebra $\hat{\mathfrak{g}}$, the latter being endowed with the relations $[at^m, bt^n] = m(a|b)\delta_{m, -n}K$.

Example 1.2.17 (Fermionic Lie conformal superalgebra). Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace. A bilinear form $\langle \bullet, \bullet \rangle : V \times V \rightarrow \mathbb{C}$ is *antisupersymmetric* if it satisfies the relation

$$\langle a, b \rangle = -(-1)^{p(a)} \langle b, a \rangle \quad \text{for } a, b \in V, a \text{ homogeneous.}$$

Alternatively, we can define an antisymmetric bilinear form as a bilinear form that vanishes on $V_{\bar{0}} \oplus V_{\bar{1}}$ and $V_{\bar{1}} \oplus V_{\bar{0}}$, is antisymmetric on the even part $V_{\bar{0}} \oplus V_{\bar{0}}$ and symmetric on the odd part $V_{\bar{1}} \oplus V_{\bar{1}}$.

The *Clifford affinization* of V is defined by

$$\widehat{V} = V[t, t^{-1}] \oplus \mathbb{C}K$$

with the superbracket defined by:

$$\begin{aligned} [at^m, bt^n] &= \delta_{m, -n-1} \langle a, b \rangle K \quad \text{for } a, b \in V \text{ and } m, n \in \mathbb{Z}, \\ [\widehat{V}, K] &= 0, \end{aligned}$$

and with parity given by $p(at^n) = p(a)$ and $p(K) = \bar{0}$. This makes \widehat{V} into a Lie superalgebra. We now construct \widehat{V} -valued formal distributions by setting

$$a(z) = \sum_{n \in \mathbb{Z}} at^n z^{-n-1} \quad \text{for } a \in V.$$

In terms of formal distributions, the commutation relations become:

$$\begin{aligned} [a(z), b(w)] &= \delta(z, w) \langle a, b \rangle K, \\ [a(z), K] &= 0. \end{aligned}$$

In terms of the λ -bracket, the commutation relations become:

$$\begin{aligned} [a(z)_\lambda b(z)] &= \langle a, b \rangle K, \\ [a(z)_\lambda K] &= 0. \end{aligned}$$

As before, we obtain a regular formal distribution Lie superalgebra $(\widehat{V}, \{a(z) \mid a \in V\} \cup \{K\}, -\partial_t)$, from which we obtain the fermionic Lie conformal superalgebra

$$F(V) = \mathbb{C}[\partial]V \oplus \mathbb{C}K.$$

1.3. Fields over vector spaces. In this subsection, we fix a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, and all formal distributions are $\text{End}(V)$ -valued, unless otherwise stated. Let $a(z)$ be a formal distribution. We set

$$\begin{aligned} a(z)_+ &= \sum_{n \leq -1} a_{(n)} z^{-n-1}, \\ a(z)_- &= \sum_{n \geq 0} a_{(n)} z^{-n-1}. \end{aligned}$$

Let $a(z), b(z)$ be two formal distributions. We define the *normal product between $a(z)$ and $b(z)$* as the following formal distribution in two variables

$$: a(z)b(w) := a(z)_+ b(w) + p(a(z), b(z)) b(w) a(z)_-.$$

Theorem 1.3.1 ([Noz08, Proposition 3.2.3]). *Let $(a(z), b(z))$ be a pair of local formal distributions. The following identities are known as the operator product expansion of $a(z)$ and $b(w)$:*

$$\begin{aligned} a(z)b(w) &= \sum_{j \in \mathbb{N}} a(w)_{(j)} b(w) i_{z,w} \left(\frac{1}{(z-w)^{j+1}} \right) + : a(z)b(w) :, \\ p(a, b)b(w)a(z) &= \sum_{j \in \mathbb{N}} a(w)_{(j)} b(w) i_{w,z} \left(\frac{1}{(z-w)^{j+1}} \right) + : a(z)b(w) :. \end{aligned}$$

A formal distribution $a(z)$ is a *field* if

$$a(z)b = \sum_{n \in \mathbb{Z}} a_{(n)} b z^{-n-1} \in V((z)) \quad \text{for } b \in V.$$

The vector space of fields over V is denoted by $\mathcal{F}(V)$. We note that

$$\mathcal{F}(V) = \text{Hom}(V, V((z))).$$

Therefore, we can define a field $a(z)$ by defining $a(z)b \in V((z))$ for $b \in V$.

Proposition 1.3.2 ([Noz08, Proposition 3.3.2]). *Let $a(z), b(z) \in \mathcal{F}(V)$ be two fields. Then $:a(z)b(z): \in \text{End}(V)[[z^{\pm 1}]]$ is again a field, where $:a(z)b(z):$ is defined by*

$$:a(z)b(z): c = a(z)_+ b(z)c + p(a, b)b(z)a(z)_- c \quad \text{for } c \in V.$$

We thus defined the notion of *normal ordered product between fields* $a(z), b(z) \in \mathcal{F}(V)$, denoted by $:a(z)b(z):$. In general, the operation of normal ordered product is neither commutative nor associative. We follow the convention that the normal ordered product is read from right to left, so that, by definition,

$$:a(z)b(z)c(z): =: a(z)(:b(z)c(z):):.$$

We define the normal ordered product of a single field as the field itself, and the normal ordered product of no fields as the identity field Id_V , so we have:

$$\begin{aligned} :a(z): &= a(z), \\ :: &= \text{Id}_V. \end{aligned}$$

The identity field Id_V acts as an identity for the normal ordered product, i.e.,

$$:\text{Id}_V a(z): =: a(z) \text{Id}_V := a(z).$$

Lemma 1.3.3 ([Noz08, Proposition 3.3.3]). *Let $a(z), b(z) \in \mathcal{F}(V)$ be two fields. Their normal ordered product is written explicitly as*

$$:a(z)b(z): = \sum_{j \in \mathbb{Z}} :a(z)b(z):_{(j)} z^{-j-1},$$

with

$$:a(z)b(z):_{(j)} c = \sum_{n \leq -1} a_{(n)} b_{(j-n-1)} c + p(a(z), b(z)) \sum_{n \geq 0} b_{(j-n-1)} a_{(n)} c \quad \text{for } c \in V.$$

Lemma 1.3.4. *Let V be a purely even vector space. We consider s fields $a^1(z), \dots, a^s(z) \in \mathcal{F}(V)$, with $s \geq 2$, and let $b \in V$. For $l \in \mathbb{Z}$,*

$$:a^1(z)a^2(z)\dots a^s(z):_{(l)} b = \sum_{n_1, \dots, n_{s-1} \in \mathbb{N}} \sum_{k=0}^{s-1} R_{n_1, \dots, n_{s-1}}^{l, k}(a^1(z), \dots, a^s(z))b,$$

where $R_{n_1, \dots, n_{s-1}}^{l, k}(a^1(z), \dots, a^s(z))$ is the sum of $\binom{s-1}{k}$ terms given by

$$R_{n_1, \dots, n_{s-1}}^{l, k}(a^1(z), \dots, a^s(z)) = \sum_{\substack{1 \leq i_1 < \dots < i_k \leq s-1 \\ 1 \leq j_1 < \dots < j_{s-1-k} \leq s-1 \\ \{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{s-1-k}\} = \{1, \dots, s-1\}}} a_{(-n_{j_1}-1)}^{j_1} \dots a_{(-n_{j_{s-1-k}}-1)}^{j_{s-1-k}} a_{(l-k-\sum_{r=1}^k n_{i_r} + \sum_{r=1}^{s-1-k} n_{j_r})}^s a_{(n_{i_k})}^{i_k} \dots a_{(n_{i_1})}^{i_1}.$$

Proof. We do this by induction on $s \geq 2$. The base case $s = 2$ follows from Lemma 1.3.4 and by replacing n by $-n - 1 \in \mathbb{N}$ in the following computation

$$\begin{aligned}
: a^1(z)a^2(z) :_{(l)} &= \sum_{n \leq -1} a_{(n)}^1 a_{(l-n-1)}^2 + \sum_{n \geq 0} a_{(l-n-1)}^2 a_{(n)}^1 \\
&= \sum_{n \in \mathbb{N}} a_{(-n-1)}^1 a_{(l+n)}^2 + \sum_{n \in \mathbb{N}} a_{(l-1-n)}^2 a_{(n)}^1 \\
&= \sum_{n \in \mathbb{N}} R_n^{l,0}(a^1(z), a^2(z)) + \sum_{n \in \mathbb{N}} R_n^{l,1}(a^1(z), a^2(z)) \\
&= \sum_{n \in \mathbb{N}} \sum_{k=0}^1 R_n^{l,k}(a^1(z), a^2(z)).
\end{aligned}$$

We now assume our desired formula for $s - 1$, with $s \geq 3$, and we prove it for s . We observe that in the sum $R_{n_1, \dots, n_{s-1}}^{l,k}(a^1(z), \dots, a^s(z))$, either $i_1 = 1$ or $j_1 = 1$. Therefore, by the induction hypothesis and the base case $s = 2$, we have

$$\begin{aligned}
: a^1(z)a^2(z) \dots a^s(z) :_{(l)} &=: a^1(z) (: a^2(z) \dots a^s(z) :)_{(l)} \\
&= \sum_{n_1 \in \mathbb{N}} a_{(-n_1-1)}^1 (: a^2(z) \dots a^s(z) :)_{(l+n_1)} \\
&\quad + \sum_{n_1 \in \mathbb{N}} (: a^2(z) \dots a^s(z) :)_{(l-1-n_1)} a_{(n_1)}^1 \\
&= \sum_{n_1 \in \mathbb{N}} a_{(-n_1-1)}^1 \sum_{n_2, \dots, n_{s-1} \in \mathbb{N}} \sum_{k=0}^{s-2} R_{n_2, \dots, n_{s-1}}^{l+n_1, k}(a^2(z), \dots, a^s(z)) \\
&\quad + \sum_{n_1 \in \mathbb{N}} \sum_{n_2, \dots, n_{s-1} \in \mathbb{N}} \sum_{k=0}^{s-2} R_{n_2, \dots, n_{s-1}}^{l-1-n_1, k}(a^2(z), \dots, a^s(z)) a_{(n_1)}^1 \\
&= \sum_{n_1, \dots, n_{s-1} \in \mathbb{N}} \sum_{k=0}^{s-2} a_{(-n_1-1)}^1 R_{n_2, \dots, n_{s-1}}^{l+n_1, k}(a^2(z), \dots, a^s(z)) \\
&\quad + \sum_{n_1, \dots, n_{s-1} \in \mathbb{N}} \sum_{k=0}^{s-2} R_{n_2, \dots, n_{s-1}}^{l-1-n_1, k}(a^2(z), \dots, a^s(z)) a_{(n_1)}^1 \\
&= \sum_{n_1, \dots, n_{s-1} \in \mathbb{N}} \sum_{k=0}^{s-1} R_{n_1, \dots, n_{s-1}}^{l, k}(a^1(z), \dots, a^s(z)). \quad \square
\end{aligned}$$

We now extend the j -products. Let $a(w), b(w) \in \mathcal{F}(V)$ be two fields. For $j \in \mathbb{N}$, $\partial_w^j a(w)$ is also a field, and we define

$$a(w)_{(-j-1)} b(w) = \frac{: (\partial_w^j a(w)) b(w) :}{j!}. \quad (1.3.1)$$

Theorem 1.3.5 ([Noz08, Proposition 3.4.3]). *The j -products (1.2.2) and (1.3.1) are special cases of the following generalized j -product defined by*

$$a(w)_{(j)} b(w) c = \text{res}_z (i_{z,w}((z-w)^j) a(z) b(w) c - p(a(z), b(z)) i_{w,z}((z-w)^j) b(w) a(z) c) \quad \text{for } c \in V.$$

The usual properties of j -products for $j \in \mathbb{N}$ carry over to the generalized j -products.

Proposition 1.3.6 ([Noz08, Proposition 3.4.4]). *Let $a(z), b(z) \in \mathcal{F}(V)$ be two fields. For $j \in \mathbb{Z}$:*

- (i) $(\partial_z a(z))_{(j)} b(z) = -j a(z)_{(j-1)} b(z)$;
- (ii) $\partial_z (a(z)_{(j)} b(z)) = (\partial_z a(z))_{(j)} b(z) + a(z)_{(j)} \partial_z b(z)$.

Lemma 1.3.7 (Dong's lemma [Kac98, Lemma 3.2]). *If $a(z), b(z)$ and $c(z)$ are pairwise mutually local fields, then $(a(z), b(z))_{(n)} c(z)$ is a local pair of fields as well for $n \in \mathbb{Z}$.*

Let $L(z)$ be a formal distribution. An *eigendistribution of weight $\Delta_{a(z)}$ with respect to $L(z)$* is a formal distribution $a(z)$ satisfying

$$[L(z)_\lambda a(z)] = (\partial + \Delta_{a(z)}\lambda)a(z) + O(\lambda^2), \quad (1.3.2)$$

where $O(\lambda^2)$ denotes sums of terms of monomials $t\lambda^j$ with $j \geq 2$ and $t \in \mathbb{C}$. If $L(z)$ is a Virasoro formal distribution, $\Delta_{a(z)}$ is called the *conformal weight*. Clearly, $a(z)$ is an eigendistribution in the sense that it is an eigenvector of the endomorphism $L_{(1)}$ whose action is defined by $L_{(1)}(b(z)) = L(z)_{(1)}b(z)$. Indeed, (1.3.2) implies $L(z)_{(1)}a(z) = \Delta_{a(z)}a(z)$. We note that, by definition, a Virasoro formal distribution $L(z)$ is an eigendistribution of conformal weight 2 with respect to itself.

Theorem 1.3.8 ([Noz08, Proposition 3.7.4]). *If $a(z)$ and $b(z)$ have weights $\Delta_{a(z)}$ and $\Delta_{b(z)}$ with an even formal distribution $L(z)$, then $\Delta_{a(z)_{(n)}b(z)} = \Delta_{a(z)} + \Delta_{b(z)} - n - 1$ with respect to $L(z)$. In particular, $\Delta_{:a(z)b(z):} = \Delta_{a(z)} + \Delta_{b(z)}$ and $\Delta_{\partial a(z)} = \Delta_{a(z)} + 1$.*

The expansion of an eigendistribution $a(z)$ of weight $\Delta_{a(z)}$ is often adapted as follows

$$a(z) = \sum_{n \in \mathbb{Z} - \Delta_{a(z)}} a_n z^{-n - \Delta_{a(z)}}.$$

This justifies the way we wrote the Virasoro formal distribution when we defined the Virasoro Lie conformal algebra. By comparison with the usual way of writing $a(z) = \sum_{m \in \mathbb{Z}} a_{(m)} z^{-m-1}$, we must have

$$a_{(m)} = a_{m - \Delta_{a(z)} + 1} \quad \text{for } m \in \mathbb{Z}$$

or the other way

$$a_n = a_{(n + \Delta_{a(z)} - 1)} \quad \text{for } n \in \mathbb{Z} - \Delta_{a(z)}.$$

One of the interesting features of this change of notation is that it reveals the grading of the superbracket.

Proposition 1.3.9 ([Noz08, Proposition 3.7.6]). *In the new notation, we can write*

$$[a_m, b_n] = \sum_{j \in \mathbb{N}} \binom{m + \Delta_{a(z)} - 1}{j} (a_{(j)}b)_{m+n}.$$

An eigendistribution $a(z)$ is called *primary* of conformal weight $\Delta_{a(z)}$ if

$$[L(z)_\lambda a(z)] = (\partial_z + \Delta_{a(z)}\lambda)a(z),$$

where $L(z)$ is a Virasoro formal distribution.

Let us fix an operator $T \in \text{End}(V)_{\bar{0}}$. A formal distribution $a(z)$ is called *covariant* with respect to T if

$$[T, a(z)] = \partial_z a(z).$$

Theorem 1.3.10 ([Kac17, Lemma 1]). *Assume that $|0\rangle \in V_{\bar{0}}$ is such that $T|0\rangle = 0$. Then:*

- (i) *For any translation covariant field $a(z)$, we have $a(z)|0\rangle \in V[[z]]$.*
- (ii) *Let $a(z)$ be a translation covariant field, and we set $a = a_{(-1)}|0\rangle$. Then*

$$a(z)|0\rangle = e^{Tz}a = \sum_{n=0}^{\infty} \frac{T^n a}{n!} z^n.$$

Let

$$\mathcal{F}_{\text{tc}} = \{a(z) \in \mathcal{F}(V) \mid [T, a(z)] = \partial_z a(z)\}$$

be the subspace of translation covariant fields.

Lemma 1.3.11 ([Kac17, Lemma 3]). *\mathcal{F}_{tc} contains Id_V , it is ∂_z -invariant and is closed under all n -products, i.e., $\partial_z a(z), a(z)_{(n)}b(z) \in \mathcal{F}_{\text{tc}}$ for $a(z), b(z) \in \mathcal{F}_{\text{tc}}$ and $n \in \mathbb{Z}$.*

By Theorem 1.3.10(i), we can define a linear map

$$\begin{aligned} \text{fs} : \mathcal{F}_{\text{tc}} &\rightarrow V, \\ \text{fs}(a(z)) &= a(z) |0\rangle|_{z=0} \end{aligned}$$

called the *field-state correspondence*.

Lemma 1.3.12 ([Noz08, Proposition 4.3.2]). *Let A be a linear operator on a linear space V . The formal differential equation*

$$\frac{df(z)}{dz} = Af(z) \quad \text{for } f(z) \in V[[z]]$$

admits a unique solution, given an initial condition $f(0) = f_0$.

Theorem 1.3.13. *Let $a(z), b(z) \in \mathcal{F}_{\text{tc}}$, $a = \text{fs}(a(z))$, $b = \text{fs}(b(z))$, and let $n \in \mathbb{Z}$. We write $a(z) = \sum_{j \in \mathbb{Z}} a_{(j)} z^{-j-1}$ and $b(z) = \sum_{j \in \mathbb{Z}} b_{(j)} z^{-j-1}$. Then:*

- (i) $\text{fs}(\text{Id}_V) = |0\rangle$;
- (ii) $\text{fs}(\partial_z a(z)) = Ta$;
- (iii) (*n -product identity*) $\text{fs}(a(z)_{(n)} b(z)) = a_{(n)} b$;
- (iv) $T(a_{(n)} b) = -na_{(n-1)} b + a_{(n)} T b$;
- (v) $e^{Tw} a(z) e^{-Tw} = i_{z,w} a(z+w)$;
- (vi) (*Borcherds identity*) *If $a(z)$ and $b(z)$ are local, then for $c \in V$,*

$$i_{z,w}(z-w)^n a(z) b(w) c - p(a, b) i_{w,z}(z-w)^n b(w) a(z) c = \sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} a(w)_{(n+j)} b(w) c;$$

- (vii) (*Skewsymmetry*) *If $a(z)$ and $b(z)$ are local, then*

$$a(z) b = p(a, b) e^{Tz} b(-z) a.$$

Proof.

- (i) Clear.
- (ii)

$$\begin{aligned} \text{fs}(\partial_z a(z)) &= [T, a(z)] |0\rangle|_{z=0} \\ &= (Ta(z) - p(a(z), T)a(z)T) |0\rangle|_{z=0} \\ &= Ta. \end{aligned}$$

- (iii) By definition, we have

$$\text{fs}(a(z)_{(n)} b(z)) = a(z)_{(n)} b(z) |0\rangle|_{z=0},$$

and the right-hand side, by Theorem 1.3.5, is equal to

$$\text{res}_w(a(w) b(z) i_{w,z}((w-z)^n) |0\rangle - p(a, b) b(z) a(w) i_{z,w}((w-z)^n) |0\rangle)|_{z=0}.$$

Now, since $a(w) |0\rangle \in V[[w]]$ and $i_{z,w}((w-z)^n)$ has only nonnegative powers of w , we have

$$\text{res}_w(b(z) a(w) i_{z,w}((w-z)^n) |0\rangle) = 0.$$

For the first term, since $b(z) |0\rangle \in V[[z]]$, we can let $z = 0$ before we calculate the residue, which gives

$$\text{res}_w(a(w) b(z) i_{w,z}((w-z)^n) |0\rangle)|_{z=0} = \text{res}_w(a(w) b w^n) = a_{(n)} b.$$

- (iv) This follows from $[T, a_{(n)}] = -na_{(n-1)}$, which is equivalent to translation covariance of the field $a(z)$.

- (v) We set $b_1(z, w) = i_{z,w}a(z+w)$ and $b_2(z, w) = e^{Tw}a(z)e^{-Tw}$. By Proposition 1.1.2 and translation covariance:

$$\begin{aligned}\frac{\partial b_1(z, w)}{\partial w} &= \sum_{j \in \mathbb{N}} \frac{\partial_z^{j+1} a(z)}{j!} w^j = \sum_{j \in \mathbb{N}} \frac{\partial^j [T, a(z)]}{j!} = [T, b_1(z, w)], \\ b_1(z, 0) &= a(z), \\ \frac{\partial b_2(z, w)}{\partial w} &= T e^{Tw} a(z) e^{-Tw} + e^{Tw} a(z) (-T) e^{-Tw} = [T, b_2(z, w)], \\ b_2(z, 0) &= a(z).\end{aligned}$$

By Lemma 1.3.12, $b_1(z, w) = b_2(z, w)$.

- (vi) The left-hand side of the Borcherds identity is a local formal distribution in z and w applied to c . Apply Theorem 1.1.1 to it to get that it is equal to

$$\sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} c^j(w) c,$$

where

$$\begin{aligned}c^j(w) c &= (\text{res}_z((z-w)^j (i_{z,w}((z-w)^n a(z) b(w) - p(a, b) i_{w,z}((z-w)^n b(w) a(z)))))) c \\ &= \text{res}_z((i_{z,w}((z-w)^{n+j} a(z) b(w) c - p(a, b) i_{w,z}((z-w)^{n+j} b(w) a(z) c))) \\ &= a(w)_{(n+j)} b(w) c.\end{aligned}$$

- (vii) By locality, there is $N \in \mathbb{Z}$ such that

$$(z-w)^N a(z) b(w) = p(a, b) (z-w)^N b(w) a(z).$$

Apply $|0\rangle$ to both sides; by Theorem 1.3.10, we get

$$(z-w)^N a(z) e^{Tz} b = p(a, b) (z-w)^N b(w) e^{Tz} a.$$

We now use (v) and Proposition 1.1.2,

$$RHS = p(a, b) (z-w)^N e^{Tz} e^{-Tz} b(w) e^{Tz} a = p(a, b) (z-w)^N e^{Tz} i_{w,z} b(w-z) a.$$

For N big enough, this is a formal power series in $(z-w)$, so we can set $w=0$ and get

$$LHS = z^N a(z) b = p(a, b) e^{Tz} z^N b(-z) a = RHS,$$

which proves the desired formula. \square

Lemma 1.3.14. *Let $\mathcal{F}' \subseteq \mathcal{F}_{tc}$ and $a(z) \in \mathcal{F}_{tc}$. We assume that:*

- (i) $\text{fs}(a(z)) = 0$;
- (ii) $a(z)$ is local with any element in \mathcal{F}' ;
- (iii) $\text{fs}(\mathcal{F}') = V$.

Then $a(z) = 0$.

Proof. Let $b(z) \in \mathcal{F}'$. By the locality of $a(z)$ and $b(z)$, we have $(z-w)^N [a(z), b(w)] = 0$, for some $N \in \mathbb{N}$. Apply $|0\rangle$ to both sides to get

$$(z-w)^N a(z) b(w) |0\rangle = \pm (z-w)^N b(w) a(z) |0\rangle.$$

By the property (i), we have $a_{(-1)} |0\rangle = 0$ and $a(z)$ is translation covariant, hence by Theorem 1.3.10(i), $b(w) |0\rangle \in V[[w]]$, so we can let $w=0$ and get $z^N a(z) b = 0$, which means $a_{(n)} b = 0$ for any $n \in \mathbb{Z}$. This is true for any $b \in V$ by the property (iii). So in fact, we have $a(z) = 0$. \square

1.4. **Vertex superalgebras.** A *vertex superalgebra* is the data consisting of four elements $(V, |0\rangle, T, Y)$ satisfying the following properties:

- (i) V is a vector superspace called the *state space*;
- (ii) $|0\rangle \in V_{\bar{0}}$ is called the *vacuum vector*;
- (iii) $T \in \text{End}(V)_{\bar{0}}$ is called the *translation operator*;
- (iv) $Y : V \rightarrow \mathcal{F}(V)$ is a linear and parity preserving map called the *state-field correspondence*, which is commonly written as $Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ for $a \in V$.

By parity preserving map we mean that for $a \in V$ homogeneous, $p(a_{(n)}) = p(a)$ for $n \in \mathbb{Z}$. The operator $Y(a) = Y(a, z) \in \text{End}(V)[[z^{\pm 1}]]$ for $a \in V$ is sometimes called a *vertex operator*. The data must satisfy the following axioms for $a \in V$:

- (i) (Vacuum axiom)

$$\begin{aligned} Y(|0\rangle, z) &= \text{Id}_V, \\ Y(a, z) |0\rangle &\in V[[z]], \\ Y(a, z) |0\rangle |_{z=0} &= a, \\ T |0\rangle &= 0; \end{aligned}$$

- (ii) (Translation covariance) $[T, Y(a, z)] = \partial_z Y(a, z)$;
- (iii) (Locality) $\{Y(b, z) \mid b \in V\}$ is a local family of fields.

Remark 1.4.1. Writing $Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ for $a \in V$, the first two vertex superalgebra axioms imply that for $a \in V$:

$$\begin{aligned} |0\rangle_{(n)} a &= \delta_{n,-1} a \quad \text{for } n \in \mathbb{Z}, \\ a_{(n)} |0\rangle &= \delta_{n,-1} a \quad \text{for } n \in \mathbb{N} \cup \{-1\}, \\ [T, a_{(n)}] &= -n a_{(n-1)} \quad \text{for } n \in \mathbb{Z}. \end{aligned}$$

A vertex superalgebra with odd subspace equal to 0 is called a *vertex algebra*.

Remark 1.4.2. The even part $V_{\bar{0}}$ of a vertex superalgebra V is a vertex algebra.

Remark 1.4.3. The translation covariance axiom together with Theorem 1.3.10(ii) permit us to express T by

$$Ta = a_{(-2)} |0\rangle. \quad (1.4.1)$$

As a consequence, the data of the translation operator T is redundant. The original definition with T appears to be more natural, though.

A *vertex superalgebra homomorphism* $f : (V_1, |0\rangle_1, T_1, Y_1) \rightarrow (V_2, |0\rangle_2, T_2, Y_2)$ is a linear and parity preserving map $f : V_1 \rightarrow V_2$ such that $f(|0\rangle_1) = |0\rangle_2$ and for $a, b \in V_1$,

$$f(Y_1(a, z)b) = \sum_{n \in \mathbb{Z}} f(a_{(n)}b)z^{-n-1} = \sum_{n \in \mathbb{Z}} f(a)_{(n)}f(b)z^{-n-1} = Y_2(f(a), z)f(b).$$

We obtain the category of vertex superalgebras.

Let V_1 and V_2 be two vertex algebras. The *tensor product* $V_1 \otimes V_2$ is a vertex algebra with the vacuum vector $|0\rangle_{V_1} \otimes |0\rangle_{V_2}$, the translation operator $T_{V_1} \otimes \text{Id}_{V_2} + \text{Id}_{V_1} \otimes T_{V_2}$ and the state-field correspondence $Y(a \otimes b, z) = Y_{V_1}(a, z) \otimes Y_{V_2}(b, z)$ for $a \in V_1$ and $b \in V_2$.

Let V_1, \dots, V_n be vertex algebras. The *direct sum* $V_1 \oplus \dots \oplus V_n$ is a vertex algebra with the vacuum vector $(|0\rangle_{V_1}, \dots, |0\rangle_{V_n})$, the translation operator $(T_{V_1}, \dots, T_{V_n})$ and the state-field correspondence $Y((a_1, \dots, a_n), z) = (Y(a_1, z), \dots, Y(a_n, z))$ for $a_1 \in V_1, \dots, a_n \in V_n$.

Example 1.4.4 (Commutative vertex algebras). A vertex algebra V is called *commutative* if all vertex operators $Y(a, z)$, $a \in V$ commute with each other.

Suppose we are given a commutative vertex algebra V . Then for $a, b \in V$,

$$Y(a, z)b = Y(a, z)Y(b, w) |0\rangle |_{w=0} = Y(b, w)Y(a, z) |0\rangle |_{w=0}.$$

But by the vacuum axiom, the last expression has no negative power of z . Therefore, $Y(a, z)b \in V[[z]]$ for $a, b \in V$, so $Y(a, z) \in \text{End}(V)[[z]]$ for $a \in V$. Conversely, suppose that we are given a vertex algebra V in which $Y(a, z) \in \text{End}(V)[[z]]$ for $a \in V$. Observe that if the equality $(z-w)^N f_1(z, w) = (z-w)^N f_2(z, w)$ holds for $f_1(z, w), f_2(z, w) \in V[[z, w]]$ and $N \in \mathbb{N}$, then necessarily $f_1(z, w) = f_2(z, w)$. Therefore, we obtain that $[Y(a, z), Y(b, w)] = 0$ for $a, b \in V$, so V is commutative.

Thus, a commutative vertex algebra may be defined as one in which all $Y(a, z)$ belong to $\text{End}(V)[[z]]$.

We denote by Y_a the endomorphism of a commutative vertex algebra V which is the constant term of $Y(a, z)$ for $a \in V$, and we define a bilinear operation \circ on V by setting $a \circ b = Y_a b$. By construction, $Y_a Y_b = Y_b Y_a$. This implies both commutativity and associativity of \circ . Furthermore, the vacuum vector $|0\rangle$ is a unit, and the operator T is a derivation with respect to this product. Thus, we have the structure of a differential commutative associative algebra with unit on V .

Conversely, let V be a differential commutative associative algebra with unit 1 and derivation T . Then V becomes a vertex algebra by setting $|0\rangle = 1$ and

$$Y(a, z) = e^{Tz} a \quad \text{for } a \in V.$$

It is straightforward to check that all the axioms of a commutative vertex algebra are satisfied.

Therefore, we obtain an isomorphism between the category of differential commutative associative algebras with unit and the category of commutative vertex algebras.

Let V be a vertex superalgebra. A *vertex subalgebra* of V is a subspace W of V , which contains $|0\rangle$, and such that $Y(a, z)b \in W((z))$ for $a, b \in W$. Because $Ta = a_{(-2)}|0\rangle$ and $|0\rangle \in W$, this implies that $T(W) \subseteq W$. Thus, $(W, |0\rangle, T|_W : W \rightarrow W, Y|_W : W \rightarrow \mathcal{F}(W))$ is a vertex superalgebra in its own right. Let $S \subseteq V$ be a subset. The *vertex subalgebra generated by S* is the smallest vertex subalgebra containing S , which is the intersection of all vertex subalgebras containing S . It is denoted by $\langle S \rangle$, and we can prove that

$$\langle S \rangle = \text{span}\{a_{(n_1)}^1 \dots a_{(n_s)}^s |0\rangle \mid s \in \mathbb{N}, a^1, \dots, a^s \in S, n_1, \dots, n_s \in \mathbb{Z}\}.$$

The vertex superalgebra V is *strongly generated by $S \subseteq V$* if

$$V = \text{span}\{a_{(-n_1-1)}^1 \dots a_{(-n_s-1)}^s |0\rangle \mid s, n_1, \dots, n_s \in \mathbb{N}, a^1, \dots, a^s \in S\}.$$

An *ideal* of V is a subspace I of V such that $Y(a, z)b \in I((z))$ and $Y(b, z)a \in I((z))$ for $a \in V$ and $b \in I$. For example, the kernel of a vertex superalgebra homomorphism is an ideal. A vertex superalgebra is *simple* or *irreducible* if 0 is the only proper ideal.

It follows that for any ideal I , V/I inherits a natural *quotient vertex superalgebra* structure $(V/I, |0\rangle + I, T_{V/I} : V/I \rightarrow V/I, Y_{V/I} : V/I \rightarrow \mathcal{F}(V/I))$. Let $S \subseteq V$ be a subset. The *ideal generated by S* is the smallest ideal containing S , which is the intersection of all ideals containing S . It is denoted by (S) , and we can prove that

$$(S) = \text{span}\{b_{(n)} T^m a \mid b \in V, n \in \mathbb{Z}, m \in \mathbb{N}, a \in S\}.$$

The theory done in §1.3 is translated into the language of vertex superalgebras, as the following theorem shows.

Theorem 1.4.5. *Let V be a vertex superalgebra. For $a, b, c \in V$ and $m, n \in \mathbb{Z}$:*

- (i) $Y : V \rightarrow \mathcal{F}(V)$ is injective;
- (ii) $Y(a, z)|0\rangle \in V[[z]]$ and $Y(a, z)|0\rangle = e^{Tz} a$, so $T^n a = n! a_{(-n-1)}|0\rangle$;
- (iii) $Y(Ta, z) = \partial_z Y(a, z)$ or, equivalently, $(Ta)_{(n)} = -n a_{(n-1)}$;
- (iv) (n -product identity) $Y(a, z)_{(n)} Y(b, z) = Y(a_{(n)} b, z)$;
- (v) $[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}$;
- (vi) $T(Y(a, z)b) = Y(Ta, z)b + Y(a, z)Tb$;
- (vii) $e^{Tw} Y(a, z) e^{-Tw} = i_{z,w} Y(a, z+w)$;

(viii) (Borcherds identity)

$$i_{z,w}((z-w)^n)Y(a,z)Y(b,w)c - p(a,b)i_{w,z}((z-w)^n)Y(b,w)Y(a,z)c = \sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z,w)}{j!} Y(a_{(n+j)}b, w)c;$$

(ix) (Skewsymmetry) $Y(a,z)b = p(a,b)e^{Tz}Y(b,-z)a;$

(x) $(a_{(m)}b)_{(n)}c = \sum_{j \in \mathbb{N}} (-1)^j \binom{m}{j} (a_{(m-j)}b_{(n+j)}c - (-1)^m p(a,b)b_{(m+n-j)}a_{(j)}c).$

Remark 1.4.6. We note that a right ideal I is automatically T -invariant ($T(I) \subseteq I$) because of (1.4.1). Also, right ideals and T -invariant left ideals are automatically two-sided ideals because of skewsymmetry. However, to prove that a subspace is an ideal, it is usually easier to check that it is T -invariant and a left ideal.

Theorem 1.4.7 (Original Borcherds identity). *Let V be a vertex superalgebra. For $a, b, c \in V$ and $m, n, k \in \mathbb{Z}$,*

$$\sum_{j \in \mathbb{N}} (-1)^j \binom{n}{j} (a_{(m+n-j)}(b_{(k+j)}c) - (-1)^n p(a,b)b_{(n+k-j)}(a_{(m+j)}c)) = \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c,$$

or, equivalently, for $a, b, c \in \mathbb{Z}$ and $F(z,w) = z^m w^n (z-w)^k$, where $m, n, k \in \mathbb{Z}$,

$$\begin{aligned} & \operatorname{res}_{z-w}(i_{w,z-w}F(z,w)Y(Y(a,z-w)b,w)c) = \\ & \operatorname{res}_z(i_{z,w}F(z,w)Y(a,z)Y(b,w)c) - \operatorname{res}_z(i_{w,z}F(z,w)Y(b,w)Y(a,z)c). \end{aligned}$$

We do not have examples of vertex superalgebras other than the ones coming from differential algebras. It turns out that it is not an easy task to construct nontrivial vertex superalgebras. We need a preliminary concept to do that task. A *pre-vertex superalgebra* is a quadruple $(V, |0\rangle, T, \mathcal{F})$, where $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace, $|0\rangle \in V_{\bar{0}}$, $T \in \operatorname{End}(V)_{\bar{0}}$, and $\mathcal{F} = \{a^j(z) = \sum_{n \in \mathbb{Z}} a_{(n)}^j z^{-n-1}\}_{j \in J}$ is a collection of $\operatorname{End}(V)$ -valued fields such that for $j \in J$, all $a_{(n)}^j$ for $n \in \mathbb{Z}$ have the same parity. The above data satisfies the following axioms:

- (i) (Vacuum axiom) $T|0\rangle = 0;$
- (ii) (Translation covariance) $[T, a^j(z)] = \partial_z a^j(z)$ for $j \in J;$
- (iii) (Locality) $a^i(z)$ and $a^j(z)$ are mutually local for $i, j \in J;$
- (iv) (Completeness) $\operatorname{span}\{a_{(n_1)}^{j_1} \dots a_{(n_s)}^{j_s} |0\rangle \mid s \in \mathbb{N}, j_i \in J, n_i \in \mathbb{Z}\} = V.$

Let $(V, |0\rangle, T, \mathcal{F})$ be a pre-vertex superalgebra. We define the following subspaces of $\mathcal{F}(V)$:

$$\mathcal{F}_{\min} = \operatorname{span}\{a^{j_1}(z)_{(n_1)}(a^{j_2}(z)_{(n_2)} \dots (a^{j_s}(z)_{(n_s)} \operatorname{Id}_V) \dots) \mid s \in \mathbb{N}, n_i \in \mathbb{Z}, j_i \in J\},$$

$$\mathcal{F}_{\max} = \{a(z) \in \mathcal{F}(V) \mid [T, a(z)] = \partial_z a(z) \text{ and for } j \in J, (a(z), a^j(z)) \text{ is a local pair}\}.$$

We have inclusions

$$\mathcal{F} \subseteq \mathcal{F}_{\min} \subseteq \mathcal{F}_{\max} \subseteq \mathcal{F}_{\text{tc}}.$$

The first inclusion is because for $a(z) \in \mathcal{F}$, $a(z)_{(-1)} \operatorname{Id}_V = a(z) \in \mathcal{F}_{\min}$. The second inclusion is by Lemma 1.3.11 and Dong's Lemma. The last inclusion is by definition.

Now we come to a very fundamental theorem, which allows us to construct noncommutative vertex superalgebras and is the backbone of several of our most important examples of vertex superalgebras.

Theorem 1.4.8 (Extension theorem). *Let $(V, |0\rangle, T, \mathcal{F})$ be a pre-vertex superalgebra, and let $\mathcal{F}_{\min}, \mathcal{F}_{\max}$ be defined as above. Then:*

- (i) $\mathcal{F}_{\min} = \mathcal{F}_{\max};$
- (ii) *The linear map*

$$\begin{aligned} \operatorname{fs} : \mathcal{F}_{\max} &\rightarrow V, \\ \operatorname{fs}(a(z)) &= a(z)|0\rangle|_{z=0} \end{aligned}$$

is well-defined and bijective, and we denote by $Y : V \rightarrow \mathcal{F}(V)$ the inverse map;

(iii) $(V, |0\rangle, T, Y)$ is a vertex superalgebra, with $Y : V \rightarrow \mathcal{F}(V)$ given explicitly by

$$Y(a_{(n_1)}^{j_1} a_{(n_2)}^{j_2} \dots a_{(n_s)}^{j_s} |0\rangle) = a^{j_1}(z)_{(n_1)} (a^{j_2}(z)_{(n_2)} \dots (a^{j_s}(z)_{(n_s)} \text{Id}_V) \dots) \quad (1.4.2)$$

for $s \in \mathbb{N}$, $j_1, \dots, j_s \in J$ and $n_1, \dots, n_s \in \mathbb{Z}$;

(iv) The vertex superalgebra V is generated by $\{a_{(-1)}^j |0\rangle \mid j \in J\}$;

(v) The only vertex superalgebra structure on V with $Y(a_{(-1)}^j |0\rangle, z) = a^j(z)$ for $j \in J$ is the one given by (1.4.2).

Proof. By Theorem 1.3.10, the map fs is well-defined as was noted already in §1.3. By Theorem 1.3.13(i) and Theorem 1.3.13(iii), $\text{fs}|_{\mathcal{F}_{\min}} : \mathcal{F}_{\min} \rightarrow V$ is given by

$$\text{fs}|_{\mathcal{F}_{\min}}(a^{j_1}(z)_{(n_1)}(a^{j_2}(z)_{(n_2)} \dots (a^{j_s}(z)_{(n_s)} \text{Id}_V) \dots)) = a_{(n_1)}^{j_1} a_{(n_2)}^{j_2} \dots a_{(n_s)}^{j_s} |0\rangle.$$

The completeness axiom of pre-vertex superalgebras implies that $\text{fs}|_{\mathcal{F}_{\min}}$ is surjective.

The map $\text{fs} : \mathcal{F}_{\max} \rightarrow V$ is injective using Lemma 1.3.14 with $\mathcal{F}' = \mathcal{F}_{\min}$. Recall the inclusion $\mathcal{F}_{\min} \subseteq \mathcal{F}_{\max}$. We know that $\text{fs}|_{\mathcal{F}_{\min}}$ is surjective and fs is injective, so we can conclude that is in fact bijective and $\mathcal{F}_{\min} = \mathcal{F}_{\max}$. This proves (i) and (ii).

For (iii), we need to show that $Y(a, z)$ is translation covariant for $a \in V$, and that each pair $(Y(a, z), Y(b, w))$ is local for $a, b \in V$. Translation covariance comes from Lemma 1.3.11, and locality comes from Dong's lemma.

We note that we have $Y(a_{(-1)}^j |0\rangle, z) = a^j(z)$ for $j \in J$. Therefore, $(a_{(-1)}^j |0\rangle)_{(n)} = a_{(n)}^j$ for $j \in J$ and $n \in \mathbb{Z}$. By the completeness axiom, we get (iv).

Uniqueness of the vertex superalgebra structure follows from the completeness axiom of pre-vertex superalgebras, the n -product identity and the fact that $|0\rangle \mapsto \text{Id}_V$ in any vertex superalgebra homomorphism. This finishes (v) and the proof of the theorem. \square

Corollary 1.4.9. *Let V be a vertex superalgebra, let $s \in \mathbb{N}$, $a^1, \dots, a^s \in V$, and let $n_1, \dots, n_s \in \mathbb{Z}$. Then*

$$Y(a_{(n_1)}^1 a_{(n_2)}^2 \dots a_{(n_s)}^s |0\rangle, z) = Y(a^j, z)_{(n_1)} (Y(a^2, z)_{(n_2)} \dots (Y(a^s, z)_{(n_s)} \text{Id}_V) \dots).$$

In particular, for $s, n_1, \dots, n_s \in \mathbb{N}$ and $a^1, \dots, a^s \in V$,

$$Y(a_{(-n_1-1)}^1 \dots a_{(-n_s-1)}^s |0\rangle, z) = \frac{:\partial_z^{n_1} Y(a^1, z) \dots \partial_z^{n_s} Y(a^s, z):}{n_1! \dots n_s!}.$$

If V is given by a pre-vertex superalgebra $(V, |0\rangle, T, \mathcal{F})$ as in the Extension Theorem, where $\mathcal{F} = \{a^j(z)\}_{j \in J}$, then for $s, n_1, \dots, n_s \in \mathbb{N}$ and $j_1, \dots, j_s \in J$,

$$Y(a_{(-n_1-1)}^{j_1} \dots a_{(-n_s-1)}^{j_s} |0\rangle, z) = \frac{:\partial_z^{n_1} a^{j_1}(z) \dots \partial_z^{n_s} a^{j_s}(z):}{n_1! \dots n_s!}.$$

Let $(\mathfrak{g}, \mathfrak{F}, T_0)$ be a regular formal distribution Lie superalgebra with $\mathfrak{F} = \{a^j(z)\}_{j \in J}$, and let \mathfrak{g}_- be the annihilation subalgebra. Since $T_0(\mathfrak{g}_-) \subseteq \mathfrak{g}_-$, $DU(T_0) : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ is a $(U(\mathfrak{g}), U(\mathfrak{g}_-))$ -bimodule homomorphism. We consider the trivial representation $0 : \mathfrak{g}_- \rightarrow \mathfrak{gl}(\mathbb{C})$, and we define:

$$V = \text{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}}(\mathbb{C}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_-)} \mathbb{C},$$

$$\pi = \text{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}}(0) : \mathfrak{g} \rightarrow \mathfrak{gl}(V),$$

$$|0\rangle = 1 \otimes 1 \in V,$$

$$T = DU(T_0) \otimes \text{Id}_{\mathbb{C}} \in \text{End}(V)_{\bar{0}},$$

$$\mathcal{F} = \left\{ \pi(a^j(z)) = \sum_{n \in \mathbb{Z}} \pi(a_{(n)}^j) z^{-n-1} \mid j \in J \right\}.$$

Theorem 1.4.10. *With the notation above, \mathcal{F} consists of fields, and $(V, |0\rangle, T, \mathcal{F})$ is a pre-vertex superalgebra.*

Proof. First, we prove that $\pi(a^j(z))$ is a field for $j \in J$. We do this by induction on s in $v = a_{(n_1)}^{j_1} \dots a_{(n_s)}^{j_s} |0\rangle$, where $s \in \mathbb{N}$, $n_1, \dots, n_s \in \mathbb{Z}_-$ and $j_1, \dots, j_s \in J$. By the PBW theorem, these elements form a spanning set of V . For $s = 0$, we have $v = |0\rangle$ and

$$\pi(a^j(z)) |0\rangle = \sum_{n \in \mathbb{Z}} a_{(n)}^j |0\rangle z^{-n-1} = \sum_{n \in \mathbb{Z}_-} a_{(n)}^j |0\rangle z^{-n-1} \in V((z)).$$

The last equality is true because $a_{(n)} |0\rangle = 0$ for $n \in \mathbb{N}$. We now proceed by proving the induction step:

$$\begin{aligned} \pi(a^j(z))v &= \sum_{n \in \mathbb{Z}} a_{(n)}^j a_{(n_1)}^{j_1} \dots a_{(n_s)}^{j_s} |0\rangle z^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} [a_{(n)}^j, a_{(n_1)}^{j_1}] a_{(n_2)}^{j_2} \dots a_{(n_s)}^{j_s} |0\rangle z^{-n-1} \pm \sum_{n \in \mathbb{Z}} a_{(n_1)}^{j_1} a_{(n)}^j a_{(n_2)}^{j_2} \dots a_{(n_s)}^{j_s} |0\rangle z^{-n-1}. \end{aligned} \quad (1.4.3)$$

By the induction hypothesis, the second sum in (1.4.3) is in $V((z))$, so we only need to show that the first sum is also in $V((z))$. By (1.2.4),

$$[a_{(n)}^j, a_{(n_1)}^{j_1}] = \sum_{k \in \mathbb{N}} \binom{n}{k} (a^j(z)_{(k)} a^{j_1}(z))_{(n+n_1-k)}. \quad (1.4.4)$$

The regularity property implies that $a^j(z)_{(k)} a^{j_1}(z) \in \mathbb{C}[\partial_z] \mathfrak{F}$, thus we can assume that

$$a^j(z)_{(k)} a^{j_1}(z) = \sum_{l \in J} f_l^k(\partial_z) a^l(z),$$

for some polynomials f_l^k . Since $(a^j(z), a^{j_1}(z))$ is a local pair, there exists $N \in \mathbb{N}$ such that $a^j(z)_{(k)} a^{j_1}(z) = 0$ for $k \geq N$. This allows us to rewrite (1.4.4) as

$$[a_{(n)}^j, a_{(n_1)}^{j_1}] = \sum_{0 \leq k \leq N} \binom{n}{k} \left(\sum_{l \in J} f_l^k(\partial_z) a^l(z) \right)_{(n+n_1-k)}.$$

Therefore, we can rewrite the first sum in (1.4.3) as

$$\sum_{0 \leq k \leq N} \sum_{n \in \mathbb{Z}} \binom{n}{k} \left(\sum_{l \in J} f_l^k(\partial_z) a^l(z) \right)_{(n+n_1-k)} a_{(n_2)}^{j_2} \dots a_{(n_s)}^{j_s} |0\rangle z^{-n-1}.$$

By the induction hypothesis, for each k ,

$$\sum_{n \in \mathbb{Z}} \binom{n}{k} \left(\sum_{l \in J} f_l^k(\partial_z) a^l(z) \right)_{(n+n_1-k)} a_{(n_2)}^{j_2} \dots a_{(n_s)}^{j_s} |0\rangle z^{-n-1} \in V((z)).$$

Finally, the first sum in (1.4.3) is also in $V((z))$.

We now verify the four axioms of a pre-vertex superalgebra:

- (i) $T|0\rangle = DU(T_0) \otimes \text{Id}_{\mathbb{C}}(1 \otimes 1) = DU(T_0)(1) \otimes 1 = 0 \otimes 1 = 0$.
- (ii) We recall that T is an even endomorphism. For $j \in J$,

$$[T, \pi(a^j(z))] = \left[T, \sum_{n \in \mathbb{Z}} \pi(a_{(n)}^j) z^{-n-1} \right] = \sum_{n \in \mathbb{Z}} [T, \pi(a_{(n)}^j)] z^{-n-1} = \sum_{n \in \mathbb{Z}} T \pi(a_{(n)}^j) - \pi(a_{(n)}^j) T.$$

For $x \in U(\mathfrak{g})$,

$$\begin{aligned}
T\pi(a_{(n)}^j)(x \otimes 1) &= T(a_{(n)}^j x \otimes 1) \\
&= DU(T_0)(a_{(n)}^j x) \otimes 1 \\
&= (T_0(a_{(n)}^j)x + a_{(n)}^j T_0(x)) \otimes 1 \\
&= (-na_{(n-1)}^j x + a_{(n)}^j T_0(x)) \otimes 1 \\
&= -na_{(n-1)}^j(x \otimes 1) + \pi(a_{(n)}^j)T(x \otimes 1).
\end{aligned}$$

The last two equalities imply that for $j \in J$, $[T, \pi(a^j(z))] = \partial_z \pi(a^j(z))$.

(iii) We note that for $i, j \in J$ and $N \in \mathbb{N}$,

$$(z - w)^N [\pi(a^i(z)), \pi(a^j(z))] = (z - w)^N \pi([a^i(z), a^j(z)]) = \pi((z - w)^N [a^i(z), a^j(z)]).$$

(iv) This was already done. \square

By Theorem 1.4.10 and the Extension theorem, V is a vertex superalgebra, denoted by $V(\mathfrak{g}, \mathfrak{F}, T_0)$, and by the PBW theorem, it is explicitly given by

$$V = \text{span}\{a_{(-n_1-1)}^{j_1} \cdots a_{(-n_s-1)}^{j_s} | 0\rangle \mid s, n_1, \dots, n_s \in \mathbb{N}, j_1, \dots, j_s \in J\}, \quad (1.4.5)$$

which means V is strongly generated by $\{a_{(-1)}^j | 0\rangle\}_{j \in J}$.

Remark 1.4.11. The PBW theorem for Lie superalgebras is actually more precise than (1.4.5). Let \leq be a total order on J . Then

$$\begin{aligned}
V = \text{span}\{a_{(-n_1-1)}^{j_1} \cdots a_{(-n_s-1)}^{j_s} | 0\rangle \mid s, n_1, \dots, n_s \in \mathbb{N}, j_1, \dots, j_s \in J \text{ and for } k = 1, \dots, s-1, \\
\text{if } a_{(n_k)}^{j_k}, a_{(n_{k+1})}^{j_{k+1}} \in V_{\bar{1}}, \text{ then } j_k < j_{k+1}\}.
\end{aligned}$$

Remark 1.4.12. We have constructed a functor

$$\begin{aligned}
VA : \{\text{regular formal distribution Lie superalgebras}\} &\rightarrow \{\text{vertex superalgebras}\}, \\
VA(\mathfrak{g}, \mathfrak{F}, T_0) &= V(\mathfrak{g}, \mathfrak{F}, T_0).
\end{aligned}$$

By Remark 1.2.12, we could have constructed a functor (see [Li04, Theorem 2.15])

$$VA : \{\text{Lie conformal superalgebras}\} \rightarrow \{\text{vertex superalgebras}\}.$$

Usually, we need to quotient the vertex superalgebras obtained this way. Let $(\mathfrak{g}, \mathfrak{F}, T_0)$ be a regular formal distribution Lie superalgebra, and let $\lambda : \mathfrak{h} \rightarrow \mathbb{C}$ be a linear functional, where \mathfrak{h} is a subalgebra of \mathfrak{g}_+ with $\mathfrak{h} \subseteq \ker(T_0)$. We denote by I^λ the \mathfrak{g} -submodule of $V(\mathfrak{g}, \mathfrak{F}, T_0)$ generated by the vectors $(a - \lambda(a)) | 0\rangle$ for $a \in \mathfrak{h}$. For $a \in \mathfrak{h}$ and $x \in U(\mathfrak{g})$,

$$\begin{aligned}
T(x(a - \lambda(a)) | 0\rangle) &= DU(T_0)(x(a - \lambda(a))) \otimes 1 \\
&= (DU(T_0)(x)(a - \lambda(a)) + xDU(T_0)(a - \lambda(a))) \otimes 1 \\
&= DU(T_0)(x)(a - \lambda(a)) \otimes 1 \\
&= DU(T_0)(x)(a - \lambda(a)) | 0\rangle.
\end{aligned}$$

Thus, I^λ is T -invariant. By Theorem 1.4.10, $\{a_{(-1)}^j | 0\rangle\}_{j \in J}$ strongly generates V , and from this, we see that I^λ is a left ideal. By Remark 1.4.6, I^λ is an ideal of $V(\mathfrak{g}, \mathfrak{F}, T_0)$. Taking the quotient, we get a vertex superalgebra, denoted by

$$V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0) = V(\mathfrak{g}, \mathfrak{F}, T_0)/I^\lambda.$$

Let \mathbb{C}_λ be the representation of $\mathfrak{g}_- \oplus \mathfrak{h}$ on which \mathfrak{g}_- acts as 0 and a acts as $\lambda(a)$ for $a \in \mathfrak{h}$. Using the universal property of the induced representation, we find a \mathfrak{g} -module homomorphism

$\text{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}}(\mathbb{C}) \rightarrow \text{Ind}_{\mathfrak{g}_- \oplus \mathfrak{h}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$. By the universal property of the quotient, we find a \mathfrak{g} -module homomorphism $f : V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0) \rightarrow \text{Ind}_{\mathfrak{g}_- \oplus \mathfrak{h}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ such that the following diagram commutes

$$\begin{array}{ccc} \text{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}}(\mathbb{C}) & \twoheadrightarrow & V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0) \\ & \searrow & \downarrow f \\ & & \text{Ind}_{\mathfrak{g}_- \oplus \mathfrak{h}}^{\mathfrak{g}}(\mathbb{C}_\lambda) \end{array}$$

Using again the universal property of the induced representation, we find a \mathfrak{g} -module homomorphism $g : \text{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}}(\mathbb{C}) \rightarrow V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0)$. We can verify that f and g are inverses of each other. Therefore, it is better to think of $V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0)$ as being $\text{Ind}_{\mathfrak{g}_- \oplus \mathfrak{h}}^{\mathfrak{g}}(\mathbb{C}_\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_- \oplus \mathfrak{h})} \mathbb{C}$. We now use this construction to obtain several important examples of vertex superalgebras.

Remark 1.4.13. The vertex superalgebra $V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0)$ comes equipped with a vertex superalgebra epimorphism $\pi : V(\mathfrak{g}, \mathfrak{F}, T_0) \twoheadrightarrow V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0)$ that satisfies $\pi(a|0\rangle) = \lambda(a)|0\rangle$ for $a \in \mathfrak{h}$ and is universal with this property, i.e., if $f : V(\mathfrak{g}, \mathfrak{F}, T_0) \rightarrow W$ is a vertex superalgebra homomorphism such that $f(a|0\rangle) = \lambda(a)|0\rangle$ for $a \in \mathfrak{h}$, then there exist a unique vertex superalgebra homomorphism $\bar{f} : V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0) \rightarrow W$ such that the following diagram commutes

$$\begin{array}{ccc} V(\mathfrak{g}, \mathfrak{F}, T_0) & \xrightarrow{\pi} & V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0) \\ & \searrow f & \downarrow \bar{f} \\ & & W \end{array}$$

Example 1.4.14 (Universal Virasoro vertex algebra of central charge c). We pick $c \in \mathbb{C}$. We take $(\text{Vir}, \{L(z), C\}, \text{ad}(L_{-1}))$ as the regular formal distribution Lie superalgebra as constructed in Example 1.2.14 and $\lambda : \mathbb{C}C \rightarrow \mathbb{C}$, $\lambda(C) = c$ as the linear functional. The resulting vertex algebra is the universal Virasoro vertex algebra of central charge c , denoted by Vir^c .

A *partition* (of $n \in \mathbb{N}$) is a sequence $\lambda = [\lambda_1, \dots, \lambda_m]$ such that $\lambda_i \in \mathbb{Z}_+$ for $i = 1, \dots, m$, $\lambda_1 \geq \dots \geq \lambda_m$ (and $\lambda_1 + \dots + \lambda_m = n$). We also consider the *empty partition* \emptyset , which is the unique partition of 0. By Remark 1.4.11, for $c \in \mathbb{C}$, a basis of Vir^c is given by

$$\{L_{-\lambda_1} \dots L_{-\lambda_m} |0\rangle \mid [\lambda_1, \dots, \lambda_m] \text{ is a partition with } \lambda_m \geq 2\}.$$

When we deal with conformal vertex superalgebras, we will explain why this vertex algebra is called universal.

Example 1.4.15 (Universal affine vertex superalgebra of level k). We pick $k \in \mathbb{C}$. We take $(\widehat{\mathfrak{g}}, \{a(z) \mid a \in \mathfrak{g}\} \cup \{K\}, -\partial_t)$ as the regular formal distribution Lie superalgebra as constructed in Example 1.2.16 and $\lambda : \mathbb{C}K \rightarrow \mathbb{C}$, $\lambda(K) = k$ as the linear functional. The resulting vertex superalgebra is the universal affine vertex superalgebra of level k , denoted by $V^k(\mathfrak{g})$.

Example 1.4.16 (Fermionic vertex superalgebra). We take $(\widehat{V}, \{a(z) \mid a \in V\} \cup \{K\}, -\partial_t)$ as the regular formal distribution Lie superalgebra as constructed in Example 1.2.17 and $\lambda : \mathbb{C}K \rightarrow \mathbb{C}$, $\lambda(K) = 1$ as the linear functional. The resulting vertex superalgebra is the fermionic vertex superalgebra, denoted by $F(V)$.

Let V be a vertex superalgebra. We can naturally endow V with the structure of a Lie conformal superalgebra by taking $\partial = T$, and by defining the following λ -bracket

$$\begin{aligned} [\bullet_\lambda \bullet] : V \times V &\rightarrow V[\lambda], \\ [a_\lambda b] &= F^\lambda(a(z)b) = \sum_{j \in \mathbb{N}} (a_{(j)}b) \frac{\lambda^j}{j!}. \end{aligned}$$

To verify the Lie conformal superalgebra axioms, we do the following. We set $\mathcal{G} = \{Y(a, z) \mid a \in V\}$. Then $\mathcal{G} \subseteq \mathcal{F}(V)$ is a Lie conformal superalgebra, and the following diagram commutes

$$\begin{array}{ccc} V \times V & \xrightarrow{Y \times Y} & \mathcal{G} \times \mathcal{G} \\ \downarrow [\bullet \lambda \bullet]_V & & \downarrow [\bullet \lambda \bullet]_{\mathcal{G}} \\ V[\lambda] & \xrightarrow{Y[\lambda]} & \mathcal{G}[\lambda] \end{array}$$

As $Y : V \rightarrow \mathcal{F}(V)$ is injective, we obtain that V itself is a Lie conformal superalgebra. We have constructed a functor

$$\begin{aligned} LCA : \{\text{Vertex superalgebras}\} &\rightarrow \{\text{Lie conformal superalgebras}\}, \\ LCA(V) &= (V, T, [\bullet \lambda \bullet]). \end{aligned}$$

Remark 1.4.17. Recall the functor VA constructed in Remark 1.4.12. We could prove that (VA, LCA) is an adjoint pair of functors, i.e., that for a vertex superalgebra V and a Lie conformal superalgebra \mathcal{R} , we have a natural isomorphism

$$\text{Hom}(VA(\mathcal{R}), V) \cong \text{Hom}(\mathcal{R}, LCA(V)).$$

1.5. Graded and conformal vertex superalgebras. Let V be a vertex superalgebra. A *Hamiltonian operator* of V is a diagonalizable operator $H \in \text{End}(V)$ such that

$$[H, Y(a, z)] = z\partial_z Y(a, z) + Y(Ha, z) \quad \text{for } a \in V. \quad (1.5.1)$$

A vertex superalgebra with a Hamiltonian is called *graded*. The *grading* of V is the eigenspace decomposition of H

$$V = \bigoplus_{\Delta \in \mathbb{C}} V_{\Delta},$$

where

$$V_{\Delta} = \ker(H - \Delta \text{Id}_V) \quad \text{for } \Delta \in \mathbb{C}.$$

If a is an eigenvector of H , it is called *homogeneous*, its eigenvalue is called the *conformal weight* of a , and it is denoted by Δ_a . Condition (1.5.1) is equivalent to

$$[H, a_{(n)}] = -(n+1)a_{(n)} + (Ha)_{(n)} \quad \text{for } a \in V \text{ and } n \in \mathbb{Z} \quad (1.5.2)$$

and to

$$[H, a_{(n)}] = (\Delta_a - n - 1)a_{(n)} \quad \text{for } a \in V \text{ homogeneous and } n \in \mathbb{Z}. \quad (1.5.3)$$

For $a \in V$ homogeneous with conformal weight Δ_a , we write, as was done with eigendistributions, $Y(a, z) = \sum_{n \in \mathbb{Z} - \Delta_a} a_n z^{-n - \Delta_a}$, which is equivalent to

$$a_{(n)} = a_{n - \Delta_a + 1} \quad \text{for } n \in \mathbb{Z}. \quad (1.5.4)$$

With this notation, (1.5.3) is equivalent to

$$[H, a_n] = -na_n \quad \text{for } a \in V \text{ homogeneous and } n \in \mathbb{Z} - \Delta_a. \quad (1.5.5)$$

Remark 1.5.1. When a formula involves Δ_a , it is assumed that a is an eigenvector of H with eigenvalue Δ_a , and the formula is extended to arbitrary a by linearity. We have two meanings of the word homogeneous, one related to the Hamiltonian, and the other related to the parity. The context will clarify the meaning.

Theorem 1.5.2 ([Kac98, §4.9]). *Let V be a graded vertex superalgebra with Hamiltonian H and grading $V = \bigoplus_{\Delta \in \mathbb{C}} V_{\Delta}$. Then:*

- (i) $H|0\rangle = 0$, which means that $\Delta_{|0\rangle} = 0$;
- (ii) $[H, T] = T$ and $HT^n = nT^n + T^n H$ for $n \in \mathbb{N}$;
- (iii) $[T, a_n] = (-n - \Delta_a + 1)a_{n-1}$ for $a \in V$ homogeneous and $n \in \mathbb{Z} - \Delta_a$;
- (iv) $a_n(V_{\Delta}) \subseteq V_{\Delta - n}$ for $a \in V$ homogeneous, $\Delta \in \mathbb{C}$ and $n \in \mathbb{Z} - \Delta_a$;
- (v) $T(V_{\Delta}) \subseteq V_{\Delta + 1}$ for $\Delta \in \mathbb{C}$ or, equivalently, $\Delta_{Ta} = \Delta_a + 1$ for $a \in V$ homogeneous;
- (vi) $\Delta_{a_{(n)}b} = \Delta_a + \Delta_b - n - 1$ for $a, b \in V$ homogeneous and $n \in \mathbb{Z}$.

A \mathbb{Z} or \mathbb{N} -graded vertex superalgebra V is a graded vertex superalgebra V such that $V_\Delta = 0$ for $\Delta \notin \mathbb{Z}$ or \mathbb{N} , respectively.

Homomorphisms of graded vertex superalgebras are assumed to respect the gradings, i.e., if $f : V_1 \rightarrow V_2$ is a homomorphism of graded vertex superalgebras, then $f \circ H^{V_1} = H^{V_2} \circ f$, where H^{V_1} is the Hamiltonian of V_1 , and H^{V_2} is the Hamiltonian of V_2 .

Let V be a vertex superalgebra. A *conformal vector of central charge $c \in \mathbb{C}$ of V* is a vector $\omega \in V$ such that $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies:

- (i) $Y(\omega, z)$ is a Virasoro formal distribution of central charge $C = c \text{Id}_V$;
- (ii) $L_{-1} = T$;
- (iii) L_0 is diagonalizable.

A *conformal vertex superalgebra (of central charge c)* is a vertex superalgebra V together with a conformal vector ω (of central charge c). We denote the conformal superalgebra by (V, ω) if we want to emphasize the conformal vector.

Remark 1.5.3. Because of property (ii), a conformal vector ω is necessarily even. We note that

$$\omega_{(n)} = L_{n-1} \quad \text{for } n \in \mathbb{Z}.$$

If (V, ω) is a conformal vertex superalgebra, and I is an ideal of V , then $(V/I, \omega + I)$ is a conformal vertex superalgebra of the same central charge.

Theorem 1.5.4. *If ω is a conformal vector of a vertex superalgebra V , then L_0 is a Hamiltonian of V , and for $a \in V$, $Y(a, z)$ is an eigendistribution of conformal weight Δ_a with respect to $Y(\omega, z)$ if and only if a is homogeneous of conformal weight Δ_a . Moreover, ω has conformal weight 2.*

Proof. By Theorem 1.4.5(iii) and Theorem 1.4.5(v), for $a \in V$ and $n \in \mathbb{Z}$,

$$\begin{aligned} [L_0, a_{(n)}] &= [\omega_{(1)}, a_{(n)}] \\ &= \sum_{j \in \mathbb{N}} \binom{1}{j} (\omega_{(j)} a)_{(1+n-j)} \\ &= (\omega_{(0)} a)_{(n+1)} + (\omega_{(1)} a)_{(n)} \\ &= (T a)_{(n+1)} + (L_0 a)_{(n)} \\ &= -(n+1)a_{(n)} + (L_0 a)_{(n)}. \end{aligned}$$

By (1.5.2), this is equivalent to L_0 being a Hamiltonian of V .

The n -product identity for vertex superalgebras and Theorem 1.4.5(iii) show that for $a \in V$,

$$\begin{aligned} [Y(\omega, z)_\lambda Y(a, z)] &= \sum_{j \in \mathbb{N}} \frac{Y(\omega, z)_{(j)} Y(a, z)}{j!} \lambda^j \\ &= \sum_{j \in \mathbb{N}} \frac{Y(\omega_{(j)} a, z)}{j!} \lambda^j \\ &= Y(T a, z) + Y(L_0 a, z) \lambda + O(\lambda^2) \\ &= \partial_z Y(a, z) + Y(L_0 a, z) \lambda + O(\lambda^2). \end{aligned}$$

Because $Y : V \rightarrow \mathcal{F}(V)$ is injective, this implies that

$$[Y(\omega, z)_\lambda Y(a, z)] = \partial_z Y(a, z) + \Delta_a Y(a, z) \lambda + O(\lambda^2) \text{ if and only if } L_0 a = \Delta_a a.$$

As $Y(\omega, z)$ has conformal weight 2 with respect to itself, ω has conformal weight 2. \square

A *conical vertex superalgebra* is an \mathbb{N} -graded vertex superalgebra with $V_0 = \mathbb{C}|0\rangle$. A *vertex operator superalgebra* is a \mathbb{Z} -graded conformal vertex superalgebra such that:

- (i) For $n \in \mathbb{Z}$, $\dim(V_n) < \infty$;
- (ii) There is $N \in \mathbb{Z}$ satisfying $V_n = 0$ for $n \leq N$.

Remark 1.5.5. A vertex superalgebra can have many different conformal vectors (see [FBZ01, Example 2.5.9]).

Example 1.5.6. It is straightforward to verify that for $c \in \mathbb{C}$, Vir^c is a conformal vertex algebra of central charge c with conformal vector $\omega = L_{-2}|0\rangle$.

Example 1.5.7 (Sugawara construction [Kac98, Theorem 5.7]). We recall the universal affine vertex superalgebra of level k of Example 1.4.15, and we assume that \mathfrak{g} is simple and finite dimensional. Let $(a_i)_{i \in I}, (a^i)_{i \in I}$ be dual bases of $(\bullet|\bullet)$, which means $(a_i|a^j) = \delta_{ij}$ for $i, j \in I$. We denote by $c_{\mathfrak{g}} = \sum_{i \in I} a^i a_i \in U(\mathfrak{g})$ the universal Casimir element of \mathfrak{g} . Let $2h^\wedge$ be the eigenvalue of $\text{ad}(c_{\mathfrak{g}}) \in \text{End}(\mathfrak{g})$ in the adjoint representation, i.e., $\text{ad}(c_{\mathfrak{g}}) = \sum_{i \in I} \text{ad}(a^i) \text{ad}(a_i) = 2h^\wedge \text{Id}_V$. We assume $k \neq -h^\wedge$ and set

$$\omega = \frac{1}{2(k+h^\wedge)} \sum_{i \in I} a_{(-1)}^i a_{i(-1)} |0\rangle \in V^k(\mathfrak{g}).$$

Then ω is a conformal vector of central charge $\frac{k \text{sdim}(\mathfrak{g})}{k+h^\wedge}$, and for $a \in \mathfrak{g}$, $a(z)$ is a primary eigendistribution of conformal weight 1.

Example 1.5.8 ([Kac98, Proposition 4.10]). We recall the fermionic vertex superalgebra of Example 1.4.16. Let $(a_i)_{i \in I}$ and $(a^i)_{i \in I}$ be a pair of dual bases of V , i.e., $\langle a_i, a^j \rangle = \delta_{ij}$ for $i, j \in I$. We set

$$\omega = \frac{1}{2} \sum_{i \in I} a_{(-2)}^i a_{i(-1)} |0\rangle \in F(V).$$

Then ω is a conformal vector of central charge $-\frac{1}{2} \text{sdim}(V)$, and all fields $a(z)$, where $a \in V$, are primary of conformal weight $1/2$.

A conformal vertex superalgebra homomorphism $(V_1, \omega_1) \rightarrow (V_2, \omega_2)$ is a homomorphism of vertex superalgebras $V_1 \rightarrow V_2$ such that $\omega_1 \mapsto \omega_2$.

Remark 1.5.9. Because homomorphisms of conformal vertex superalgebras respect the conformal vectors, they automatically respect the gradings as well.

Theorem 1.5.10 (Universal property of Vir^c). *Let V be a conformal vertex superalgebra with conformal vector ω of central charge c . There exists a unique homomorphism of conformal vertex superalgebras $(\text{Vir}^c, L_{-2}|0\rangle) \rightarrow (V, \omega)$.*

Proof. Let Vir also denote the Virasoro Lie conformal algebra. Because V is conformal, the map

$$\begin{aligned} g : \text{Vir} &\rightarrow \text{LCA}(V), \\ g(L) &= \omega, \\ g(C) &= c|0\rangle \end{aligned}$$

gives a homomorphism of Lie conformal superalgebras. By Remark 1.4.17, we obtain a homomorphism

$$\begin{aligned} f : VA(\text{Vir}) &\rightarrow V, \\ f(L_{-2}|0\rangle) &= \omega, \\ f(C|0\rangle) &= c|0\rangle. \end{aligned}$$

Using Remark 1.4.13 with Vir^c , we obtain our desired homomorphism of vertex superalgebras $\bar{f} : \text{Vir}^c \rightarrow V$ such that $\bar{f}(L_{-2}|0\rangle) = \omega$. As $\{L_{-2}|0\rangle\}$ strongly generates Vir^c , and a homomorphism of conformal vertex superalgebras is required to send $L_{-2}|0\rangle$ to ω , \bar{f} is the only homomorphism of conformal vertex superalgebras $(\text{Vir}^c, L_{-2}|0\rangle) \rightarrow (V, \omega)$.

Alternatively, we could use Theorem 1.6.5 ahead to obtain a state-field correspondence $f : \text{Vir}^c \rightarrow \mathcal{F}(V)$. Actually, the image of f is contained in the image of the state-field correspondence $Y^V : V \rightarrow \mathcal{F}(V)$. Because Y^V is injective, we can simply define $\bar{f} = (Y^V)^{-1} \circ f$. \square

We recall some basic facts about diagonalizable operators.

Lemma 1.5.11 ([HK71, §6.2]). *Let V be a vector space, $H \in \text{End}(V)$, and for $\Delta \in \mathbb{C}$, we set $V_\Delta = \ker(H - \Delta \text{Id}_V)$. Then the family of subspaces $(V_\Delta)_{\Delta \in \mathbb{C}}$ is linearly independent.*

Lemma 1.5.12 ([KRR13, Corollary 1.1]). *Let V be a vector space (not necessarily a vertex superalgebra), and let $H \in \text{End}(V)$ be a diagonalizable operator with eigenspace decomposition $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$. Let U be an H -invariant subspace of V , which means $H(U) \subseteq U$. Then U is graded, i.e.,*

$$U = \sum_{\Delta \in \mathbb{C}} U \cap V_\Delta.$$

We can generalize and combine the previous two lemmas as follows. Let k be a field, let $\mathcal{P} \subseteq k[x]$ be a set of representatives of the irreducible polynomials in $k[x]$ (for example, if k is algebraically closed, then we may take $\mathcal{P} = \{x - \Delta \mid \Delta \in k\}$), let V be a vector space over k , and let $H \in \text{End}(V)$. For $p(x) \in \mathcal{P}$, we define the *generalized eigenspace of H with respect to $p(x)$* as

$$V_{p(x)}^{\text{Gen}} = \{a \in V \mid \text{there is } k \in \mathbb{N} \text{ such that } p(H)^k a = 0\}.$$

We also define the subspace

$$V^{\text{Gen}} = \sum_{p(x) \in \mathcal{P}} V_{p(x)}^{\text{Gen}}.$$

When k is algebraically closed, we write V_Δ^{Gen} instead of $V_{x-\Delta}^{\text{Gen}}$ for $\Delta \in k$, and we call that subspace the *generalized eigenspace of H with eigenvalue Δ* .

Lemma 1.5.13. *With notation as above, we have:*

- (i) *The family of subspaces $(V_{p(x)}^{\text{Gen}})_{p(x) \in \mathcal{P}}$ is linearly independent;*
- (ii) *$H(V_{p(x)}^{\text{Gen}}) \subseteq V_{p(x)}^{\text{Gen}}$ for $p(x) \in \mathcal{P}$ and $H(V^{\text{Gen}}) \subseteq V^{\text{Gen}}$;*
- (iii) *If U is an H -invariant subspace of V^{Gen} , then U is graded, i.e.,*

$$U = \sum_{p(x) \in \mathcal{P}} U \cap V_{p(x)}^{\text{Gen}}.$$

An ideal I of a graded vertex superalgebra V with Hamiltonian H is called *graded* if it is H -invariant or, equivalently, $I = \sum_{\Delta \in \mathbb{C}} I \cap V_\Delta$ (see Lemma 1.5.12).

Lemma 1.5.14. *Let V be a graded vertex superalgebra with grading $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$ such that $V_0 = \mathbb{C}|0\rangle$. Then V has a unique maximal proper graded ideal I_{\max} and V/I_{\max} is the unique simple graded quotient of V , this means that for a proper graded ideal I of V , V/I is simple if and only if $I = I_{\max}$.*

Proof. All proper ideals of V do not contain $|0\rangle$ because an ideal containing $|0\rangle$ is all of V . Let I_{\max} be the sum of all graded proper ideals. Then I_{\max} is a graded ideal of V , and $I_{\max} \neq V$ because $|0\rangle \notin I_{\max}$. Therefore, I_{\max} is the unique maximal proper graded ideal of V . \square

Lemma 1.5.15. *In a conformal vertex superalgebra V such that $V_0 = \mathbb{C}|0\rangle$, all ideals are graded, and there is a unique maximal proper ideal I_{\max} and a unique simple quotient V/I_{\max} .*

Proof. Let ω be the conformal vector of V , and let I be an ideal of V . As $L_0(I) = \omega_{(1)}(I) \subseteq I$, I is a graded ideal. The result follows from Lemma 1.5.14. \square

The conformal vertex algebra Vir^c satisfies $\text{Vir}_0^c = \mathbb{C}|0\rangle$. We denote by Vir_c the unique simple quotient, and we call it the *simple Virasoro vertex algebra of central charge c* . Let $p, q \geq 2$ be relatively prime integers, and we set

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq}.$$

Let V be a Vir -module. We say V is *smooth* if $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \in \text{End}(V)[[z^{\pm 1}]]$ is a field. We say V is of *central charge $c \in \mathbb{C}$* if the central element C acts as multiplication by c . A vector u in V is called *singular* if it is nonzero and

$$L_n u = 0 \quad \text{for } n \in \mathbb{Z}_+.$$

Theorem 1.5.16 ([GK07]). *The following are equivalent:*

- (i) Vir^c is not simple, i.e., $\text{Vir}^c \neq \text{Vir}_c$;
- (ii) c is of the form $c_{p,q}$ for some $p, q \geq 2$ relatively prime integers.

Moreover, the maximal proper ideal of $\text{Vir}^{c_{p,q}}$ is generated by a singular vector of conformal weight $(p-1)(q-1)$, denoted by $a_{p,q}$. In the expression

$$a_{p,q} = \sum_{\substack{i_1 \geq \dots \geq i_k \geq 2 \\ i_1 + \dots + i_k = (p-1)(q-1)}} c_{i_1 \dots i_k} L_{-i_1} \dots L_{-i_k} |0\rangle,$$

where $c_{i_1 \dots i_k} \in \mathbb{Q}$, the coefficient of $L_{-2}^{(p-1)(q-1)/2}$ is nonzero.

We assume $p, q \geq 2$ are relatively prime integers and denote $\text{Vir}_{c_{p,q}}$ by $\text{Vir}_{p,q}$. For $p = 2$ and $q = 3$, we have $c_{2,3} = 0$ and $\text{Vir}_0 = \mathbb{C}|0\rangle$ (the one dimensional vertex algebra) because in this case, $\omega = L_{-2}|0\rangle$ is the singular vector generating the maximal proper ideal. Thus, the smallest case is not very interesting. The next case is the *Ising model* $\text{Vir}_{3,4} = \text{Vir}_{c_{3,4}} = \text{Vir}_{1/2}$. In this thesis, we study the Ising model and its irreducible admissible modules called *Ising modules*. By Theorem 1.5.16, the maximal proper ideal of $\text{Vir}_{3,4}$ is generated by the singular vector

$$a_{3,4} = (L_{-2}^3 + \frac{93}{64}L_{-3}^2 - \frac{27}{16}L_{-6} - \frac{33}{8}L_{-4}L_{-2})|0\rangle \quad (1.5.6)$$

of conformal weight $(3-1)(4-1) = 6$ (see §2.3 ahead for more details).

1.6. Modules over vertex superalgebras. Let V be a vertex superalgebra. A *module over V* , V -module or *representation of V* is a vector superspace M together with a linear and parity preserving map

$$Y^M(\bullet, z) : V \rightarrow \mathcal{F}(M),$$

$$a \mapsto Y^M(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-n-1}$$

satisfying:

- (i) $Y(|0\rangle, z) = \text{Id}_M$;
- (ii) (Borcherds identity) For $a, b \in V$, $u \in M$ and $m, n, k \in \mathbb{Z}$,

$$\sum_{j \in \mathbb{N}} (-1)^j \binom{n}{j} (a_{(m+n-j)}^M (b_{(k+j)}^M u) - (-1)^n p(a, b) b_{(n+k-j)}^M (a_{(m+j)}^M u)) = \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(n+j)} b)_{(m+k-j)}^M u.$$

Remark 1.6.1. What we call V -modules are sometimes called weak V -modules by some authors (for example, in [DLM98] and [ABD03]). Since this is the minimum requirement for a module, I think it is a better idea to just call them modules and add adjectives as we require more assumptions.

Let V be a vertex superalgebra, and let M_1, M_2 be V -modules. A V -module homomorphism $f : M_1 \rightarrow M_2$ is a linear and parity preserving map such that for $a \in V$ and $u \in M_1$,

$$f(Y^{M_1}(a, z)u) = \sum_{n \in \mathbb{Z}} f(a_{(n)}^{M_1} u) z^{-n-1} = \sum_{n \in \mathbb{Z}} a_{(n)}^{M_2} f(u) z^{-n-1} = Y^{M_2}(a, z)f(u).$$

We obtain the category $V\text{-Mod}$ of V -modules, which is abelian.

The vertex superalgebra V is clearly a V -module, and it is called the *adjoint representation of V* . A *submodule of M* is a subspace N of M such that $Y^M(a, z)u \in N((z))$ for $a \in V$ and $u \in N$, i.e., $a_{(n)}^M u \in N$ for $n \in \mathbb{Z}$. The *quotient module M/N* is defined in the usual way. A module whose only proper submodule is 0 is called *simple* or *irreducible*. Let $T \subseteq M$ be a subset. The *submodule generated by T* is the smallest submodule containing T , which is the intersection of all submodules containing T . It is denoted by (T) , and we can prove that

$$(T) = \text{span}\{a_{(n_1)}^{1M} \dots a_{(n_s)}^{sM} u \mid s \in \mathbb{N}, a^1, \dots, a^s \in V, n_1, \dots, n_s \in \mathbb{Z}, u \in T\}.$$

We say M is strongly generated over V by $T \subseteq M$ if

$$M = \text{span}\{a_{(-n_1-1)}^{1M} \cdots a_{(-n_s-1)}^{sM} u \mid s, n_1, \dots, n_s \in \mathbb{N}, a^1, \dots, a^s \in V, u \in T\}.$$

Most of the theorems about vertex superalgebras and their proofs carry over to modules over vertex superalgebras (cf. Theorem 1.4.5 and Corollary 1.4.9), as the following theorem shows.

Theorem 1.6.2. *Let V be a vertex superalgebra, and let $Y^M : V \rightarrow \mathcal{F}(M)$ be a V -module. For $a, b \in V$, $u \in M$ and $m, n \in \mathbb{Z}$:*

- (i) $(|0\rangle)_{(n)}^M = \delta_{n,-1} \text{Id}_M$;
- (ii) $Y^M(Ta, z) = \partial_z Y^M(a, z)$ or, equivalently, $(Ta)_{(n)}^M = -na_{(n-1)}^M$;
- (iii) All fields $\{Y^M(a, z)\}_{a \in V}$ are mutually local;
- (iv) (n -product identity) $Y^M(a, z)_{(n)} Y^M(b, z) = Y^M(a_{(n)}b, z)$;
- (v) $[a_{(m)}^M, b_{(n)}^M] = \sum_{j \in \mathbb{N}} (a_{(j)}b)_{(m+n-j)}^M$;
- (vi) $e^{Tw} Y^M(a, z) e^{-Tw} = i_{z,w} Y^M(a, z+w)$;
- (vii) (Borcherds identity)

$$i_{z,w}((z-w)^n) Y^M(a, z) Y^M(b, w) u - p(a, b) i_{w,z}((z-w)^n) Y^M(b, w) Y^M(a, z) u = \sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} Y^M(a_{(n+j)}b, w) u;$$

- (viii) $(a_{(n)}b)_{(m)}^M u = \sum_{j \in \mathbb{N}} (-1)^j \binom{n}{j} (a_{(n-j)}^M b_{(m+j)}^M) u - (-1)^n p(a, b) b_{(n+m-j)}^M a_{(j)}^M u$;
- (ix) For $s, n_1, \dots, n_s \in \mathbb{N}$ and $a^1, \dots, a^s \in V$,

$$Y^M(a_{(-n_1-1)}^1 \cdots a_{(-n_s-1)}^s |0\rangle, z) = \frac{\partial_z^{n_1} Y^M(a^1, z) \cdots \partial_z^{n_s} Y^M(a^s, z)}{n_1! \cdots n_s!}.$$

Remark 1.6.3. In contrast to vertex superalgebras, the map $Y^M : V \rightarrow \mathcal{F}(M)$ is in general not injective for a module M over a vertex superalgebra V . Moreover, we do not have the property of skewsymmetry for modules.

Proposition 1.6.4 ([DLM98, §3]). *Let V be a vertex superalgebra. A V -module is equivalently a superspace M together with a linear and parity preserving map $Y^M : V \rightarrow \mathcal{F}(M)$, written as $Y^M(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-n-1}$, such that for $a, b \in V$, $u \in M$ and $m, n \in \mathbb{Z}$:*

$$(|0\rangle)_{(n)}^M = \delta_{n,-1} \text{Id}_M, \tag{1.6.1}$$

$$[a_{(m)}^M, b_{(n)}^M] = \sum_{j \in \mathbb{N}} (a_{(j)}b)_{(m+n-j)}^M, \tag{1.6.2}$$

$$(a_{(n)}b)_{(m)}^M u = \sum_{j \in \mathbb{N}} (-1)^j \binom{n}{j} (a_{(n-j)}^M b_{(m+j)}^M) u - (-1)^n p(a, b) b_{(n+m-j)}^M a_{(j)}^M u. \tag{1.6.3}$$

Let V be a vertex superalgebra, and let M be a V -module. We say $T^M \in \text{End}(M)$ is a differential of M if

$$[T^M, a_{(n)}^M] = -na_{(n-1)}^M \quad \text{for } a \in V \text{ and } n \in \mathbb{Z}$$

or, equivalently,

$$[T^M, Y^M(a, z)] = Y^M(Ta, z) \quad \text{for } a \in V.$$

A differential V -module is a V -module equipped with a differential.

Let $(\mathfrak{g}, \mathfrak{F}, T_0)$ be a regular formal distribution Lie superalgebra with $\mathfrak{F} = \{a^j(z)\}_{j \in J}$. A smooth $(\mathfrak{g}, \mathfrak{F}, T_0)$ -module is a \mathfrak{g} -module M such for $j \in J$ and $u \in M$, $a^j(z)u \in M((z))$. Let $\lambda : \mathfrak{h} \rightarrow \mathbb{C}$ be a linear functional, where \mathfrak{h} is a subalgebra of \mathfrak{g}_+ with $\mathfrak{h} \subseteq \ker(T_0)$. We say \mathfrak{h} acts as λ if for $h \in \mathfrak{h}$ and $u \in M$, $hu = \lambda(h)u$.

Theorem 1.6.5 ([Li04, Theorem 2.15]). *Let $(\mathfrak{g}, \mathfrak{F}, T_0)$ be a regular formal distribution Lie superalgebra with $\mathfrak{F} = \{a^j(z)\}_{j \in J}$. Given a smooth $(\mathfrak{g}, \mathfrak{F}, T_0)$ -module M , there is a unique module structure $Y^M : V(\mathfrak{g}, \mathfrak{F}, T_0) \rightarrow \mathcal{F}(M)$ such that*

$$Y^M(a_{(-1)}^j |0\rangle, z) = a^j(z) \quad \text{for } j \in J.$$

Let $\lambda : \mathfrak{h} \rightarrow \mathbb{C}$ be a linear functional, where \mathfrak{h} is a subalgebra of \mathfrak{g}_+ with $\mathfrak{h} \subseteq \ker(T_0)$. Given a smooth $(\mathfrak{g}, \mathfrak{F}, T_0)$ -module M where \mathfrak{h} acts as λ , there is a unique module structure $Y_\lambda^M : V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0) \rightarrow \mathcal{F}(M)$ such that Y_λ^M factors through Y^M , i.e., such that the following diagram commutes

$$\begin{array}{ccc} V(\mathfrak{g}, \mathfrak{F}, T_0) & \twoheadrightarrow & V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0) \\ & \searrow^{Y^M} & \downarrow^{Y_\lambda^M} \\ & & \mathcal{F}(M) \end{array}$$

where the horizontal arrow is the quotient map.

Remark 1.6.6. We can define naturally the categories $\{\text{smooth } (\mathfrak{g}, \mathfrak{F}, T_0)\text{-modules}\}$ and $\{\text{smooth } (\mathfrak{g}, \mathfrak{F}, T_0)\text{-modules where } \mathfrak{h} \text{ acts as } \lambda\}$. Theorem 1.6.5 constructs a pair of functors:

$$\begin{aligned} & \{\text{smooth } (\mathfrak{g}, \mathfrak{F}, T_0)\text{-modules}\} \rightarrow V(\mathfrak{g}, \mathfrak{F}, T_0)\text{-Mod}, \\ & \{\text{smooth } (\mathfrak{g}, \mathfrak{F}, T_0)\text{-modules where } \mathfrak{h} \text{ acts as } \lambda\} \rightarrow V^\lambda(\mathfrak{g}, \mathfrak{F}, T_0)\text{-Mod}, \end{aligned}$$

which are actually isomorphisms.

Let M be a module over a graded vertex superalgebra V with Hamiltonian H . A *Hamiltonian operator* of M is a diagonalizable operator $H^M \in \text{End}(M)$ such that

$$[H^M, Y^M(a, z)] = z\partial_z Y^M(a, z) + Y^M(Ha, z) \quad \text{for } a \in V. \quad (1.6.4)$$

A V -module together with a Hamiltonian is called *graded*. The *grading* of M is the eigenspace decomposition of H^M

$$M = \bigoplus_{\Delta \in \mathbb{C}} M_\Delta,$$

where

$$M_\Delta = \ker(H^M - \Delta \text{Id}_M).$$

If u is an eigenvector of H^M , it is called *homogeneous*, its eigenvalue is called the *conformal weight* of u , and it is denoted by Δ_u . Condition (1.6.4) is equivalent to

$$[H^M, a_{(n)}^M] = -(n+1)a_{(n)}^M + (Ha)_{(n)}^M \quad \text{for } a \in V \text{ and } n \in \mathbb{Z}$$

and to

$$[H^M, a_{(n)}^M] = (\Delta_a - n - 1)a_{(n)}^M \quad \text{for } a \in V \text{ homogeneous and } n \in \mathbb{Z}. \quad (1.6.5)$$

We usually write $Y^M(a, z) = \sum_{n \in \mathbb{Z} - \Delta_a} a_n^M z^{-n - \Delta_a}$ when $a \in V$ is homogeneous with conformal weight Δ_a , which is equivalent to

$$a_{(n)}^M = a_{n - \Delta_a + 1}^M \quad \text{for } n \in \mathbb{Z}.$$

With this notation, (1.6.5) is equivalent to

$$[H^M, a_n^M] = -na_n^M \quad \text{for } a \in V \text{ homogeneous and } n \in \mathbb{Z} - \Delta_a. \quad (1.6.6)$$

Theorem 1.6.7. *Let V be a graded vertex superalgebra with Hamiltonian H , and let M be a graded V -module with Hamiltonian H^M . Then:*

- (i) $a_n^M M_\Delta \subseteq M_{\Delta - n}$ for $a \in V$ homogeneous and $n \in \mathbb{Z} - \Delta_a$;
- (ii) $\Delta_{a_{(n)}^M u} = \Delta_a + \Delta_u - n - 1$ for $a \in V$ homogeneous, $u \in M$ and $n \in \mathbb{Z}$;

(iii) (Graded Borcherds identity) For $a, b \in V$ homogeneous, $u \in M$, $m \in \mathbb{Z} - \Delta_a$, $k \in \mathbb{Z} - \Delta_b$ and $n \in \mathbb{Z}$,

$$\sum_{j \in \mathbb{N}} (-1)^j \binom{n}{j} (a_{m+n-j}^M (b_{k+j-n}^M u) - (-1)^n p(a, b) b_{k-j}^M (a_{m+j}^M u)) = \sum_{j \in \mathbb{N}} \binom{m + \Delta_a - 1}{j} (a_{(n+j)b}^M)_{m+k}^M u.$$

Proof.

- (i) The proof of Theorem 1.5.2(iv) also works here.
- (ii) The proof of Theorem 1.5.2(vi) also works here.
- (iii) The graded Borcherds identity is simply a change of notation in the usual Borcherds identity. \square

A homomorphism of graded modules over a graded vertex superalgebra is assumed to respect the gradings.

Theorem 1.6.8 ([LL04, Proposition 4.1.5 and (4.1.18)]). *Let V be a conformal vertex algebra of central charge c with conformal vector ω , and let M be a V -module. We write $Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^M z^{-n-2}$. For $a \in V$ and $m, n \in \mathbb{Z}$:*

$$\begin{aligned} [L_{-1}^M, Y^M(a, z)] &= Y^M(L_{-1}a, z) = \partial_z Y^M(a, z), \\ [L_m^M, L_n^M] &= (m - n)L_{m+n}^M + \delta_{m, -n} \frac{m^3 - m}{12} c \text{Id}_M, \\ [L_0^M, Y^M(a, z)] &= z \partial_z Y^M(a, z) + Y^M(L_0 a, z). \end{aligned}$$

In particular, M is a differential V -module with L_{-1}^M as differential, and M is a smooth Vir-module of central charge c . If L_0^M is diagonalizable, then M is graded by L_0^M and:

$$\begin{aligned} [L_0^M, L_{-1}^M] &= L_{-1}^M, \\ L_{-1}^M(M_\Delta) &\subseteq M_{\Delta+1} \quad \text{for } \Delta \in \mathbb{C}. \end{aligned}$$

Remark 1.6.9. Theorem 1.6.8 is analogous to Theorem 1.5.2 with L_{-1}^M in place of T and L_0^M in place of H .

Let V be a conformal vertex algebra with conformal vector ω . It is better to use the notation (V, ω) for now because of Remark 1.5.5. By Theorem 1.6.8, the following definition makes sense. A (V, ω) -module is a V -module M such that L_0^M is diagonalizable, where we write $Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^M z^{-n-2}$. The (V, ω) -modules form a subcategory of $V\text{-Mod}$. In particular, M is a differential module with differential L_{-1}^M and is graded by L_0^M . A (V, ω) -module M is called a *positive energy representation* of (V, ω) if $M = \bigoplus_{n \in \mathbb{N}} M_{h+n}$ for some $h \in \mathbb{C}$, and in this case, we also say that M is $h + \mathbb{N}$ -graded. The subspace M_h is called a *top degree component* of M . When we require $M_h \neq 0$, the number h and the top degree component are well-defined.

Remark 1.6.10. Let (V, ω) be a conformal vertex algebra, and let M_1 and M_2 be (V, ω) -modules. A homomorphism $f : M_1 \rightarrow M_2$ automatically respects the gradings of M_1 and M_2 given by $L_0^{M_1}$ and $L_0^{M_2}$.

Let V be a \mathbb{Z} -graded vertex superalgebra. An *admissible* V -module is a V -module M together with a grading $M = \bigoplus_{n \in \mathbb{Z}} M(n)$, such that:

- (i) $M(n) = 0$ for $n \in \mathbb{Z}_-$;
- (ii) $a_{(n)}^M M(m) \subseteq M(m + \Delta_a - n - 1)$ for $a \in V$ homogeneous and $m, n \in \mathbb{Z}$ or, equivalently, $a_n^M M(m) \subseteq M(m - n)$ for $a \in V$ homogeneous and $m, n \in \mathbb{Z}$.

Thus, an admissible V -module is a pair $(M, \bigoplus_{n \in \mathbb{Z}} M(n))$ satisfying the above conditions, though we often omit the grading from the notation. If $M \neq 0$, we set $n_0 = \min\{n \in \mathbb{N} \mid M(n) \neq 0\}$. The subspace $M(n_0)$ is called the *top degree component* of M . The top degree component of the zero module 0 is 0 .

A *homomorphism of admissible V -modules* is a homomorphism of V -modules, no condition on the grading is imposed. We have defined the category of admissible V -modules, which is abelian. Arbitrary direct sums of admissible V -modules always exist.

We note that shifting the grading of an admissible V -module gives an isomorphic admissible V -module, and that submodules of admissible V -modules are not necessarily graded. When $M \neq 0$ is an admissible V -module, we can always shift the grading so that $M(0) \neq 0$.

A *simple* or *irreducible* admissible V -module is an admissible V -module M such that 0 is the only proper graded submodule of M . We note that we ask for graded submodules, not just submodules, in the definition of irreducible. An object in the category of admissible V -modules is *completely reducible* if it is a direct sum of irreducible admissible V -modules. The completely reducible admissible V -modules form a full subcategory of the category of admissible V -modules.

Let V be a \mathbb{Z} -graded conformal vertex algebra, and let M be a module over V . As usual, we write $Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^M z^{-n-2}$. We consider the endomorphism $L_0^M \in \text{End}(M)$, which in general is not diagonalizable. By Lemma 1.5.13(i), we have a direct sum decomposition

$$M^{\text{Gen}} = \bigoplus_{\Delta \in \mathbb{C}} M_{\Delta}^{\text{Gen}},$$

where

$$M_{\Delta}^{\text{Gen}} = \{a \in V \mid \text{there is } k \in \mathbb{N} \text{ such that } (L_0^M - \Delta \text{Id}_M)^k a = 0\}.$$

We define:

$$\begin{aligned} \Lambda^M &= \{\Delta \in \mathbb{C} \mid M_{\Delta}^{\text{Gen}} \neq 0 \text{ and for } n \in \mathbb{Z}_+, M_{\Delta-n}^{\text{Gen}} = 0\}, \\ M^{\text{Gen}}(n) &= \begin{cases} \bigoplus_{\Delta \in \Lambda^M} M_{\Delta+n}^{\text{Gen}} & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases} \end{aligned} \quad (1.6.7)$$

We often consider V -modules M satisfying the following condition

$$\text{for } \Delta \in \mathbb{C}, \text{ there is } N \in \mathbb{N} \text{ such that for } n \in \mathbb{N} \text{ with } n \geq N, M_{\Delta-n}^{\text{Gen}} = 0. \quad (1.6.8)$$

This condition resembles the definition of vertex operator superalgebras.

Proposition 1.6.11. *Let V be a \mathbb{Z} -graded conformal vertex algebra, and let M be a module over V . Then:*

- (i) *If M is admissible with grading $\bigoplus_{n \in \mathbb{Z}} M(n)$ such that for $n \in \mathbb{Z}$, $\dim(M(n)) < \infty$, then $M = M^{\text{Gen}}$;*
- (ii) *If $a \in V$ is homogeneous, $n \in \mathbb{Z}$ and $u \in M_{\Delta}^{\text{Gen}}$, then $a_n^M u \in M_{\Delta-n}^{\text{Gen}}$;*
- (iii) *M^{Gen} is a submodule of M ;*
- (iv) *If M satisfies (1.6.8), then M^{Gen} is an admissible V -module with grading given by (1.6.7);*
- (v) *If $f : M_1 \rightarrow M_2$ is a homomorphism of V -modules, then $f(M_{1\Delta}^{\text{Gen}}) \subseteq M_{2\Delta}^{\text{Gen}}$ for $\Delta \in \mathbb{C}$ and $f(M_1^{\text{Gen}}) \subseteq M_2^{\text{Gen}}$.*

Proof. We omit superscripts in this proof.

- (i) For $n \in \mathbb{Z}$, $L_0(M(n)) \subseteq M(n)$. By [HK71, §6.8 Theorem 12], for $n \in \mathbb{Z}$, $M(n) = M(n)^{\text{Gen}}$, and this implies the assertion.
- (ii) From the formula $[L_0, a_n] = -na_n$ given in (1.6.6), we inductively get the formula

$$L_0^k a_n = \sum_{j \in \mathbb{N}} \binom{k}{j} (-n)^j a_n L_j^{k-j} \quad \text{for } k \in \mathbb{N}.$$

By definition, there is $k \in \mathbb{N}$ such that $(L_0 - \Delta \text{Id}_M)^k u = 0$. We now compute

$$\begin{aligned}
(L_0 - (\Delta - n) \text{Id}_M)^k a_n u &= \left(\sum_{i=0}^k \binom{k}{i} L_0^{k-i} (-(\Delta - n))^i \text{Id}_M \right) a_n u \\
&= \sum_{i=0}^k \binom{k}{i} (-(\Delta - n))^i L_0^{k-i} a_n u \\
&= \sum_{i=0}^k \binom{k}{i} (-(\Delta - n))^i \sum_{j=0}^{k-i} \binom{k-i}{j} (-n)^j a_n L_0^{k-i-j} u \\
&= a_n \left(\sum_{i=0}^k \sum_{j=0}^{k-i} \frac{k!}{i! j! (k-i-j)!} (-(\Delta - n))^i (-n)^j L_0^{k-i-j} u \right) \\
&= a_n ((L_0 - (\Delta - n) \text{Id}_M - n \text{Id}_M)^k u) \\
&= a_n ((L_0 - \Delta \text{Id}_M)^k u) \\
&= 0.
\end{aligned}$$

This means $a_n u \in M_{\Delta-n}^{\text{Gen}}$.

- (iii) From (ii), we immediately get that M^{Gen} is a submodule of M .
- (iv) If this condition is satisfied, then we have $M^{\text{Gen}} = \bigoplus_{n \in \mathbb{Z}} M^{\text{Gen}}(n)$. From (ii), we get that this grading makes M^{Gen} into an admissible V -module.
- (v) This follows straight from the definitions. □

We have constructed a functor

$$\text{Gen} : \{V\text{-modules satisfying (1.6.8)}\} \rightarrow \{\text{admissible } V\text{-modules}\}.$$

For a \mathbb{Z} -graded vertex algebra V , we have a natural forgetful functor

$$\text{For} : \{\text{admissible } V\text{-modules}\} \rightarrow V\text{-Mod}.$$

Remark 1.6.12. We followed [DLM98] when we defined admissible modules and their homomorphisms. In [Zhu96], homomorphisms of admissible modules are required to shift the grading by a fixed integer k , i.e., a morphism $f : M_1 \rightarrow M_2$ between admissible modules M_1 and M_2 must satisfy $f(M_1(n)) \subseteq M_2(n+k)$ for $n \in \mathbb{Z}$. However, with Zhu's definition, direct sums of admissible modules might not exist. For example, if $M \neq 0$ is an admissible module, then we may define the admissible module M' , which is equal to M but has grading shifted by 1. Then we run into trouble while defining $M \oplus M'$ because the identity map $M \oplus M \rightarrow M \oplus M'$ does not shift the grading by a fixed integer.

Furthermore, our definition of irreducible admissible V -module is not the same as the usual definition of irreducible object in an abelian category. As we will see in §4.5, under certain conditions on V , both Gen and For are equivalences of categories, and an admissible module is irreducible if and only if it is irreducible as an object in the category of V -modules.

Let V be a \mathbb{Z} -graded conformal vertex algebra, and let M be a (V, ω) -module. As usual, we write $Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^M z^{-n-2}$. We say M is *ordinary* if:

- (i) $\dim(M_\Delta) < \infty$ for $\Delta \in \mathbb{C}$;
- (ii) For $\Delta \in \mathbb{C}$, there is $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$ with $n \geq N$, $M_{\Delta-n} = 0$.

The ordinary V -modules form a subcategory of $V\text{-Mod}$, and they satisfy (1.6.8). By Proposition 1.6.11, we can make an ordinary module into an admissible module. Thus, we can think of the following inclusions of categories

$$\{\text{ordinary } V\text{-modules}\} \subseteq \{\text{admissible } V\text{-modules}\} \subseteq \{V\text{-modules}\}.$$

A positive energy representation M of the conformal vertex algebra (V, ω) is ordinary if for $n \in \mathbb{N}$, $\dim(M_{h+n}) < \infty$, where M_h is a top degree component of M . For an ordinary positive

energy representation M , the *character* of M is a formal power series defined by

$$\text{ch}_M(q) = \sum_{n \in \mathbb{N}} \dim(M_{h+n})q^{h+n} \in q^h \mathbb{C}[[q]].$$

Example 1.6.13. Let M be a highest weight representation of Vir with highest weight (c, h) (see §2 ahead) and highest weight vector v . Then M is a smooth Vir -module of central charge c . We note that L_0 is diagonalizable. By Theorem 1.6.5 and Theorem 1.6.8, M is an $h + \mathbb{N}$ -graded $(\text{Vir}^c, L_{-2} | 0)$ -module. We set $M(n) = M_{h+n}$ for $n \in \mathbb{Z}$. By Theorem 1.6.7(ii), $M = \bigoplus_{n \in \mathbb{Z}} M(n)$ is an admissible Vir^c -module with top degree component $\mathbb{C}v$. Furthermore, M is an ordinary positive energy representation of Vir^c .

Let V be a vertex operator algebra. We say that V is *rational* if every admissible V -module is completely reducible, i.e., a direct sum of simple admissible V -modules. We say that V is *regular* if every V -module is a direct sum of simple ordinary V -modules. Regularity is a stronger form of complete reducibility of modules.

The definition of ordinary and admissible modules is similar to that of vertex operator algebras (in the sense that we required them to be lower truncated) except that modules are not required to be \mathbb{Z} -graded. Indeed, one of our objectives is to study $L(1/2, 1/2)$ and $L(1/2, 1/16)$, which are \mathbb{Q} -graded.

Theorem 1.6.14 ([DLM98]). *If V is a rational vertex operator algebra, then there are only finitely many simple admissible modules up to isomorphism, and any simple admissible module is an ordinary module.*

Example 1.6.15. It was proved in [Wan93] that $\text{Vir}_{p,q}$ is rational when $p, q \geq 2$ are relatively prime integers. In fact, as we will see in Theorem 3.3.4, a complete list of all the irreducible modules over $\text{Vir}_{p,q}$ is given in that article. Actually, we will see later that in that case, $\text{Vir}_{p,q}$ is regular.

1.7. Lie algebras associated to vertex algebras.

Lemma 1.7.1. *For a vertex algebra V , V/TV is a Lie algebra with bracket*

$$[a + TV, b + TV] = a_{(0)}b + TV \quad \text{for } a, b \in V.$$

Proof. The skewsymmetry of the bracket follows from skewsymmetry of vertex algebras. The Jacobi identity follows from Theorem 1.4.5(x) with $m = n = 0$. \square

Lemma 1.7.2. *Let V be a vertex algebra, and let (R, ∂) be a differential commutative associative algebra with unit. Then*

$$\text{Lie}(V, R) = (V \otimes R) / (T \otimes \text{Id}_R + \text{Id}_V \otimes \partial)(V \otimes R)$$

is a Lie algebra with bracket

$$[a \otimes r, b \otimes s] = \sum_{j \in \mathbb{N}} a_{(j)}b \otimes \left(\frac{\partial^j r}{j!} \right) s \quad \text{for } a, b \in V \text{ and } r, s \in R.$$

Proof. Since R is a commutative vertex algebra (see Example 1.4.4), $V \otimes R$ is a vertex algebra with translation operator $T \otimes \text{Id}_R + \text{Id}_V \otimes \partial$. The assertion follows by applying Lemma 1.7.1 to the vertex algebra $V \otimes R$. \square

The *Borcherds Lie algebra associated with a vertex algebra V* is the Lie algebra

$$[V] = \text{Lie}(V, \mathbb{C}[t, t^{-1}]),$$

where $\mathbb{C}[t, t^{-1}]$ is viewed as a differential algebra with the derivation ∂_t . For $a \in V$ and $n \in \mathbb{Z}$, let $a(n)$ be the class of $a \otimes t^n \in V \otimes \mathbb{C}[t, t^{-1}]$. By definition, we have

$$[a(m), b(n)] = \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(j)}b)(m+n-j) \quad \text{for } a, b \in V \text{ and } m, n \in \mathbb{Z}, \quad (1.7.1)$$

$$(Ta)(n) = -na(n-1) \quad \text{for } a \in V \text{ and } n \in \mathbb{Z}. \quad (1.7.2)$$

We note that these are exactly the same formulas as in Theorem 1.4.5(iii), (v).

Therefore, we have constructed a functor

$$[\bullet] : \{\text{Vertex algebras}\} \rightarrow \{\text{Lie algebras}\}.$$

Lemma 1.7.3. *Any V -module M is a $[V]$ -module by setting*

$$\begin{aligned} [V] &\rightarrow \mathfrak{gl}(M), \\ a(n) &\mapsto a_{(n)}^M \quad \text{for } a \in V \text{ and } n \in \mathbb{Z}. \end{aligned}$$

Proof. First, this map is well-defined because of Theorem 1.6.2(ii). It is a Lie algebra homomorphism because of Theorem 1.6.2(v). \square

Therefore, we have constructed a functor

$$[\bullet] : V\text{-Mod} \rightarrow [V]\text{-Mod}.$$

We now assume that V is a \mathbb{Z} -graded vertex algebra with Hamiltonian H . Then $[V]$ is a graded Lie algebra by defining $H \in \text{End}([V])$ as

$$H(a(n)) = -(n+1)a(n) + (Ha)(n) \quad \text{for } a \in V \text{ and } n \in \mathbb{Z}.$$

This operator is diagonalizable because for $a \in V$ homogeneous and $n \in \mathbb{Z}$, $H(a(n)) = (\Delta_a - n - 1)a(n)$. This means that

$$\Delta_{a(n)} = \Delta_a - n - 1 \quad \text{for } a \in V \text{ homogeneous and } n \in \mathbb{Z}.$$

Moreover, we can verify that H is a derivation of $[V]$. We have a \mathbb{Z} -grading

$$[V] = \bigoplus_{n \in \mathbb{Z}} [V]_n$$

and a triangular decomposition

$$[V] = [V]_+ \oplus [V]_0 \oplus [V]_-,$$

where

$$[V]_{\pm} = \bigoplus_{n \in \mathbb{Z}_{\pm}} [V]_{\pm n}.$$

Remark 1.7.4. We note that $([V], \{\sum_{n \in \mathbb{Z}} a(n)z^{-n-1}\}_{a \in V}, T)$ is a regular formal distribution Lie algebra. I have not found any use of this, though.

We observe that $[V]_0$ is spanned by elements of the form $a(\Delta_a - 1)$ for $a \in V$ homogeneous, and it is a Lie subalgebra of $[V]$. By (1.7.1) and Theorem 1.5.2(vi), the bracket in $[V]_0$ is given by

$$[a(\Delta_a - 1), b(\Delta_b - 1)] = \sum_{j \in \mathbb{N}} \binom{\Delta_a - 1}{j} (a_{(j)}b)(\Delta_{a_{(j)}b} - 1) \quad \text{for } a, b \in V \text{ homogeneous.} \quad (1.7.3)$$

We consider the surjective linear map

$$\begin{aligned} \psi : V &\twoheadrightarrow [V]_0, \\ \psi(a) &= a(\Delta_a - 1). \end{aligned}$$

Lemma 1.7.5. *We have*

$$\ker(\psi) = (T + H)V.$$

Proof. If $a \in V$ is homogeneous, then by Theorem 1.5.2(v) and (1.7.2),

$$\psi((T + H)(a)) = \psi(Ta + \Delta_a a) = (Ta)(\Delta_{Ta} - 1) + \Delta_a a(\Delta_a - 1) = (Ta)(\Delta_a) + \Delta_a a(\Delta_a - 1) = 0.$$

Thus, $(T + H)V \subseteq \ker(\psi)$.

We now assume $a = a_m + a_{m+1} + \cdots + a_n \in \ker(\psi)$, where $a_i \in V_i$ for $i = m, m+1, \dots, n$, $a_m \neq 0$ and $a_n \neq 0$. Unwinding the definitions, we see that $a \in \ker(\psi)$ means that there exist $b_r, b_{r+1}, \dots, b_s \in V$ such that $b_r \neq 0$, $b_s \neq 0$ and

$$a_m \otimes t^{m-1} + a_{m+1} \otimes t^m + \cdots + a_n \otimes t^{n-1} = (T \otimes \text{Id}_{\mathbb{C}[t, t^{-1}]} + \text{Id}_V \otimes \partial_t)(b_r \otimes t^r + b_{r+1} \otimes t^{r+1} + \cdots + b_s \otimes t^s).$$

The last equation implies $r \leq m$ and $s \geq n-1$, and it is equivalent to the following system of equations in V :

$$\begin{aligned} rb_r &= 0 \\ Tb_r + (r+1)b_{r+1} &= 0 \\ &\dots \\ Tb_m + (m+1)b_{m+1} &= a_m \\ Tb_{m+1} + (m+2)b_{m+2} &= a_{m+1} \\ &\dots \\ Tb_{n-1} + nb_n &= a_n \\ &\dots \\ Tb_{s-1} + sb_s &= 0 \\ Tb_s &= 0. \end{aligned}$$

For $i = r, r+1, \dots, s$, let c_i be the component of b_i in V_i . The last system of equations in V implies:

$$\begin{aligned} Hc_r &= 0 \\ Tc_r + Hc_{r+1} &= 0 \\ &\dots \\ Tc_m + Hc_{m+1} &= a_m \\ Tc_{m+1} + Hc_{m+2} &= a_{m+1} \\ &\dots \\ Tc_{n-1} + Hc_n &= a_n \\ &\dots \\ Tc_{s-1} + Hc_s &= 0 \\ Tc_s &= 0. \end{aligned}$$

Adding these equations, we get

$$(T+H)(c_r + c_{r+1} + \cdots + c_s) = a_m + a_{m+1} + \cdots + a_n = a,$$

which implies $a \in (T+H)V$. □

By Lemma 1.7.5, we are led to define

$$\text{Lie}_0(V) = V/(T+H)V,$$

whose bracket is given by

$$[a + (T+H)V, b + (T+H)V] = \sum_{j \in \mathbb{N}} \binom{\Delta_a - 1}{j} a_{(j)} b + (T+H)V \quad \text{for } a, b \in V.$$

From what we have done, there is a natural Lie algebra isomorphism

$$\begin{aligned} \text{Lie}_0(V) &\xrightarrow{\sim} [V]_0, \\ a + (T+H)V &\mapsto a(\Delta_a - 1) \quad \text{for } a \in V \text{ homogeneous,} \end{aligned}$$

and we have another functor

$$\text{Lie}_0 : \{\mathbb{Z}\text{-graded vertex algebras}\} \rightarrow \{\text{Lie algebras}\}.$$

2. HIGHEST WEIGHT REPRESENTATIONS OF THE VIRASORO LIE ALGEBRA

In this section, we start by reviewing some well-known facts about forms on complex vector spaces. Then, we consider highest weight representations of the Virasoro Lie algebra with particular interest on the Verma modules $M(c, h)$ and the irreducible representations $L(c, h)$. Afterward, we construct a contravariant form on an arbitrary highest weight representation. Next, we present the Kac determinant formula, which allows us to determine $L(c, h)$ explicitly by determining the generating singular vectors of the maximal proper subrepresentation of $M(c, h)$.

2.1. Unitary and contravariant representations of Lie algebras. Let V be a vector space. A (*sesquilinear*) form on V is a function

$$\begin{aligned} \langle \bullet | \bullet \rangle : V \times V &\rightarrow \mathbb{C}, \\ (u, v) &\mapsto \langle u | v \rangle \end{aligned}$$

such that for $u, v, w \in V$ and $t \in \mathbb{C}$:

- (i) $\langle tu + v | w \rangle = \bar{t}\langle u | w \rangle + \langle v | w \rangle$;
- (ii) $\langle u | tv + w \rangle = t\langle u | v \rangle + \langle u | w \rangle$.

All forms are assumed to be sesquilinear.

A *Hermitian* form on V is a form $\langle \bullet | \bullet \rangle$ satisfying

$$\langle u | v \rangle = \overline{\langle v | u \rangle} \quad \text{for } u, v \in V.$$

Let $S \subseteq V$ be a subset of a vector space V equipped with a form $\langle \bullet | \bullet \rangle$. We define the *orthogonal complement* of S as the subspace

$$S^\perp = \{v \in V \mid \text{for } u \in S, \langle u | v \rangle = 0\}.$$

A form $\langle \bullet | \bullet \rangle$ on V is called *nondegenerate* if $V^\perp = 0$, and it is called *positive-definite* if it is Hermitian and

$$\langle v | v \rangle > 0 \quad \text{for } v \in V \text{ with } v \neq 0.$$

An *inner product space* is a vector space together with a positive-definite form.

Lemma 2.1.1 ([HK71, §8.2 Theorem 5]). *Let V be an inner product space, and let W be a finite dimensional subspace of V . Then $W \oplus W^\perp = V$.*

Let V be a vector space. A map $\omega : V \rightarrow V$ is *antilinear* if for $u, v \in V$ and $t \in \mathbb{C}$:

$$\begin{aligned} \omega(u + v) &= \omega(u) + \omega(v), \\ \omega(tu) &= \bar{t}\omega(u). \end{aligned}$$

Let \mathfrak{g} be a Lie algebra. A map $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ is an *antilinear anti-involution* if ω is antilinear and for $a, b \in \mathfrak{g}$:

$$\begin{aligned} \omega([a, b]) &= [\omega(b), \omega(a)], \\ \omega(\omega(a)) &= a. \end{aligned}$$

We note that this means we can extend ω to the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} , obtaining a map $\omega : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ that is still an antilinear anti-involution, i.e., for $a, b \in U(\mathfrak{g})$:

$$\begin{aligned} \omega(ab) &= \omega(b)\omega(a), \\ \omega(\omega(a)) &= a. \end{aligned}$$

We shall be mostly interested in the Virasoro Lie algebra Vir , which has the following antilinear anti-involution

$$\begin{aligned} \omega : \text{Vir} &\rightarrow \text{Vir}, \\ \omega(L_n) &= L_{-n} \quad \text{for } n \in \mathbb{Z}, \\ \omega(C) &= C, \end{aligned}$$

which is extended by antilinearity.

Let \mathfrak{g} be a Lie algebra with an antilinear anti-involution $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$, and let V be a \mathfrak{g} -module with a Hermitian form $\langle \bullet | \bullet \rangle$. We say $\langle \bullet | \bullet \rangle$ is *contravariant* if

$$\langle au | v \rangle = \langle u | \omega(a)v \rangle \quad \text{for } a \in \mathfrak{g} \text{ and } u, v \in V.$$

We further say this representation is *unitary* if, in addition, it is positive-definite.

2.2. Verma modules. A *highest weight representation* of Vir is a Vir -module V , which has a nonzero vector v such that there exist complex numbers $c, h \in \mathbb{C}$ satisfying:

- (i) $Cv = cv$;
- (ii) $L_0v = hv$;
- (iii) $V = \text{span}\{L_{-i_k} \dots L_{-i_1}v \mid i_k \geq \dots \geq i_1 > 0\}$.

The numbers c and h are uniquely determined, and the pair (c, h) is called the *highest weight of V* . The vector v is not uniquely determined; it can be replaced by any nonzero scalar multiple of it, and we call v a *highest weight vector of V* .

For $\Delta \in \mathbb{C}$, we set $V_\Delta = \ker(L_0 - \Delta \text{Id}_V)$. We observe that all vectors of the form $L_{-i_k} \dots L_{-i_1}v$ with $i_k \geq \dots \geq i_1 > 0$ and a fixed value of $j = i_1 + \dots + i_k$ belong to V_{h+j} . By axiom (iii), $V = \sum_{j \in \mathbb{N}} V_{h+j}$, and by Lemma 1.5.11, this is in fact a direct sum

$$V = \bigoplus_{j \in \mathbb{N}} V_{h+j}. \quad (2.2.1)$$

By axiom (ii),

$$V_h = \mathbb{C}v.$$

We note that

$$L_n V_{h+j} \subseteq V_{h+j-n} \quad \text{for } n \in \mathbb{Z} \text{ and } j \in \mathbb{N}.$$

In particular, we have

$$L_n v = 0 \quad \text{for } n \in \mathbb{Z}_+. \quad (2.2.2)$$

For $j \in \mathbb{N}$, let $p(j)$ denote the number of partitions of n . It is clear that

$$\dim(V_{h+j}) \leq p(j) \quad \text{for } j \in \mathbb{N}, \quad (2.2.3)$$

with equality if and only if all vectors of the form $L_{-i_k} \dots L_{-i_1}v$, with $i_k \geq \dots \geq i_1 > 0$ and $j = i_1 + \dots + i_k$, are linearly independent. By axiom (i), C acts on V as multiplication by c because it commutes with every L_n for $n \in \mathbb{Z}$. Thus, a highest weight representation of Vir with highest weight (c, h) is necessarily smooth and of central charge c .

Lemma 2.2.1. *Let V be a highest weight representation of Vir with highest weight (c, h) , highest weight vector v and grading $V = \bigoplus_{j \in \mathbb{N}} V_{h+j}$. Then:*

- (i) *Any subrepresentation U of V is graded*

$$U = \bigoplus_{j \in \mathbb{N}} U \cap V_{h+j};$$

- (ii) *V is indecomposable, i.e., we cannot find nontrivial subrepresentations U, W such that*

$$V = U \oplus W;$$

- (iii) *V has a unique maximal proper subrepresentation J_{\max} , and V/J_{\max} is the unique irreducible quotient of V , which is also a highest weight representation with highest weight (c, h) .*

Proof.

- (i) This is Lemma 1.5.12 with $H = L_0$.
- (ii) We assume we have found such a decomposition. Both U and W are graded subrepresentations, and therefore we must have either $v \in U$ or $v \in W$, which implies either $U = V$ or $W = V$.
- (iii) We just take J_{\max} as the sum of all proper subrepresentations of V . □

Let V be a highest weight representation of Vir with highest weight (c, h) . The *character* of V is a formal power series defined by

$$\text{ch}_V(q) = \sum_{j \in \mathbb{N}} \dim(V_{h+j})q^{h+j} \in q^h \mathbb{C}[[q]].$$

The character of V satisfies the inequality

$$\text{ch}_V(q) \leq \frac{q^h}{\prod_{k \in \mathbb{Z}_+} (1 - q^k)} = \frac{q^h}{(q)_\infty}.$$

A *Verma representation* is a highest weight representation of Vir in which all vectors of the form $L_{-i_k} \dots L_{-i_1} v$, with $i_k \geq \dots \geq i_1 > 0$, are linearly independent. Since all vectors of the form $L_{-i_k} \dots L_{-i_1} v$, with $i_k \geq \dots \geq i_1 > 0$, in a Verma representation V are linearly independent, it follows that they form a basis of V . Consequently, there is a homomorphism from V to any other highest weight representation of Vir with the same highest weight (c, h) , mapping a highest weight vector to a highest weight vector. In particular, for any pair (c, h) of complex numbers, there is at most one Verma representation having (c, h) as its highest weight.

We now show we do have a Verma representation for each pair (c, h) of complex numbers using standard Lie algebra techniques. We set $\text{Vir}_{\geq 0} = \bigoplus_{n \in \mathbb{N}} \mathbb{C}L_n$. The subalgebra $\text{Vir}_{\geq 0} \oplus \mathbb{C}C$ of Vir acts on \mathbb{C} as follows:

$$L_n 1 = 0 \text{ for } n \in \mathbb{Z}_+, L_0 1 = h \text{ and } C 1 = c.$$

Then

$$M(c, h) = \text{Ind}_{\text{Vir}_{\geq 0} \oplus \mathbb{C}C}^{\text{Vir}}(\mathbb{C}) = U(\text{Vir}) \otimes_{U(\text{Vir}_{\geq 0} \oplus \mathbb{C}C)} \mathbb{C}$$

is a Vir -module, where Vir acts by left multiplication. We set $|c, h\rangle = 1 \otimes 1$.

For a partition $\lambda = [\lambda_1, \dots, \lambda_m]$, we define

$$L_\lambda = L_{-\lambda_1} \dots L_{-\lambda_m} \in U(\text{Vir}).$$

By the PBW theorem, the set

$$\{L_\lambda |c, h\rangle \mid \lambda \text{ is a partition}\}$$

is a vector space basis of $M(c, h)$. Therefore, $M(c, h)$ is a Verma representation of Vir with highest weight (c, h) and highest weight vector $|c, h\rangle$. We usually simplify $|c, h\rangle$ to just $|h\rangle$ when c is understood. By (2.2.3), the character of a Verma representation is given by

$$\text{ch}_{M(c, h)}(q) = \frac{q^h}{(q)_\infty}.$$

Any other highest weight representation V with highest weight (c, h) is a quotient of $M(c, h)$: we simply map v to a highest weight vector of V , the resulting homomorphism is surjective, and V is isomorphic to $M(c, h)$ quotiented by the kernel of this homomorphism.

By Lemma 2.2.1(iii), $M(c, h)$ has a unique maximal proper subrepresentation, denoted by $J(c, h)$. The quotient

$$L(c, h) = M(c, h)/J(c, h)$$

is an irreducible highest weight representation with highest weight (c, h) . Actually, this is the unique irreducible highest weight representation with highest weight (c, h) because if V is such a representation, then V is isomorphic to $M(c, h)/U$ for some proper subrepresentation U of $M(c, h)$ which has to be maximal, implying $U = J(c, h)$. We wish to determine when $L(c, h)$ is equal to $M(c, h)$.

Lemma 2.2.2. *Let V be a Vir -module. If $u \in V$ is nonzero and $L_1 u = L_2 u = 0$, then u is a singular vector.*

Proof. The condition $L_1 u = L_2 u = 0$ implies, using induction, that $L_n u = 0$ for $n \geq 3$. \square

Remark 2.2.3. By (2.2.2), a highest weight vector is singular. By (2.2.1), we can write a singular vector u as $u = \sum_{j \in \mathbb{N}} u_j$, where $u_j \in V_{h+j}$ for $j \in \mathbb{N}$. Then all the nonzero vectors u_j are also singular vectors. Therefore, we can focus on homogeneous singular vectors.

Theorem 2.2.4. *Let V be a highest weight representation of Vir with a contravariant form $\langle \bullet | \bullet \rangle$. Then all eigenspaces of L_0 are pairwise orthogonal. Let U be a subrepresentation of V . Then U^\perp is a subrepresentation of V . Moreover, if $\langle \bullet | \bullet \rangle$ is unitary, then $U \oplus U^\perp = V$.*

Proof. Let (c, h) be the highest weight of V , and let v be a highest weight vector of V . We have the decomposition $V = \bigoplus_{j \in \mathbb{N}} V_{h+j}$. First, we note that if $j_1 \neq j_2$, then $\langle V_{h+j_1} | V_{h+j_2} \rangle = 0$. This is because if $u \in V_{h+j_1}$ and $w \in V_{h+j_2}$, then

$$(\bar{h} + j_1) \langle u | w \rangle = \langle L_0 u | w \rangle = \langle u | L_0 w \rangle = (h + j_2) \langle u | w \rangle$$

which implies $(h - \bar{h} + j_2 - j_1) \langle u | w \rangle = 0$ and $\langle u | w \rangle = 0$. We know $\langle U | U^\perp \rangle = 0$ and $L_j U \subseteq U$ for $j \in \mathbb{Z}$. Then $0 = \langle L_j U | U^\perp \rangle = \langle U | L_{-j} U^\perp \rangle$ which implies $L_j U^\perp \subseteq U^\perp$ for $j \in \mathbb{Z}$, so U^\perp is a subrepresentation of V . By Lemma 2.2.1(i),

$$U^\perp = \bigoplus_{j \in \mathbb{N}} U^\perp \cap V_{h+j} = \bigoplus_{j \in \mathbb{N}} (U \cap V_{h+j})^{\perp | V_{h+j}},$$

where $(U \cap V_{h+j})^{\perp | V_{h+j}}$ denotes the orthogonal subspace of $U \cap V_{h+j}$ in V_{h+j} . If $\langle \bullet | \bullet \rangle$ is unitary, then by Lemma 2.1.1, $U \cap V_{h+j} \oplus (U \cap V_{h+j})^{\perp | V_{h+j}} = V_{h+j}$ for $j \in \mathbb{N}$ because all the vector spaces V_{h+j} are finite dimensional. It is now clear that $U \oplus U^\perp = V$. \square

Corollary 2.2.5. *A highest weight representation V with a unitary form is irreducible.*

Proof. Let U be a subrepresentation of V . By Theorem 2.2.4, U^\perp is a subrepresentation of V and $U \oplus U^\perp = V$. By Lemma 2.2.1(ii), either $U = V$ or $U = 0$. We conclude that V is irreducible. \square

2.3. Kac determinant formula and singular vectors. Let V be a highest weight representation with highest weight (c, h) , and we pick a highest weight vector v . We wish to define a contravariant form $\langle \bullet | \bullet \rangle$ on V such that $\langle v | v \rangle = 1$. We now show we do not have much choice. Since $V_h = \mathbb{C}v$, it makes sense to define the *expectation value* of $u \in V$, denoted by $\langle u \rangle$, as the coefficient of u with respect to the direct sum $V = \bigoplus_{j \in \mathbb{N}} V_{h+j}$, i.e., as the unique $t \in \mathbb{C}$ such that $u - tv \in \bigoplus_{j \in \mathbb{Z}_+} V_{h+j}$. We consider $L_\lambda \in U(\text{Vir})$ for a partition λ and $u \in V$. By contravariance, we must have $\langle L_\lambda v | u \rangle = \langle v | \omega(L_\lambda) u \rangle$. But $\langle v | \omega(L_\lambda) u \rangle = \langle \omega(L_\lambda) u \rangle$ because all eigenspaces of L_0 are pairwise orthogonal by Theorem 2.2.4. Therefore, if $\langle \bullet | \bullet \rangle$ is a contravariant Hermitian nonzero form on V such that $\langle v | v \rangle = 1$, we are forced to have

$$\langle L_\lambda v | u \rangle = \langle \omega(L_\lambda) u \rangle \quad \text{for a partition } \lambda \text{ and } u \in V. \quad (2.3.1)$$

Contravariance of $\langle \bullet | \bullet \rangle$ and (2.3.1) impose conditions on the highest weight (c, h) . Let $a \in U(\text{Vir})$. Then

$$\overline{\langle av \rangle} = \overline{\langle \omega(\omega(a))v \rangle} = \overline{\langle \omega(a)v | v \rangle} = \langle v | \omega(a)v \rangle = \langle av | v \rangle = \langle \omega(a)v \rangle.$$

Taking $a = C$ and $a = L_0$, we obtain $c, h \in \mathbb{R}$.

We note, however, that initially we cannot define the form using (2.3.1) for a general highest weight representation because there may be linear dependences between terms of the form $L_\lambda v$ for partitions λ , so we may ask if it is well-defined in the first place. That is only possible for $M(c, h)$, and later we will see that it is also possible for any highest weight representation.

Theorem 2.3.1 ([KRR13, Proposition 3.4]). *Let $M(c, h)$ be the Verma representation of Vir with highest weight (c, h) , where $c, h \in \mathbb{R}$. We pick a highest weight vector v , and we define the form $\langle \bullet | \bullet \rangle$ on $M(c, h)$ using equation (2.3.1). Then:*

- (i) $\langle \bullet | \bullet \rangle$ is a contravariant Hermitian form on $M(c, h)$ such that $\langle v | v \rangle = 1$.
- (ii) The eigenspaces of L_0 are pairwise orthogonal.
- (iii) $M(c, h)^\perp = J(c, h)$.

- (iv) Let V be a highest weight representation with highest weight vector v' . There is a unique contravariant form $\langle \bullet | \bullet \rangle$ on V such that $\langle v' | v' \rangle = 1$ and is defined by equation (2.3.1) with v' in place of v . This form satisfies properties (i)–(iii) of this Theorem. If we pick other highest weight vector, the resulting contravariant form is the previously defined form times a nonzero constant. In particular, for $L(c, h)$, this form is nondegenerate.

Let V be a highest weight representation with highest weight (c, h) , where $c, h \in \mathbb{R}$, and we pick a highest weight vector v . Let $\langle \bullet | \bullet \rangle$ be the contravariant form on V defined by (2.3.1). We say V is *unitary* if this form is unitary. By Theorem 2.3.1, this is independent of the choice of the highest weight vector v , and there is essentially one form. By Corollary 2.2.5, a unitary highest weight representation is necessarily irreducible, and it is of the form $L(c, h)$ for some real numbers $c, h \in \mathbb{R}$. We wish to study when is $L(c, h)$ unitary. A simple necessary condition is given by the next theorem.

Theorem 2.3.2. *If $L(c, h)$ is unitary then $c \geq 0$ and $h \geq 0$.*

Proof. We pick a highest weight vector v and assume $\langle v | v \rangle = 1$. A necessary condition for unitarity is that

$$c_n = \langle L_{-n}v | L_{-n}v \rangle > 0 \quad \text{for } n \in \mathbb{N}.$$

But contravariance and the commutation rules of Virasoro show that

$$c_n = 2nh + c(n^3 - n)/12. \quad (2.3.2)$$

Putting $n = 1$, we get $c_1 = 2h$ so that we must have $h \geq 0$. Moreover, (2.3.2) shows that c_n is dominated by cn^3 for large n , so that $c \geq 0$ is also necessary. \square

Let $M(c, h)$ be the Verma representation with highest weight (c, h) . For $n \in \mathbb{N}$, the subspace $M(c, h)_{h+n}$ is finite dimensional. Therefore, we can consider the determinant $\det_n(c, h)$ of the contravariant Hermitian form $\langle \bullet | \bullet \rangle$ restricted to $M(c, h)_{h+n}$. This is well-defined up to a nonzero constant.

Theorem 2.3.3. *The Verma representation $M(c, h)$ is irreducible if and only if for $n \in \mathbb{Z}_+$, $\det_n(c, h) \neq 0$.*

Proof. By Theorem 2.3.1, $M(c, h)$ is irreducible if and only if $M(c, h)^\perp = 0$. By Theorem 2.2.4, $M(c, h)^\perp = \bigoplus_{j \in \mathbb{N}} M(c, h)^\perp \cap M(c, h)_{h+j} = \bigoplus_{j \in \mathbb{N}} M(c, h)_{h+j}^\perp$. Therefore, $M(c, h)$ is irreducible if and only if for $j \in \mathbb{N}$, $M(c, h)_{h+j}^\perp = 0$, which is equivalent to for $n \in \mathbb{Z}_+$, $\det_n(c, h) \neq 0$. \square

Thus, to determine when $M(c, h)$ is irreducible, it is worthwhile to study the number $\det_n(c, h)$ for $n \in \mathbb{N}$. Fortunately, there is a formula for this.

Theorem 2.3.4 (Kac determinant formula [IK11, Theorem 4.2]). *For $n \in \mathbb{N}$,*

$$\det_n(c, h) = \text{constant} \cdot \prod_{\substack{k, l \in \mathbb{Z}_+ \\ k > l \\ 1 \leq kl \leq n}} \phi_{k, l}(c, h)^{p(n-kl)},$$

where

$$\phi_{k, l}(c, h) = \begin{cases} \left(h + \frac{(k^2-1)(c-13)}{24} + \frac{kl-1}{2} \right) \left(h + \frac{(l^2-1)(c-13)}{24} + \frac{kl-1}{2} \right) + \frac{(k^2-l^2)^2}{16} & \text{if } k \neq l; \\ h + \frac{(k^2-1)(c-13)}{24} + \frac{k^2-1}{2} & \text{if } k = l. \end{cases}$$

Remark 2.3.5. Using Kac determinant formula, it is possible to prove that $L(1/2, h)$ unitary implies $h = 0, 1/2$ or $1/16$ (see [KRR13, §3]). This is one of the reasons we study $L(1/2, 1/2)$ and $L(1/2, 1/16)$ in this thesis.

We will need to find the maximal proper submodules of $M(1/2, 1/2)$ and $M(1/2, 1/16)$ explicitly. Kac determinant formula also helps with this, allowing us to compute $J(c, h)$ for rational numbers c and h .

It turns out that $J(c, h)$ is generated by at most two singular vectors, which can be computed explicitly for isolated cases. It is always possible to write (nonuniquely)

$$c = \frac{(3r+2s)(3s+2r)}{rs}, h = \frac{(r+s)^2 - t^2}{4rs}$$

for some $r, s \in \mathbb{C} \setminus \{0\}$ and $t \in \mathbb{C}$. Then

$$\phi_{k,l}(c, h) = \begin{cases} \frac{(rk+sl+t)(sk+rl+t)(rk+sl-t)(sk+rl-t)}{rs} & \text{if } k \neq l; \\ \frac{(rk+sk+t)(rk+sk-t)}{4rs} & \text{if } k = l. \end{cases}$$

Therefore, to find singular vectors in $M(c, h)$, we have to study integral solutions to the linear equation $rk + sl + t = 0$. This line is real if and only if $c \leq 1$ or $c \geq 25$. Let $l_{c,h}$ denote the solutions to this linear equation.

Theorem 2.3.6 ([Ast97]). *Let c and h be real numbers such that $c \leq 1$ or $c \geq 25$. The integral points (the points in \mathbb{Z}^2) of $l_{c,h}$ determine the maximal proper subrepresentation of $M(c, h)$ completely according to the following three cases:*

Case I: *The line $l_{c,h}$ contains no integral points. In this case, $J(c, h) = 0$.*

Case II: *The line $l_{c,h}$ contains exactly one integral point (k, l) . We have three subcases:*

Subcase II₊: *The product $kl > 0$. Let u be a singular vector in $M(c, h + kl)$. Then $J(c, h) = U(\text{Vir})\{u\}$.*

Subcase II₀: *The product $kl = 0$. In this subcase, $J(c, h) = 0$.*

Subcase II₋: *The product $kl < 0$. In this subcase, $J(c, h) = 0$.*

Case III: *The line $l_{c,h}$ contains infinitely many integral points. Let $(k_1, l_1), (k_2, l_2), \dots$ be all integral points on the line $l_{c,h}$ up to equivalence relation $(k, l) \sim (k', l')$ if and only if $kl = k'l'$ and such that $kl > 0$. We ordered them in such a way that $k_i l_i < k_{i+1} l_{i+1}$ for $i \in \mathbb{Z}_+$. We have two subcases:*

Subcase $c \leq 1$: *We have three subsubcases:*

Subsubcase III₀₀⁰⁰: *Line $l_{c,h}$ intersects both axes at integral points. Let u be a singular vector in $M(c, h + k_1 l_1)$. Then $J(c, h) = U(\text{Vir})\{u\}$.*

Subsubcase III₀⁰: *Line $l_{c,h}$ intersects only one axis at an integral point. Let u be a singular vector in $M(c, h + k_1 l_1)$. Then $J(c, h) = U(\text{Vir})\{u\}$.*

Subsubcase III₋⁰: *Line $l_{c,h}$ intersects both axes at nonintegral points. Let u and w be singular vectors in $M(c, h + k_1 l_1)$ and $M(c, h + k_2 l_2)$, respectively. Then $J(c, h) = U(\text{Vir})\{u, w\}$.*

Subcase $c \geq 25$: *We have three subsubcases:*

Subsubcase III₊⁰⁰: *Line $l_{c,h}$ intersects both axes at integral points. Let u be a singular vector in $M(c, h + k_1 l_1)$. Then $J(c, h) = U(\text{Vir})\{u\}$.*

Subsubcase III₊⁰: *Line $l_{c,h}$ intersects only one axis at an integral point. Let u be a singular vector in $M(c, h + k_1 l_1)$. Then $J(c, h) = U(\text{Vir})\{u\}$.*

Subsubcase III₊⁺: *Line $l_{c,h}$ intersects both axes at nonintegral points. Let u and w be singular vectors in $M(c, h + k_1 l_1)$ and $M(c, h + k_2 l_2)$, respectively. Then $J(c, h) = U(\text{Vir})\{u, w\}$.*

Remark 2.3.7. Theorem 2.3.4 gives us an algorithm to find $J(c, h)$ for a given highest weight (c, h) , with c, h rational numbers. We merely need to find the levels at which the singular vectors that generate $J(c, h)$ lie, and then by Lemma 2.2.2, we have to solve the linear equations $L_1 u = L_2 u = 0$ assuming u lies in the right level to obtain our desired singular vector.

Remark 2.3.8. For $c \in \mathbb{C}$, $\text{Vir}^c = M(c, 0)/U(\text{Vir})\{L_{-1}|0\rangle\}$ and $\text{Vir}_c = L(c, 0)$ because by Remark 1.6.6, the ideals of Vir^c are in bijection with the submodules of $M(c, 0)/U(\text{Vir})\{L_{-1}|0\rangle\}$. If $c = c_{p,q}$ for some $p, q \geq 2$ relatively prime integers, then the singular vector of Vir^c can be computed by solving the linear equation $L_1 u = L_2 u = 0$ for $u \in \text{Vir}_{(p-1)(q-1)}^c$ (see Theorem 1.5.16, Lemma 2.2.2 and Theorem 3.3.6 ahead).

One of the objectives of this thesis is to find a PBW basis of $L(1/2, 1/2)$ and $L(1/2, 1/16)$ similar to what was done for the Ising model $\text{Vir}_{3,4} = L(1/2, 0)$ in [AVEH22]. The Vir-modules

$L(1/2, 1/2)$ and $L(1/2, 1/16)$ are not vertex algebras, but they are modules (vertex algebra modules) over both $\text{Vir}^{1/2}$ and $\text{Vir}_{3,4}$, as we will see later.

Remark 2.3.9. The irreducible highest weight representations $L(1/2, 1/2)$ and $L(1/2, 1/16)$ can be constructed explicitly as the even or odd part of some induced representations, without using Verma representations and passing to the quotient as was done here (cf. [KRR13, §3]). Moreover, $L(1/2, 1/2)$ is isomorphic to $F_{\overline{1}}$, where $F = F(\mathbb{C}a)$ is the fermionic vertex superalgebra associated to a purely odd one dimensional superspace $\mathbb{C}a$ with an antisupersymmetric form defined by $\langle a|a \rangle = 1$ (see Example 1.5.8).

3. MODULES OVER THE SIMPLE VIRASORO VERTEX ALGEBRAS

In this section, the objective is to describe the irreducible admissible modules over Vir^c and Vir_c . To do that, we first study the Zhu algebra $\text{Zhu}(V)$ of a \mathbb{Z} -graded vertex algebra V introduced in [Zhu96]. This naturally leads to a functor $\Omega : V\text{-Mod} \rightarrow \text{Zhu}(V)\text{-Mod}$. Next, we construct an inverse $L : \text{Zhu}(V)\text{-Mod} \rightarrow \{\text{admissible } V\text{-modules}\}$ of the functor Ω using the associated Lie algebras of a vertex algebra constructed in §1.7. We end this section by describing the irreducible admissible modules over Vir^c and Vir_c explicitly in terms of the irreducible highest weight representations $L(c, h)$ for certain values of h .

3.1. The Zhu algebra. Let V be a \mathbb{Z} -graded vertex algebra with Hamiltonian H . First, we define two operations on V . For $a, b \in V$ with a homogeneous, we set:

$$\begin{aligned} a \circ b &= \text{res}_z \left(\frac{(1+z)^{\Delta_a}}{z^2} Y(a, z)b \right) \in V, \\ a * b &= \text{res}_z \left(\frac{(1+z)^{\Delta_a}}{z} Y(a, z)b \right) \in V. \end{aligned}$$

We extend these operations linearly to obtain bilinear products on V , and we define

$$O(V) = \text{span}\{a \circ b \mid a, b \in V\}.$$

Lemma 3.1.1. *For $a, b \in V$:*

- (i) $a \circ |0\rangle = (T + H)a$;
- (ii) $a * |0\rangle = a$;
- (iii) $(T + H)V \subseteq O(V)$.

Proof.

- (i) We just expand the definition of $a \circ |0\rangle$ recalling that $Ta = a_{(-2)}|0\rangle$ and $Ha = \Delta_a a$ when a is homogeneous.
- (ii) Again, we just expand the definition recalling that $a = a_{(-1)}|0\rangle$.
- (iii) This follows from (i). □

Lemma 3.1.2 ([DLM98, Lemma 2.2]).

- (i) *We assume $a \in V$ is homogeneous, $b \in V$ and $m \geq n \geq 0$. Then*

$$\text{res}_z \left(\frac{(1+z)^{\Delta_a+n}}{z^{m+2}} Y(a, z)b \right) \in O(V).$$

- (ii) *We assume $a, b \in V$ are homogeneous. Then*

$$a * b - \text{res}_z \left(\frac{(1+z)^{\Delta_b-1}}{z} Y(b, z)a \right) \in O(V).$$

- (iii) *We assume $a \in V$ is homogeneous. Then*

$$a * b - b * a - \text{res}_z((1+z)^{\Delta_a-1} Y(a, z)b) \in O(V).$$

Theorem 3.1.3 ([DLM98, Proposition 2.3 and Theorem 2.4]). *Let V be a \mathbb{Z} -graded vertex algebra. Then $O(V)$ is a two-sided ideal of V under the operation $*$, and $V/O(V)$ becomes an associative algebra with unit $|0\rangle + O(V)$.*

The quotient

$$\text{Zhu}(V) = V/O(V)$$

is called the *Zhu algebra of V* . The Zhu algebra is closely related to the Borcherds Lie algebra, as the following theorem shows.

Theorem 3.1.4. *Let V be a \mathbb{Z} -graded vertex algebra. The following map*

$$\begin{aligned} [V]_0 &\rightarrow [\text{Zhu}(V)], \\ a(\Delta_a - 1) &\mapsto a + O(V) \quad \text{for } a \in V \text{ homogeneous} \end{aligned}$$

defines an epimorphism of Lie algebras.

Proof. We know $\text{Lie}_0(V)$ is isomorphic to $[V]_0$ (see §1.7) and by Lemma 3.1.1(iii), there is a natural linear epimorphism

$$\begin{aligned} \text{Lie}_0(V) &\rightarrow [\text{Zhu}(V)], \\ a + (T + H)V &\mapsto a + O(V). \end{aligned}$$

It remains to show this is a Lie algebra homomorphism. Let $a, b \in V$ with a homogeneous. By Lemma 3.1.2(iii) and the definition of the bracket of $\text{Lie}_0(V)$, we have

$$\begin{aligned} [a + (T + H)V, b + (T + H)V] &= \sum_{j \in \mathbb{N}} \binom{\Delta_a - 1}{j} a_{(j)}b + (T + H)V \\ &\mapsto \sum_{j \in \mathbb{N}} \binom{\Delta_a - 1}{j} a_{(j)}b + O(V) \\ &= \text{res}_z((1 + z)^{\Delta_a - 1} Y(a, z)b) + O(V) \\ &= a * b - b * a + O(V) \\ &= [a + O(V), b + O(V)]. \quad \square \end{aligned}$$

We still assume V is a \mathbb{Z} -graded vertex algebra. We recall from §1.7 that $[V]$ is a \mathbb{Z} -graded Lie algebra with a triangular decomposition $[V] = [V]_+ \oplus [V]_0 \oplus [V]_-$. Let W be a $[V]$ -module. We denote by $\Omega(W)$ the subspace of *lowest weight vectors of W* with respect to the triangular decomposition of $[V]$, that is,

$$\Omega(W) = \{u \in W \mid [V]_- u = 0\}.$$

If M is a V -module, then by Lemma 1.7.3, $[M]$ is a $[V]$ -module, and we define the subspace of *lowest weight vectors of M* as

$$\Omega(M) = \Omega([M]).$$

By (1.5.4), we have

$$\Omega(M) = \{u \in M \mid \text{for } a \in V \text{ homogeneous and } n \in \mathbb{Z}_+, a_n u = 0\}.$$

One of the most important properties of the Zhu algebra is that it simplifies the theory of representations of vertex algebras, as the following theorem shows.

Theorem 3.1.5 ([DLM98, Theorem 5.3]). *Let V be a \mathbb{Z} -graded vertex algebra, and let M be a V -module. Then the following map*

$$\begin{aligned} \text{Zhu}(V) &\rightarrow \text{End}(\Omega(M)), \\ a + O(V) &\mapsto a_0^M = a_{(\Delta_a - 1)}^M \quad \text{for } a \in V \text{ homogeneous} \end{aligned}$$

defines a $\text{Zhu}(V)$ -module structure on $\Omega(M)$.

Because our constructions are natural, it is evident that we have constructed a functor

$$\Omega : V\text{-Mod} \rightarrow \text{Zhu}(V)\text{-Mod}.$$

To be more precise, if $f : M_1 \rightarrow M_2$ is a homomorphism of V -modules, then $[f] : [M_1] \rightarrow [M_2]$ is a homomorphism of $[V]$ -modules. This implies that $\Omega(M_1) \subseteq \Omega(M_2)$. Therefore, we can define $\Omega(f) = f|_{\Omega(M_1)} : \Omega(M_1) \rightarrow \Omega(M_2)$, which is evidently a $\text{Zhu}(V)$ -module homomorphism.

Example 3.1.6. If $V = \text{Vir}^c$, and M is a Vir^c -module, then

$$\Omega(M) = \{u \in M \mid \text{for } n \in \mathbb{Z}_+, L_n u = 0\}$$

is the subspace of singular vectors of M .

Theorem 3.1.7 ([DLM98, Proposition 5.4]). *Let V be a \mathbb{Z} -graded vertex algebra, and let M be an irreducible admissible V -module. Then:*

- (i) $\Omega(M) = M(0)$;
- (ii) $\Omega(M)$ is an irreducible $\text{Zhu}(V)$ -module.

3.2. The inverse of the lowest weight subspace functor. Our objective is to construct an inverse of some sort of the functor Ω . However, it turns out that this is not possible, and we have to restrict Ω to the full subcategory of completely reducible objects.

Given the $\text{Zhu}(V)$ -module U , it is a fortiori a module over the Lie algebra $[\text{Zhu}(V)]$. Thanks to Theorem 3.1.4, we can consider U as a module over the Lie algebra $[V]_0$, and then U is a module over the Lie algebra $[V]_- \oplus [V]_0$ by letting $[V]_-$ act trivially. Thus, we define

$$M(U) = \text{Ind}_{[V]_- \oplus [V]_0}^{[V]}(U) = U([V]) \otimes_{U([V]_- \oplus [V]_0)} U.$$

We recall from §1.7 that H is a derivation of $[V]$. The operator $DU(H) \in \text{End}(U([V]))$ is diagonalizable and makes $U([V])$ into a \mathbb{Z} -graded associative algebra. This means

$$U([V])_m U([V])_n \subseteq U([V])_{m+n} \quad \text{for } m, n \in \mathbb{Z},$$

where the subindex denotes the respective eigenspace, as usual. We consider the operator

$$H^{M(U)} = DU(H) \otimes \text{Id}_U \in \text{End}(M(U)).$$

We can verify that $H^{M(U)}$ is diagonalizable and makes $M(U)$ into a \mathbb{Z} -graded $[V]$ -module with grading

$$M(U) = \bigoplus_{n \in \mathbb{Z}} M(U)_n$$

such that $M(U)_n = 0$ if $n < 0$. This means that

$$[V]_m M(U)_n \subseteq M(U)_{m+n} \quad \text{for } m, n \in \mathbb{Z}.$$

The PBW theorem implies that

$$M(U)_n = U([V])_n U$$

and, in particular,

$$M(U)_0 = U.$$

Taking a hint from Lemma 1.7.3, we define

$$Y^{M(U)}(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-1-n} \in \mathcal{F}(M(U)) \quad \text{for } a \in V,$$

where $a(n) \in [V]$ acts on $M(U)$ by left multiplication. We want $M(U)$ becoming an admissible V -module by setting

$$M(U)(n) = M(U)_n \quad \text{for } n \in \mathbb{Z}.$$

However, we are not done yet because all the conditions in Proposition 1.6.4 are met except (1.6.3). Therefore, we need to divide by the respective relations.

In the following, we let $U^* = \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$ and extend U^* to $M(U)$ by letting U^* annihilate $\bigoplus_{n \in \mathbb{Z}_+} M(U)(n)$. We now define:

$$J = \{u \in M(U) \mid \text{for } u' \in U^* \text{ and } x \in U([V]), \langle u', xu \rangle = 0\},$$

$$L(U) = M(U)/J,$$

and let the grading of $L(U)$ and $Y^{L(U)}$ be inherited from $M(U)$ in the obvious way.

Theorem 3.2.1 ([DLM98, Theorem 6.3]). *With notation as above, $L(U)$ is an admissible V -module, and we have defined a functor*

$$L : \text{Zhu}(V)\text{-Mod} \rightarrow \{\text{admissible } V\text{-modules}\}$$

such that $\Omega \circ L$ is naturally equivalent to the identity.

Remark 3.2.2. Since $U([V])U = M(U)$, we see that $U([V])U = L(U)$. This implies that $L(U)$ is strongly generated by U as a vertex algebra.

At this point, we have a pair of functors Ω, L defined on appropriate module categories

$$\text{Zhu}(V)\text{-Mod} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{\Omega} \end{array} \{\text{admissible } V\text{-modules}\}$$

Although $\Omega \circ L$ is equivalent to the identity, one cannot expect that $L \circ \Omega$ is also equivalent to the identity in general. This is essentially because there are examples of \mathbb{Z} -graded vertex algebras V for which the category of admissible V -modules contains objects which are not completely reducible.

Lemma 3.2.3 ([DLM98, Lemma 7.1]). *Suppose that U is an irreducible $\text{Zhu}(V)$ -module. Then $L(U)$ is an irreducible admissible V -module.*

Theorem 3.2.4 ([DLM98, Theorem 7.2]). *L and Ω are equivalences when restricted to the full subcategories of completely reducible $\text{Zhu}(V)$ -modules and completely reducible admissible V -modules respectively. In particular, L and Ω induces mutually inverse bijections on the isomorphism classes of irreducible objects in the category of $\text{Zhu}(V)$ -modules and admissible V -modules respectively.*

Remark 3.2.5. The functor L plays well with vertex algebra homomorphism. To elaborate a bit more, let $f : V_1 \rightarrow V_2$ be a homomorphism of \mathbb{Z} -graded vertex algebras. We naturally define the functors $\circ f : V_2\text{-Mod} \rightarrow V_1\text{-Mod}$ and $\circ \text{Zhu}(f) : \text{Zhu}(V_2)\text{-Mod} \rightarrow \text{Zhu}(V_1)\text{-Mod}$. Then the following diagram of functors commutes

$$\begin{array}{ccc} \text{Zhu}(V_2)\text{-Mod} & \xrightarrow{L_{V_2}} & \{\text{completely reducible admissible } V_2\text{-modules}\} \\ \downarrow \circ \text{Zhu}(f) & & \downarrow \circ f \\ \text{Zhu}(V_1)\text{-Mod} & \xrightarrow{L_{V_1}} & \{\text{completely reducible admissible } V_1\text{-modules}\} \end{array}$$

By commuting, we mean that the functors $\circ f \circ L_{V_2}$ and $L_{V_1} \circ (\circ \text{Zhu}(f))$ are isomorphic.

Remark 3.2.6. In some articles such as [DSK06, §2], the functor Ω is defined as $\Omega(M) = M(0)$, where M is an admissible V -module. There are some difficulties with this approach:

- (i) This is not even a functor. For example, if $V = \text{Vir}^{1/2}$, $N = U(\text{Vir})\{a_{3,4}\}$ and $\text{inc} : N \hookrightarrow V$ is the inclusion, there is no sensible way of defining $\Omega(\text{inc})$ if $\Omega(M) = M(0)$. On the other hand, if we use the definition of Ω given in [DLM98], the problem disappears (see Example 3.1.6).
- (ii) It is slightly less general because [DLM98] defines $\Omega(M)$ when M is any V -module (called weak V -module there).

Remark 3.2.7. It is not necessary to consider conformal vertex algebras to do the theory of the Zhu algebra, \mathbb{Z} -graded vertex algebras are enough.

3.3. Modules over the Virasoro vertex algebras. We wish to determine all irreducible admissible modules over both Vir^c and Vir_c . This is why we studied the Zhu algebra. We start with Vir^c , which is easier.

Theorem 3.3.1 ([Wan93]). *For $c \in \mathbb{C}$, we have the following isomorphism of algebras*

$$\begin{aligned} \text{Zhu}(\text{Vir}^c) &\xrightarrow{\sim} \mathbb{C}[x], \\ L_{-2}|0\rangle + O(\text{Vir}^c) &\mapsto x. \end{aligned}$$

Theorem 3.3.2. *For $c \in \mathbb{C}$, the irreducible admissible modules over Vir^c are $L(c, h)$ for $h \in \mathbb{C}$.*

Proof. The irreducible modules over $\mathbb{C}[x]$ are $\mathbb{C}_h = \mathbb{C}$ for $h \in \mathbb{C}$, where x acts by multiplication by h . By Theorem 3.2.4 and Theorem 3.3.1, the irreducible admissible modules over Vir^c are $L(\mathbb{C}_h)$ for $h \in \mathbb{C}$. By Theorem 1.6.8, for $h \in \mathbb{C}$, $L(\mathbb{C}_h)$ is a smooth Vir -module of central charge c . Also, $L_0 = (L_{-2}|0\rangle)_{(1)}^{L(\mathbb{C}_h)}$ acts by multiplication by h on \mathbb{C}_h and by Remark 3.2.2, $L(\mathbb{C}_h)$ is strongly generated by \mathbb{C}_h as a vertex algebra. Therefore, $L(\mathbb{C}_h)$ is an irreducible highest weight representation of Vir , i.e., $L(\mathbb{C}_h) = L(c, h)$. \square

We now move on to Vir_c . If c is not of the form $c_{p,q}$ for some $p, q \geq 2$ relatively prime integers, then $\text{Vir}_c = \text{Vir}^c$ by Theorem 1.5.16, and we have already solved the problem.

Theorem 3.3.3 ([Wan93]). *We set $c = c_{p,q}$ for some $p, q \geq 2$ relatively prime integers. We have the following isomorphism of algebras*

$$\begin{aligned} \text{Zhu}(\text{Vir}_c) &\xrightarrow{\sim} \mathbb{C}[x]/(G_{p,q}(x)), \\ L_{-2}|0\rangle + O(\text{Vir}_c) &\mapsto x + (G_{p,q}(x)), \end{aligned}$$

where $G_{p,q}(x)^2 = \prod_{m=1}^{p-1} \prod_{n=1}^{q-1} (x - h_{m,n})$, and $h_{m,n} \in \mathbb{Q}$ is defined by

$$h_{m,n} = \frac{(np - mq)^2 - (p - q)^2}{4pq}. \quad (3.3.1)$$

Theorem 3.3.4. *We set $c = c_{p,q}$ for some $p, q \geq 2$ relatively prime integers. Then the irreducible admissible modules over Vir_c are $L(c, h_{m,n})$ for integers m, n such that $0 < m < p$ and $0 < n < q$. Let $Y_{\text{Vir}^c}^{L(c, h_{m,n})} : \text{Vir}^c \rightarrow \mathcal{F}(L(c, h_{m,n}))$ be the state-field correspondence of $L(c, h_{m,n})$ as a module over Vir^c , and let $Y_{\text{Vir}_c}^{L(c, h_{m,n})} : \text{Vir}_c \rightarrow \mathcal{F}(L(c, h_{m,n}))$ be the state-field correspondence of $L(c, h_{m,n})$ as a module over Vir_c . Then $Y_{\text{Vir}^c}^{L(c, h_{m,n})}$ factors through $Y_{\text{Vir}_c}^{L(c, h_{m,n})}$, i.e., the following diagram commutes*

$$\begin{array}{ccc} \text{Vir}^c & \longrightarrow & \text{Vir}_c \\ & \searrow Y_{\text{Vir}^c}^{L(c, h_{m,n})} & \downarrow Y_{\text{Vir}_c}^{L(c, h_{m,n})} \\ & & \mathcal{F}(L(c, h_{m,n})) \end{array}$$

where the horizontal arrow is the quotient map.

Proof. The irreducible modules over $\mathbb{C}[x]/(G_{p,q}(x))$ are $\mathbb{C}_{h_{m,n}} = \mathbb{C}$ for integers m, n such that $0 < m < p$ and $0 < n < q$, where $x + (G_{p,q}(x))$ acts by multiplication by $h_{m,n}$. We conclude from Theorem 3.3.2 that the Vir_c -module corresponding to $\mathbb{C}_{h_{m,n}}$ is $L(c, h_{m,n})$, and the commutativity of the diagram follows from Remark 3.2.5. \square

We are also interested in the refined character of these irreducible modules, as defined in §4 ahead. The ordinary character is fortunately already known.

Theorem 3.3.5 ([FF84]). *Let $p, q \geq 2$ be relatively prime integers, and let m, n be integers such that $0 < m < p$ and $0 < n < q$. Then*

$$\begin{aligned} \text{ch}_{L(c_{p,q}, h_{m,n})}(q) &= \frac{1}{(q)_\infty} \sum_{k \in \mathbb{Z}} q^{\frac{(2kpq + mq - np)^2 - (p-q)^2}{4pq}} - q^{\frac{(2kpq + mq + np)^2 - (p-q)^2}{4pq}} \\ &= \frac{q^{h_{m,n}}}{(q)_\infty} \sum_{k \in \mathbb{Z}} q^{k^2 pq + k(mq - np)} - q^{k^2 pq + k(mq + np) + mn}. \end{aligned}$$

It follows from Theorem 3.3.4 that the irreducible admissible modules over the Ising model $\text{Vir}_{3,4}$ are $\text{Vir}_{3,4} = L(1/2, 0)$, $L(1/2, 1/2)$ and $L(1/2, 1/16)$.

Theorem 3.3.6. *We set $c = c_{p,q}$ for some $p, q \geq 2$ relatively prime integers, and we consider the rational number $h_{m,n}$ for integers m, n such that $0 < m < p$ and $0 < n < q$ as defined in (3.3.1). Then the maximal proper subrepresentation $J(c, h_{m,n})$ of $M(c, h_{m,n})$ is generated by two singular vectors u and w in $M(c, h_{m,n})_{h_{m,n} + mn}$ and $M(c, h_{m,n})_{h_{m,n} + (p-m)(q-n)}$, respectively.*

Proof. Solving the equations

$$c = \frac{(3r+2s)(3s+2r)}{rs}, h = \frac{(r+s)^2 - t^2}{4rs}$$

for $r, s \in \mathbb{C} \setminus \{0\}$ and $t \in \mathbb{C}$, we obtain

$$r = p, s = -q, t = np - mq.$$

The solutions $(k, l) \in \mathbb{Z}^2$ of the linear equation $pk - ql + np - mq = 0$ are

$$\{(n + qj, m + pj) \mid j \in \mathbb{Z}\}. \quad (3.3.2)$$

We are in Subsubcase III₋ of Theorem 2.3.6, and we see that the first two elements of the solutions in (3.3.2) are (n, m) and $(n - q, m - p)$ (not necessarily in that order). \square

We pick a highest weight (c, h) . For $\tau \in \mathbb{H}$, we set $q = e^{2\pi i\tau}$ and define the *normalized character of $L(c, h)$* by

$$\chi_{L(c,h)}(\tau) = q^{-c/24} \text{ch}_{L(c,h)}(q) \in \mathbb{C}.$$

We express $\chi_{L(c,h)}(\tau)$ by using the *Dedekind η -function* $\eta(\tau)$ defined by

$$\eta(\tau) = q^{1/24} \prod_{n \in \mathbb{Z}_+} (1 - q^n),$$

and the *classical theta function* $\Theta_{n,m}(\tau)$ for $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}/2m\mathbb{Z}$ defined by

$$\Theta_{n,m}(\tau) = \sum_{k \in \mathbb{Z}} q^{m(k + \frac{n}{2m})^2}.$$

Theorem 3.3.7 ([IK11, Corollary 6.1]). *Let $p, q \geq 2$ be relatively prime integers, and let m, n be integers such that $0 < m < p$ and $0 < n < q$. Then*

$$\chi_{m,n}(\tau) = \chi_{L(c_{p,q}, h_{m,n})}(\tau) = (\Theta_{mq-np, pq}(\tau) - \Theta_{mq+np, pq}(\tau))\eta(\tau)^{-1}.$$

The modular group $SL_2(\mathbb{Z})$ acts on the upper half-plane \mathbb{H} by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Theorem 3.3.8 ([IK11, Proposition 6.3] and [Zhu96]). *Let us take integers p, q, m, n as in Theorem 3.3.7. Then:*

$$\begin{aligned} \chi_{m,n}(\tau + 1) &= e^{\{\frac{(mq-np)^2}{2pq} - \frac{1}{12}\}\pi i} \chi_{m,n}(\tau), \\ \chi_{m,n}\left(-\frac{1}{\tau}\right) &= \sum_{(m', n') \in K_{p,q}} S_{(m,n), (m', n')} \chi_{m', n'}(\tau), \end{aligned}$$

where:

$$\begin{aligned} S_{(m,n), (m', n')} &= \sqrt{\frac{8}{pq}} (-1)^{(m+n)(m'+n')} \sin\left(\frac{\pi mm'}{p}(p-q)\right) \sin\left(\frac{\pi nn'}{q}(p-q)\right), \\ K_{p,q} &= \{(m, n) \in \mathbb{Z}^2 \mid 0 < m < p, 0 < n < q, mq + np \leq pq\}. \end{aligned}$$

4. VERTEX POISSON ALGEBRAS AND FILTRATIONS OF VERTEX ALGEBRAS

In this section, we review the theory of filtrations for vertex algebras. We can consider either the Li filtration $(F_p V)_{p \in \mathbb{Z}}$ or the standard filtration $(G^p V)_{p \in \mathbb{Z}}$. Both of these filtrations lead to a vertex Poisson algebra, namely $\text{gr}_F(V)$ and $\text{gr}^G(V)$, as was shown by Haisheng Li in [Li05] and [Li04]. In [Ara12], it was shown by Arakawa that the resulting vertex Poisson algebra is the same. Then, we introduce the Li filtration $(F_p M)_{p \in \mathbb{Z}}$ and the standard filtration $(G^p M)_{p \in \mathbb{Z}}$ for modules. This leads to modules $\text{gr}_F(M)$ and $\text{gr}^G(M)$ that are again isomorphic. Finally, we introduce the Zhu C_2 -algebra and derive consequences of the C_2 -cofiniteness condition.

4.1. Vertex Lie superalgebras and their modules. Let V be a vector space. Given a formal distribution $f(x_1, \dots, x_n) \in V[[x_1^{\pm 1}, \dots, x_n^{\pm 1}]]$, we can write it as

$$f(x_1, \dots, x_n) = \sum_{m_1, \dots, m_n \in \mathbb{Z}} f_{(m_1, \dots, m_n)} x_1^{-m_1-1} \dots x_n^{-m_n-1}.$$

We set

$$\text{sing}(f(x_1, \dots, x_n)) = \sum_{m_1, \dots, m_n \in \mathbb{N}} f_{(m_1, \dots, m_n)} x_1^{-m_1-1} \dots x_n^{-m_n-1}.$$

A *vertex Lie superalgebra* is the data (V, T, Y_-) , where V is a vector superspace, $Y_- : V \rightarrow \mathcal{F}(V)$ is a linear and parity preserving map such that $Y_-(a, z) = \text{sing}(Y_-(a, z))$ (i.e., $Y_- : V \rightarrow \text{Hom}(V, z^{-1}V[z^{-1}])$) for $a \in V$ and $T \in \text{End}(V)$. The data must satisfy the following axioms for $a, b \in V$:

- (i) $Y_-(Ta, z) = \partial_z Y_-(a, z)$;
- (ii) $Y_-(a, z)b = \text{sing}(p(a, b)e^{Tz}Y_-(b, -z)a)$;
- (iii) $[Y_-(a, z), Y_-(b, w)] = \text{sing}(\sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} Y_-(a_{(j)}b, w))$, where $Y_-(a, z) = \sum_{n \in \mathbb{N}} a_{(n)}z^{-n-1}$, $a_{(n)} \in \text{End}(V)$.

Remark 4.1.1. Axiom (i) of vertex Lie superalgebras implies that $T \in \text{End}(V)_{\bar{0}}$.

Concepts like homomorphisms, vertex Lie subalgebras and ideals are defined in the usual way. We obtain the category of vertex Lie superalgebras.

Let V be vertex Lie superalgebra. We can make V into a $C[\partial]$ -module by declaring $\partial = T$. Then axioms (i) and (ii) are the respective axioms of Lie conformal superalgebras given in §1.2. We take $a, b \in V$. Since

$$\sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} Y_-(a_{(j)}b, w) = \sum_{m, n \in \mathbb{Z}} \left(\sum_{\substack{j, k \in \mathbb{N} \\ j+k=m+n}} \binom{m}{j} (a_{(j)}b)_{(k)} \right) z^{-m-1} w^{-n-1},$$

we have

$$\text{sing} \left(\sum_{j \in \mathbb{N}} \frac{\partial_w^j \delta(z, w)}{j!} Y_-(a_{(j)}b, w) \right) = \sum_{m, n \in \mathbb{N}} \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)_{(m+n-j)} z^{-m-1} w^{-n-1}.$$

On the other hand,

$$[Y_-(a, z), Y_-(b, w)] = \sum_{m, n \in \mathbb{N}} [a_{(m)}, b_{(n)}] z^{-m-1} w^{-n-1}.$$

Therefore, axiom (iii) of vertex Lie superalgebras is equivalent to

$$[a_{(m)}, b_{(n)}] = \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)_{(m+n-j)} \quad \text{for } a, b \in V \text{ and } m, n \in \mathbb{N}.$$

This is just axiom (iii) of Lie conformal superalgebras. Thus, the category of vertex Lie superalgebras and the category of Lie conformal superalgebras are isomorphic. Consequently, we have three equivalent concepts: vertex Lie superalgebras, Lie conformal superalgebras and regular formal distribution Lie superalgebras.

Theorem 4.1.2. *Let V be a vertex Lie superalgebra. For $a, b \in V$ and $m, j \in \mathbb{N}$:*

- (i) $[T, Y_-(a, z)] = Y_-(Ta, z) = \partial_z Y_-(a, z)$;
- (ii) $(a_{(j)}b)_{(m)} = \sum_{k=0}^j \binom{j}{k} (-1)^k [a_{(j-k)}, b_{(m+k)}]$.

Proof.

- (i) This is Proposition 1.2.8.
- (ii) This is Theorem 1.2.13. □

Let V be a vertex Lie superalgebra. A *Hamiltonian operator* of V is a diagonalizable operator $H \in \text{End}(V)$ such that:

- (i) $[H, T] = T$;
- (ii) $[H, Y_-(a, z)] = z\partial_z Y_-(a, z) + Y_-(Ha, z)$ for $a \in V$.

A vertex Lie superalgebra with a Hamiltonian is called *graded*. The *grading* of V is the eigenspace decomposition of H

$$V = \bigoplus_{\Delta \in \mathbb{C}} V_{\Delta}.$$

A *module over V* is a vector superspace M together with a linear and parity preserving map $Y_-^M : V \rightarrow \text{Hom}(M, z^{-1}M[[z^{-1}]])$, written as $Y_-^M(a, z) = \sum_{n \in \mathbb{N}} a_{(n)}^M z^{-n-1}$, $a_{(n)}^M \in \text{End}(M)$, such that for $a, b \in V$ and $m, n \in \mathbb{N}$:

- (i) $(Ta)_{(n)}^M = -na_{(n-1)}^M$;
- (ii) $[a_{(m)}^M, b_{(n)}^M] = \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)_{(m+n-j)}^M$.

Concepts like homomorphisms and vertex Lie submodules are defined in the usual way. Given a vertex Lie superalgebra V , we obtain the abelian category $V\text{-Mod}$ of modules over V .

Let M be a module over a graded vertex Lie superalgebra V with Hamiltonian H . A *Hamiltonian operator* of M is a diagonalizable operator $H^M \in \text{End}(M)$ such that

$$[H^M, Y_-^M(a, z)] = z\partial_z Y_-^M(a, z) + Y_-^M(Ha, z) \quad \text{for } a \in V.$$

A V -module together with a Hamiltonian is called *graded*. The *grading* of M is the eigenspace decomposition of H^M

$$M = \bigoplus_{\Delta \in \mathbb{C}} M_{\Delta}.$$

Let V be a vertex Lie superalgebra, and let M be a V -module. We say $T^M \in \text{End}(M)$ is a *differential* of M if

$$[T^M, Y_-^M(a, z)] = Y_-^M(Ta, z) \quad \text{for } a \in V.$$

A *differential V -module* is a module equipped with a differential.

4.2. Vertex Poisson algebras and their modules. A *vertex Poisson algebra* is the data consisting of three elements (V, T, Y_-) such that:

- (i) (V, T) is a differential commutative associative algebra with unit 1;
- (ii) (V, T, Y_-) is a vertex Lie algebra;
- (iii) The left Leibniz rule holds

$$Y_-(a, z)(bc) = (Y_-(a, z)b)c + b(Y_-(a, z)c) \quad \text{for } a, b, c \in V.$$

The left Leibniz rule is equivalent to

$$a_{(n)}(bc) = (a_{(n)}b)c + b(a_{(n)}c) \quad \text{for } a, b, c \in V \text{ and } n \in \mathbb{N}.$$

The left Leibniz rule implies that $a_{(n)} \in \text{Der}(A)$ for $a \in A$ and $n \in \mathbb{N}$. Therefore, we have

$$Y_-(a, z) \in z^{-1} \text{Der}(A)[[z^{-1}]] \quad \text{for } a \in A.$$

In particular, $a_{(n)}1 = 0$ for $a \in A$ and $n \in \mathbb{N}$. Therefore,

$$Y_-(a, z)1 = 0 \quad \text{for } a \in A.$$

By skewsymmetry,

$$Y_-(1, z) = 0.$$

Theorem 4.2.1 (Right Leibniz rule). *Let V be a vertex Poisson algebra. For $a, b, c \in V$ and $n \in \mathbb{N}$,*

$$(ab)_{(n)}c = \sum_{j \in \mathbb{N}} \left(\frac{T^j a}{j!} b_{(n+j)}c + \frac{T^j b}{j!} a_{(n+j)}c \right).$$

Proof. This follows from skewsymmetry and the left Leibniz rule. □

Let V_1 and V_2 be vertex Poisson algebras. A *vertex Poisson algebra homomorphism* $f : V_1 \rightarrow V_2$ is simultaneously a differential algebra and a vertex Lie algebra homomorphism. We obtain the category of vertex Poisson algebras.

Let V be a vertex Poisson algebra. A *module over V* is a module (M, Y_-^M) over V as a vertex Lie algebra and a module over V as a commutative associative algebra such that for $a, b \in V$, $u \in M$ and $n \in \mathbb{N}$:

- (i) (Left Leibniz rule) $a_{(n)}^M(bu) = (a_{(n)}b)u + b(a_{(n)}^M u)$;
- (ii) (Right Leibniz rule) $(ab)_{(n)}^M u = \sum_{j \in \mathbb{N}} \left(\frac{T^j a}{j!} b_{(n+j)}^M u + \frac{T^j b}{j!} a_{(n+j)}^M u \right)$.

Concepts like homomorphisms and vertex Poisson submodules are defined in the usual way. Given a vertex Poisson algebra V , we obtain the abelian category $V\text{-Mod}$ of modules over V .

Remark 4.2.2. We probably cannot deduce the right Leibniz rule from the left Leibniz rule (c.f. Remark B.3)

Proposition 4.2.3 ([Ara12, Proposition 2.3.1] and Appendix C). *Let R be a (graded) Poisson algebra. Then there is a unique (graded) vertex Poisson algebra structure on JR such that*

$$a_{(n)}b = \delta_{n,0}\{a, b\} \quad \text{for } a, b \in R \text{ and } n \in \mathbb{N}.$$

The (graded) vertex Poisson algebra structure on JR given in Proposition 4.2.3 for a (graded) Poisson algebra R will be called the *level 0 vertex Poisson algebra structure of JR* .

If R is a (graded) Poisson algebra, and M is a (graded) R -module, then we can verify that $JR \otimes_R M$ is a (graded) JR -module by defining the Poisson structure as

$$a_{(n)}(b \otimes u) = (a_{(n)}b) \otimes u + \delta_{n,0}b \otimes \{a, u\} \quad \text{for } a \in R, b \in JR, u \in M \text{ and } n \in \mathbb{N}$$

and $H^{JR \otimes_R M}$ as in Appendix C.

We have a natural (graded) inclusion $\text{inc} : M \hookrightarrow JR \otimes_R M$, which is a (graded) R -module homomorphism. The (graded) JR -module M together with the (graded) R -module inclusion $\text{inc} : M \hookrightarrow JR \otimes_R M$ satisfy a universal property similar to that of $\text{inc} : R \hookrightarrow JR$ (see Appendix C), as we now show.

Let N be a (graded) module over the (graded) vertex Poisson algebra JR , and let $f : M \rightarrow N$ be a (graded) homomorphism of modules over the (graded) Poisson algebra R . Thus, N is a module over the (graded) algebra JR and in particular, a module over the (graded) algebra R because we have an inclusion $\text{inc} : R \hookrightarrow JR$. Also, f can be considered as a (graded) homomorphism of (graded) modules over the (graded) algebra R . Therefore, there is a unique (graded) homomorphism $\bar{f} : JR \otimes_R M \rightarrow N$ of (graded) modules over the (graded) algebra JR such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\text{inc}} & JR \otimes_R M \\ & \searrow f & \downarrow \bar{f} \\ & & N \end{array}$$

Proposition 4.2.4. *The homomorphism \bar{f} as defined above is a (graded) JR -module homomorphism.*

Proof. We just expand the definitions and verify the axioms of (graded) modules over (graded) vertex Poisson algebras. \square

We have constructed a functor

$$JR \otimes_R \bullet : \{(\text{graded}) R\text{-modules}\} \rightarrow \{(\text{graded}) JR\text{-modules}\},$$

which is left adjoint to the forgetful functor $\{(\text{graded}) JR\text{-modules}\} \rightarrow \{(\text{graded}) R\text{-modules}\}$, i.e., for $M \in \{(\text{graded}) R\text{-modules}\}$ and $N \in \{(\text{graded}) JR\text{-modules}\}$, we have a natural isomorphism

$$\text{Hom}_{\{(\text{graded}) JR\text{-modules}\}}(JR \otimes_R M, N) \cong \text{Hom}_{\{(\text{graded}) R\text{-modules}\}}(M, N).$$

4.3. Filtrations of vertex algebras. Let V be a vertex algebra, and let $(a^i)_{i \in I}$ be a family of strong generators of V . For $p \in \mathbb{Z}$, we set

$$F_p V = \text{span}\{a_{(-n_1-1)}^{i_1} \cdots a_{(-n_s-1)}^{i_s} | 0\rangle \mid s, n_1, \dots, n_s \in \mathbb{N}, i_1, \dots, i_s \in I, n_1 + \cdots + n_s \geq p\}.$$

Proposition 4.3.1 ([Li05]). *The filtration $(F_p V)_{p \in \mathbb{Z}}$ satisfies:*

- (i) $F_p V = V$ for $p \leq 0$;
- (ii) $|0\rangle \in F_0 V \supseteq F_1 V \supseteq \cdots$;
- (iii) $T(F_p V) \subseteq F_{p+1} V$ for $p \in \mathbb{Z}$;
- (iv) $a_{(n)} F_q V \subseteq F_{p+q-n} V$ for $p, q \in \mathbb{Z}$, $a \in F_p V$ and $n \in \mathbb{Z}$;
- (v) $a_{(n)} F_q V \subseteq F_{p+q-n} V$ for $p, q \in \mathbb{Z}$, $a \in F_p V$ and $n \in \mathbb{N}$.

Let

$$\text{gr}_F(V) = \bigoplus_{p \in \mathbb{N}} F_p V / F_{p+1} V$$

be the associated graded vector space. By [Li05], the vector space $\text{gr}_F(V)$ is a vertex Poisson algebra with operations given as follows. For $p, q \in \mathbb{N}$, $a \in F_p V$ and $b \in F_q V$, we set:

$$\begin{aligned} \sigma_p(a)\sigma_q(b) &= \sigma_{p+q}(a_{(-1)}b), \\ T(\sigma_p(a)) &= \sigma_{p+1}(Ta), \\ Y_-(\sigma_p(a), z)\sigma_q(b) &= \sum_{n \in \mathbb{N}} \sigma_{p+q-n}(a_{(n)}b)z^{-n-1}, \end{aligned}$$

where $\sigma_p : F_p V \rightarrow \text{gr}_F(V)$ is the *principal symbol map*, which is the composition of the natural maps $F_p V \twoheadrightarrow F_p V / F_{p+1} V$ and $F_p V / F_{p+1} V \hookrightarrow \text{gr}_F(V)$. The unit is $\sigma^0(|0\rangle)$. The filtration $(F_p V)_{p \in \mathbb{Z}}$ is called the *Li filtration* of V .

Lemma 4.3.2 ([Li05, Lemma 2.9]). *Let V be a vertex algebra. Then*

$$F_p V = \text{span}\{a_{(-i-1)}b \mid a \in V, i \in \mathbb{Z}_+, b \in F_{p-i} V\} \quad \text{for } p \in \mathbb{Z}_+.$$

By Lemma 4.3.2, the Li filtration depends only on V and not on the choice of the strong generators. If V is graded with Hamiltonian H , then $H(F_p V) \subseteq F_p V$ because in that case, for $p \in \mathbb{Z}$,

$$F_p V = \text{span}\{a_{(-n_1-1)}^1 \cdots a_{(-n_s-1)}^s | 0\rangle \mid s, n_1, \dots, n_s \in \mathbb{N}, a^1, \dots, a^s \in V \text{ homogeneous, } n_1 + \cdots + n_s \geq p\}.$$

Therefore, we can define an operator $H \in \text{End}(\text{gr}_F(V))$ as $H(\sigma_p(a)) = \sigma_p(Ha)$ for $p \in \mathbb{N}$ and $a \in F_p V$. For $p \in \mathbb{Z}$ and $\Delta \in \mathbb{C}$, we define $F_p V_\Delta = F_p V \cap V_\Delta$. Since $H(F_p V) \subseteq F_p V$ for $p \in \mathbb{Z}$, Lemma 1.5.12 implies that

$$F_p V = \bigoplus_{\Delta \in \mathbb{C}} F_p V_\Delta \quad \text{for } p \in \mathbb{Z}.$$

For $\Delta \in \mathbb{C}$, we define $\text{gr}_F(V)_\Delta = \bigoplus_{p \in \mathbb{N}} \sigma_p(F_p V_\Delta)$. Then $Ha = \Delta a$ for $a \in \text{gr}_F(V)_\Delta$. The family of subspaces $(\text{gr}_F(V)_\Delta)_{\Delta \in \mathbb{C}}$ satisfies $\text{gr}_F(V) = \bigoplus_{\Delta \in \mathbb{C}} \text{gr}_F(V)_\Delta$. Therefore, the operator $H \in \text{End}(\text{gr}_F(V))$ is diagonalizable with $\text{gr}_F(V)_\Delta = \ker(H - \Delta \text{Id}_{\text{gr}_F(V)})$. In fact, more is true.

Theorem 4.3.3. *This diagonalizable operator H is a Hamiltonian of $\text{gr}_F(V)$.*

Proof. For $p, q \in \mathbb{N}$, $a \in F_p V$ and $b \in F_q V$,

$$\begin{aligned} [H, Y_-(\sigma_p(a), z)]\sigma_q(b) &= \sum_{n \in \mathbb{N}} \sigma_{p+q-n}([H, a_{(n)}]b)z^{-n-1} \\ &= \sum_{n \in \mathbb{N}} \sigma_{p+q-n}((-n+1)a_{(n)} + (Ha)_{(n)})b)z^{-n-1} \\ &= (z\partial_z Y_-(\sigma_p(a), z) + Y_-(\sigma_p(Ha), z))\sigma_q(b), \end{aligned}$$

$$\begin{aligned}
H(\sigma_p(a)\sigma_q(b)) &= H(\sigma_{p+q}(a_{(-1)}b)) \\
&= \sigma_{p+q}(H(a_{(-1)}b)) \\
&= \sigma_{p+q}(a_{(-1)}(Hb) + (Ha)_{(-1)}b) \\
&= \sigma_p(a)H(\sigma_q(b)) + H(\sigma_p(a))\sigma_q(b),
\end{aligned}$$

$$\begin{aligned}
[H, T]\sigma_p(a) &= H(T(\sigma_p(a))) - T(H(\sigma_p(a))) \\
&= \sigma_{p+1}(HTa) - \sigma_{p+1}(THa) \\
&= \sigma_{p+1}([H, T]a) \\
&= \sigma_{p+1}(Ta) \\
&= T\sigma_p(a).
\end{aligned}$$

□

We have the natural vector space isomorphisms

$$\sigma_p(F_p V_\Delta) \cong F_p V_\Delta / F_{p+1} V_\Delta \quad \text{for } p \in \mathbb{Z} \text{ and } \Delta \in \mathbb{C}$$

and the refined grading

$$\text{gr}_F(V) = \bigoplus_{\substack{p \in \mathbb{N} \\ \Delta \in \mathbb{C}}} \sigma_p(F_p V_\Delta). \quad (4.3.1)$$

By (4.3.1), when $\dim(V_\Delta) < \infty$ for $\Delta \in \mathbb{C}$, it is natural to define the *refined character of V with respect to the Li filtration* as

$$\text{ch}_{\text{gr}_F(V)}(t, q) = \sum_{\substack{p \in \mathbb{N} \\ \Delta \in \mathbb{C}}} \dim(\sigma_p(F_p V_\Delta)) t^p q^\Delta.$$

If $f : V_1 \rightarrow V_2$ is a homomorphism of vertex algebras, then

$$\begin{aligned}
\text{gr}_F(f) : \text{gr}_F(V_1) &\rightarrow \text{gr}_F(V_2), \\
\text{gr}_F(f)(\sigma_p^{V_1}(a)) &= \sigma_p^{V_2}(f(a)) \quad \text{for } p \in \mathbb{N} \text{ and } a \in F_p V_1
\end{aligned}$$

defines a homomorphism of vertex Poisson algebras. If V_1 and V_2 are graded, then we require that f respects the gradings of V_1 and V_2 , and this implies that $\text{gr}_F(f)$ also respects the gradings of $\text{gr}_F(V_1)$ and $\text{gr}_F(V_2)$. Therefore, we obtain a functor

$$\text{gr}_F : \{(\text{graded}) \text{ vertex algebras}\} \rightarrow \{(\text{graded}) \text{ vertex Poisson algebras}\}.$$

We now assume that V is an \mathbb{N} -graded vertex algebra with Hamiltonian H . Let $(a^i)_{i \in I}$ be a family of homogeneous strong generators of V . For $p \in \mathbb{Z}$, we set

$$G^p V = \text{span}\{a_{(-n_1-1)}^{i_1} \cdots a_{(-n_s-1)}^{i_s} | 0\rangle \mid s, n_1, \dots, n_s \in \mathbb{N}, i_1, \dots, i_s \in I, \Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_s}} \leq p\}.$$

Proposition 4.3.4 ([Li04]). *The filtration $(G^p V)_{p \in \mathbb{Z}}$ satisfies:*

- (i) $G^p V = 0$ for $p < 0$;
- (ii) $|0\rangle \in G^0 V \subseteq G^1 V \subseteq \dots$;
- (iii) $V_n \subseteq G^n V$ for $n \in \mathbb{Z}$;
- (iv) $V = \bigcup_{p \in \mathbb{N}} G^p V$;
- (v) $a_{(n)} G^q V \subseteq G^{p+q} V$ for $p, q \in \mathbb{Z}$, $a \in G^p V$ and $n \in \mathbb{Z}$;
- (vi) $a_{(n)} G^q V \subseteq G^{p+q-1} V$ for $p, q \in \mathbb{Z}$, $a \in G^p V$ and $n \in \mathbb{N}$;
- (vii) $H(G^p V) \subseteq G^p V$ and $T(G^p V) \subseteq G^p V$ for $p \in \mathbb{Z}$.

Let

$$\text{gr}^G(V) = \bigoplus_{p \in \mathbb{N}} G^p V / G^{p-1} V$$

be the associated graded vector space. By [Li04], the vector space $\text{gr}^G(V)$ is a vertex Poisson algebra with operations given as follows. For $p, q \in \mathbb{N}$, $a \in G^pV$ and $b \in G^qV$, we set:

$$\begin{aligned}\alpha^p(a)\alpha^q(b) &= \alpha^{p+q}(a_{(-1)}b), \\ T(\alpha^p(a)) &= \alpha^p(Ta), \\ Y_-(\alpha^p(a), z)\alpha^q(b) &= \sum_{n \in \mathbb{N}} \alpha^{p+q-1}(a_{(n)}b)z^{-n-1},\end{aligned}$$

where $\alpha^p : G^pV \rightarrow \text{gr}^G(V)$ is the principal symbol map. The unit is $\alpha^0(|0\rangle)$. The filtration $(G^pV)_{p \in \mathbb{Z}}$ is called the *standard filtration of V* . By Proposition 4.3.6 ahead, the standard filtration does not depend on the choice of the strong generators of V .

By Proposition 4.3.4(vii), we can define an operator $H \in \text{End}(\text{gr}^G(V))$ as $H(\alpha^p(a)) = \alpha^p(Ha)$ for $p \in \mathbb{Z}$ and $a \in G^pV$. For $p \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define $G^pV_n = G^pV \cap V_n$. Since $H(G^pV) \subseteq G^pV$ for $p \in \mathbb{Z}$, Lemma 1.5.12 implies that

$$G^pV = \bigoplus_{n \in \mathbb{N}} G^pV_n \quad \text{for } p \in \mathbb{Z}.$$

For $n \in \mathbb{N}$, we define $\text{gr}^G(V)_n = \bigoplus_{p \in \mathbb{N}} \alpha^p(G^pV_n)$. Then $Ha = na$ for $a \in \text{gr}^G(V)_n$. The family of subspaces $(\text{gr}^G(V)_n)_{n \in \mathbb{N}}$ satisfies $\text{gr}^G(V) = \bigoplus_{n \in \mathbb{N}} \text{gr}^G(V)_n$. Therefore, the operator $H \in \text{End}(\text{gr}^G(V))$ is diagonalizable with $\text{gr}^G(V)_n = \ker(H - n \text{Id}_{\text{gr}^G(V)})$. In fact, more is true.

Theorem 4.3.5. *This diagonalizable operator H is a Hamiltonian of $\text{gr}^G(V)$.*

Proof. The proof of Theorem 4.3.3 also works here. □

We have the natural vector space isomorphisms

$$\alpha^p(G^pV_n) \cong G^pV_n / G^{p-1}V_n \quad \text{for } p \in \mathbb{Z} \text{ and } n \in \mathbb{N}$$

and the refined grading

$$\text{gr}^G(V) = \bigoplus_{p, n \in \mathbb{N}} \alpha^p(G^pV_n). \quad (4.3.2)$$

By (4.3.2), when $\dim(V_n) < \infty$ for $n \in \mathbb{N}$, it is natural to define the *refined character of V with respect to the standard filtration* as

$$\text{ch}_{\text{gr}^G(V)}(t, q) = \sum_{p, n \in \mathbb{N}} \dim(\alpha^p(G^pV_n)) t^p q^n \in \mathbb{C}[[t, q]].$$

If $f : V_1 \rightarrow V_2$ is a homomorphism of \mathbb{N} -graded vertex algebras, then

$$\begin{aligned}\text{gr}^G(f) : \text{gr}^G(V_1) &\rightarrow \text{gr}^G(V_2), \\ \text{gr}^G(f)(\alpha_{V_1}^p(a)) &= \alpha_{V_2}^p(f(a)) \quad \text{for } p \in \mathbb{N} \text{ and } a \in G^pV_1\end{aligned}$$

defines a homomorphism of \mathbb{N} -graded vertex Poisson algebras. Therefore, we obtain a functor

$$\text{gr}^G : \{\mathbb{N}\text{-graded vertex algebras}\} \rightarrow \{\mathbb{N}\text{-graded vertex Poisson algebras}\}.$$

Proposition 4.3.6 ([Ara12, Proposition 2.6.1]). *Let V be an \mathbb{N} -graded vertex algebra. Then the Li filtration and standard filtration satisfy*

$$F_pV_n = G^{n-p}V_n \quad \text{for } p, n \in \mathbb{N}.$$

An explicit isomorphism $\text{gr}_F(V) \xrightarrow{\sim} \text{gr}^G(V)$ of vertex Poisson algebras is defined by extending linearly the isomorphisms of vector spaces given by

$$\begin{aligned}\sigma_p(F_pV_n) &\xrightarrow{\sim} \alpha^{n-p}(G^{n-p}V_n), \\ \sigma_p(a) &\mapsto \alpha^{n-p}(a) \quad \text{for } p, n \in \mathbb{N} \text{ and } a \in F_pV_n.\end{aligned}$$

Remark 4.3.7. In [Ara12], it is suggested in a footnote that it is possible to consider more general $\frac{1}{r_0}\mathbb{N}$ -graded vertex algebras instead of \mathbb{N} -graded vertex algebras. However, we encounter difficulties doing this. For example, if $V = \text{Vir}^{1/2}$, which is the case we are interested here, then V is $\frac{1}{2}\mathbb{N}$ -graded. According to the definition of $\text{gr}^G(V)$ given in [Ara12], we should have $\text{gr}^G(V) = \bigoplus_{p \in \frac{1}{2}\mathbb{N}} G_p V / G_{p-1} V = G_0 V / G_{-1} V \oplus G_{1/2} V / G_{-1/2} V \oplus \dots = \mathbb{C}|0\rangle \oplus \mathbb{C}|0\rangle \oplus \dots$, which means “the vacuum is doubled”. That is probably not intended, and also the proof of [Ara12, Proposition 2.6.1] does not work for $\frac{1}{r_0}\mathbb{N}$ -graded vertex algebras. But the hypothesis $V_0 = \mathbb{C}|0\rangle$ can be removed, just as Arakawa wrote.

Theorem 4.3.8. *Let V be an \mathbb{N} -graded vertex algebra with $\dim(V_n) < \infty$ for $n \in \mathbb{N}$. The refined characters of V are related as follows:*

- (i) $\text{ch}_V(q) = \text{ch}_{\text{gr}_F(V)}(q) = \text{ch}_{\text{gr}^G(V)}(q) = \text{ch}_{\text{gr}_F(V)}(1, q) = \text{ch}_{\text{gr}^G(V)}(1, q);$
- (ii) $\text{ch}_{\text{gr}^G(V)}(t^{-1}, tq) = \text{ch}_{\text{gr}_F(V)}(t, q).$

Proof. See the proof of Proposition 4.4.9 ahead. □

Example 4.3.9 ($\text{gr}^G(\text{Vir}^c)$). We pick any $c \in \mathbb{C}$ and define the subalgebra $\text{Vir}_{\leq -2} = \bigoplus_{n \leq -2} \mathbb{C}L_n$ of Vir . By the PBW theorem and Example 1.4.14, for $s \in \mathbb{N}$, a basis of $U(\text{Vir}_{\leq -2})^s$ (see Appendix A for the definition of $U(\text{Vir}_{\leq -2})^s$) is given by

$$\{L_\lambda \mid t \leq s \text{ and } [\lambda_1, \dots, \lambda_t] \text{ is a partition with } \lambda_t \geq 2\}.$$

We recall that $\Delta_{L_{-2}|0\rangle} = 2$. From the definition of the standard filtration, we see that for $s \in \mathbb{Z}$, $G^{2s} \text{Vir}^c = G^{2s+1} \text{Vir}^c \cong U(\text{Vir}_{\leq -2})^s$. This implies that the quotients $G^{2s} \text{Vir}^c / G^{2s+1} \text{Vir}^c$ are 0 for $s \in \mathbb{N}$. Therefore, we have a vector space isomorphism

$$\text{gr}^G(\text{Vir}^c) \xrightarrow{\sim} \text{gr}(U(\text{Vir}_{\leq -2})),$$

$$\alpha^{2s}(L_\lambda | 0\rangle) \mapsto \gamma^s(L_\lambda) \quad \text{for } s \in \mathbb{N} \text{ and } \lambda = [\lambda_1, \dots, \lambda_s] \text{ a partition with } \lambda_s \geq 2.$$

We now show this is an algebra homomorphism. We need to show that for $s, t \in \mathbb{N}$ and partitions $[\lambda_1, \dots, \lambda_s], [\eta_1, \dots, \eta_t]$ with $\lambda_s, \eta_t \geq 2$,

$$\alpha^{2s}(L_\lambda | 0\rangle) \alpha^{2t}(L_\eta | 0\rangle) \mapsto \gamma^{s+t}(L_\lambda L_\eta),$$

which is equivalent to

$$\alpha^{2s+2t}((L_\lambda | 0\rangle)_{(-1)}(L_\eta | 0\rangle)) \mapsto \gamma^{s+t}(L_\lambda L_\eta).$$

Therefore, we have to show that

$$(L_\lambda | 0\rangle)_{(-1)}(L_\eta | 0\rangle) = (L_\lambda L_\eta | 0\rangle) + a \quad \text{for some } a \in G^{2s+2t-1} \text{Vir}^c. \quad (4.3.3)$$

If $s = 0$ or $s = 1$, (4.3.3) is clear. We assume $s \geq 2$. By Corollary 1.4.9,

$$\begin{aligned} (L_\lambda | 0\rangle)_{(-1)}(L_\eta | 0\rangle) &= \left(\frac{:\partial_z^{\lambda_1-2} L(z) \dots \partial_z^{\lambda_s-2} L(z):}{(\lambda_1-2)! \dots (\lambda_s-2)!} \right)_{(-1)} (L_\eta | 0\rangle) \\ &= \frac{:\partial_z^{\lambda_1-2} L(z) \dots \partial_z^{\lambda_s-2} L(z):_{(-1)}(L_\eta | 0\rangle)}{(\lambda_1-2)! \dots (\lambda_s-2)!}, \end{aligned}$$

where $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \in \mathcal{F}(\text{Vir}^c)$. We now use Lemma 1.3.4 with $V = \text{Vir}^c$, $a^k(z) = \partial_z^{\lambda_k-2} L(z)$ for $k = 1, \dots, s$ and $b = L_\eta | 0\rangle$. First, we note that for $k = 1, \dots, s$,

$$\begin{aligned} a^k(z) &= \partial_z^{\lambda_k-2} L(z) \\ &= \sum_{n \in \mathbb{Z}} (-n-2) \dots (-n-2-(\lambda_k-3)) L_n z^{-n-2-(\lambda_k-2)} \\ &= \sum_{n \in \mathbb{Z}} (-n+\lambda_k-3) \dots (-n) L_{n+1-\lambda_k} z^{-n-1}, \end{aligned}$$

which says that

$$a^k_{(n)} = (-n+\lambda_k-3) \dots (-n) L_{n+1-\lambda_k} \quad \text{for } k = 1, \dots, s \text{ and } n \in \mathbb{Z}.$$

We note that each expression $R_{n_1, \dots, n_{s-1}}^{-1, k} L_\eta |0\rangle$ in Lemma 1.3.4, where we omitted the fields $a^k(z)$ from the notation, is the sum of elements of the form

$$\text{scalar} \cdot L_{\kappa_1} \dots L_{\kappa_s}(L_\eta |0\rangle),$$

where $\kappa_k \in \mathbb{Z}$ for $k = 1, \dots, s$. Furthermore, we note that if $\kappa_k \geq -1$ for some $k = 1, \dots, s$, then $L_{\kappa_1} \dots L_{\kappa_s}(L_\eta |0\rangle) \in G^{2s+2t-1} \text{Vir}^c$. We now study these expressions $R_{n_1, \dots, n_{s-1}}^{-1, k} L_\eta |0\rangle$ by considering the elements that appear in the sum defining it. We consider several disjoint cases:

(i) $k > 0$. If $n_{i_1} \geq \lambda_{i_1} - 2$, then $n_{i_1} + 1 - \lambda_{i_1} \geq -1$, so we get

$$a_{(-n_{j_1}-1)}^{j_1} \dots a_{(-n_{j_{s-1-k}}-1)}^{j_{s-1-k}} a_{(l-k-\sum_{r=1}^k n_{i_r} + \sum_{r=1}^{s-1-k} n_{j_r})}^s a_{(n_{i_k})}^{i_k} \dots a_{(n_{i_1})}^{i_1} L_\eta |0\rangle \in G^{2s+2t-1} \text{Vir}^c.$$

If $n_{i_1} \leq \lambda_{i_1} - 3$, then $a^{i_1}(z)_{(n_{i_1})} L_\eta |0\rangle = 0$ because $(-n_{i_1} + \lambda_k - 3) \dots (-n_{i_1}) = 0$, so we get

$$a_{(-n_{j_1}-1)}^{j_1} \dots a_{(-n_{j_{s-1-k}}-1)}^{j_{s-1-k}} a_{(l-k-\sum_{r=1}^k n_{i_r} + \sum_{r=1}^{s-1-k} n_{j_r})}^s a_{(n_{i_k})}^{i_k} \dots a_{(n_{i_1})}^{i_1} L_\eta |0\rangle = 0.$$

Therefore, we obtain

$$R_{n_1, \dots, n_{s-1}}^{-1, k} L_\eta |0\rangle \in G^{2s+2t-1} \text{Vir}^c.$$

(ii) $k = 0$ and there exists some $j = 1, \dots, s-1$ such that $n_j > 0$. In this case, we get

$$R_{n_1, \dots, n_{s-1}}^{-1, 0} L_\eta |0\rangle = a_{(-n_1-1)}^1 \dots a_{(-n_{s-1}-1)}^{s-1} a_{(-1+\sum_{r=1}^{s-1} n_r)}^s L_\eta |0\rangle \in G^{2s+2t-1} \text{Vir}^c$$

because we can repeat the reasoning of (i) with $-1 + \sum_{r=1}^{s-1} n_r$ instead of n_{i_1} .

(iii) $k = 0$ and $n_1 = \dots = n_{s-1} = 0$. In this case, we get

$$R_{0, \dots, 0}^{-1, 0} = a_{(-1)}^1 \dots a_{(-1)}^{s-1} a_{(-1)}^s L_\eta |0\rangle = (\lambda_1 - 2)! \dots (\lambda_s - 2)! L_\lambda L_\eta |0\rangle.$$

From these three cases, we obtain (4.3.3).

We conclude that we have an algebra isomorphism $\text{gr}^G(\text{Vir}^c) \xrightarrow{\sim} \text{gr}(U(\text{Vir}_{\leq -2}))$. Composing this with the inverse of the isomorphism $S(\text{Vir}_{\leq -2}) \xrightarrow{\sim} \text{gr}(U(\text{Vir}_{\leq -2}))$ in Appendix A and taking $(L_n)_{n \leq -2}$ as the basis of $\text{Vir}_{\leq -2}$, we obtain the following isomorphism of commutative associative algebras with unit

$$\begin{aligned} \text{gr}^G(\text{Vir}^c) &\xrightarrow{\sim} \mathbb{C}[L_{-2}, L_{-3}, \dots], \\ \alpha^{2s}(L_{-n_1-2} \dots L_{-n_s-2} |0\rangle) &\mapsto L_{-n_1-2} \dots L_{-n_s-2} \quad \text{for } s, n_1, \dots, n_s \in \mathbb{N}. \end{aligned}$$

In particular, the isomorphism does not depend on c .

The derivation $T \in \text{Der}(\mathbb{C}[L_{-2}, L_{-3}, \dots])$ is given by $T(L_{-n}) = (n-1)L_{-n-1}$ for $n \geq 2$, which is extended to a derivation.

The Poisson structure of $\text{gr}^G(\text{Vir}^c)$ is trivial (i.e., the map Y_- is zero) because for $s \in \mathbb{Z}$, $G^{2s} \text{Vir}^c = G^{2s+1} \text{Vir}^c$.

Since $\Delta_{L_{-2}|0} = 2$,

$$\text{ch}_{\text{gr}^G(\text{Vir}^c)}(t, q) = \frac{1}{\prod_{k \geq 2} (1 - t^2 q^k)}.$$

Remark 4.3.10. It is worth noting that we do not use the Poisson structure of $\text{gr}(U(\text{Vir}_{\leq -2}))$ given in Example B.2.

4.4. Filtrations of modules over vertex algebras. Let V be a vertex algebra, let $(a^i)_{i \in I}$ be a family of strong generators of V , and let M be a V -module. For $p \in \mathbb{Z}$, we set

$$F_p M = \text{span}\{a_{(-n_1-1)}^{i_1 M} \dots a_{(-n_s-1)}^{i_s M} u \mid s, n_1, \dots, n_s \in \mathbb{N}, i_1, \dots, i_s \in I, u \in M, n_1 + \dots + n_s \geq p\}.$$

Proposition 4.4.1 ([Li05]). *The filtration $(F_p M)_{p \in \mathbb{Z}}$ satisfies:*

- (i) $M = F_p M$ for $p \leq 0$;
- (ii) $F_0 M \supseteq F_1 M \supseteq \dots$;
- (iii) $a_{(n)} F_q M \subseteq F_{p+q-n-1} M$ for $p, q \in \mathbb{Z}$, $a \in F_p V$ and $n \in \mathbb{Z}$;
- (iv) $a_{(n)} F_q M \subseteq F_{p+q-n} M$ for $p, q \in \mathbb{Z}$, $a \in F_p V$ and $n \in \mathbb{N}$.

Let

$$\mathrm{gr}_F(M) = \bigoplus_{p \in \mathbb{N}} F_p M / F_{p+1} M$$

be the associated graded vector space. By [Li05], the vector space $\mathrm{gr}_F(M)$ is a module over $\mathrm{gr}_F(V)$ with operations given as follows. For $p, q \in \mathbb{N}$, $a \in F_p V$ and $u \in F_q M$, we set:

$$\begin{aligned} \sigma_p(a) \sigma_q^M(u) &= \sigma_{p+q}^M(a_{(-1)}^M u), \\ Y_-^M(\sigma_p(a), z) \sigma_q^M(u) &= \sum_{n \in \mathbb{N}} \sigma_{p+q-n}^M(a_{(n)}^M u) z^{-n-1}, \end{aligned}$$

where $\sigma_p^M : F_p M \rightarrow \mathrm{gr}_F(M)$ is the principal symbol map. The filtration $(F_p M)_{p \in \mathbb{Z}}$ is called the Li filtration of M .

We note that if V were conformal with conformal vector ω , then we do not always have the property $L_{-1}^M(F_p M) \subseteq F_{p+1} M$ for $p \in \mathbb{Z}$, where $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^M z^{-n-2}$. For example, for $V = \mathrm{Vir}^{1/2}$, $M = M(1/2, 1/2)$ and $|1/2\rangle \in F_0 M$, we have $L_{-1}|1/2\rangle \notin F_1 M$.

Remark 4.4.2. The observation above makes [Ara12, Lemma 3.1.2] incorrect because the expression $\sigma_{p-1}(\omega_{(0)} m)$ that is written there (that should be $\sigma_{p+1}(\omega_{(0)} m)$, but it still does not work) is not well-defined. It is probably not possible to define a differential on $\mathrm{gr}_F(M)$ in a meaningful way. We will not need that differential in this thesis, though.

Lemma 4.4.3 ([Li05, Lemma 2.9]). *Let V be a vertex algebra, and let M be a V -module. Then*

$$F_p M = \mathrm{span}\{a_{(-i-1)}^M u \mid a \in V, i \in \mathbb{Z}_+, u \in F_{p-i} M\} \quad \text{for } p \in \mathbb{Z}_+.$$

By Lemma 4.4.3, the Li filtration depends only on M and not on the choice of the strong generators of V . If V is graded with Hamiltonian H , and M is graded with Hamiltonian H^M , then $H^M(F_p M) \subseteq F_p M$ because in that case, for $p \in \mathbb{Z}$,

$$\begin{aligned} F_p M &= \mathrm{span}\{a_{(-n_1-1)}^{1M} \cdots a_{(-n_s-1)}^{sM} u \mid s, n_1, \dots, n_s \in \mathbb{N}, a^1, \dots, a^s \in V \text{ homogeneous,} \\ &\quad u \in M \text{ homogeneous, } n_1 + \cdots + n_s \geq p\}. \end{aligned}$$

Therefore, we can define an operator $H^M \in \mathrm{End}(\mathrm{gr}_F(M))$ as $H^M(\sigma_p^M(u)) = \sigma_p^M(H^M u)$ for $p \in \mathbb{N}$ and $u \in F_p M$. For $p \in \mathbb{Z}$ and $\Delta \in \mathbb{C}$, we define $F_p M_\Delta = F_p M \cap M_\Delta$. Since $H^M(F_p M) \subseteq F_p M$ for $p \in \mathbb{Z}$, Lemma 1.5.12 implies that

$$F_p M = \bigoplus_{\Delta \in \mathbb{C}} F_p M_\Delta \quad \text{for } p \in \mathbb{Z}.$$

For $\Delta \in \mathbb{C}$, we define $\mathrm{gr}_F(M)_\Delta = \bigoplus_{p \in \mathbb{N}} \sigma_p^M(F_p M_\Delta)$. Then $H^M u = \Delta u$ for $u \in \mathrm{gr}_F(M)_\Delta$. The family of subspaces $(\mathrm{gr}_F(M)_\Delta)_{\Delta \in \mathbb{C}}$ satisfies $\mathrm{gr}_F(M) = \bigoplus_{\Delta \in \mathbb{C}} \mathrm{gr}_F(M)_\Delta$. Therefore, the operator $H^M \in \mathrm{End}(\mathrm{gr}_F(M))$ is diagonalizable with $\mathrm{gr}_F(M)_\Delta = \ker(H^M - \Delta \mathrm{Id}_{\mathrm{gr}_F(M)})$. In fact, more is true.

Theorem 4.4.4. *This diagonalizable operator H^M is a Hamiltonian of $\mathrm{gr}_F(M)$.*

Proof. The proof of Theorem 4.3.3 also works here. □

We have the natural vector space isomorphisms

$$\sigma_p^M(F_p M_\Delta) \cong F_p M_\Delta / F_{p+1} M_\Delta \quad \text{for } p \in \mathbb{Z} \text{ and } \Delta \in \mathbb{C}$$

and the refined grading

$$\mathrm{gr}_F(M) = \bigoplus_{\substack{p \in \mathbb{N} \\ \Delta \in \mathbb{C}}} \sigma_p(F_p M_\Delta). \quad (4.4.1)$$

By (4.4.1), when $\dim(M)_\Delta < \infty$ for $\Delta \in \mathbb{C}$, it is natural to define the refined character of M with respect to the Li filtration as

$$\mathrm{ch}_{\mathrm{gr}_F(M)}(t, q) = \sum_{\substack{p \in \mathbb{N} \\ \Delta \in \mathbb{C}}} \dim(\sigma_p(F_p M_\Delta)) t^p q^\Delta.$$

If $f : M_1 \rightarrow M_2$ is a homomorphism of V -modules, then

$$\begin{aligned} \mathrm{gr}_F(f) : \mathrm{gr}_F(M_1) &\rightarrow \mathrm{gr}_F(M_2), \\ \mathrm{gr}_F(f)(\sigma_p^{M_1}(u)) &= \sigma_p^{M_2}(f(u)) \quad \text{for } p \in \mathbb{N} \text{ and } u \in F_p M_1 \end{aligned}$$

defines a homomorphism of $\mathrm{gr}_F(V)$ -modules. If M_1 and M_2 are graded, then we require that f respects the gradings of M_1 and M_2 , and this implies that $\mathrm{gr}_F(f)$ also respects the gradings of $\mathrm{gr}_F(M_1)$ and $\mathrm{gr}_F(M_2)$. Therefore, we obtain a functor

$$\mathrm{gr}_F : \{(\text{graded}) V\text{-modules}\} \rightarrow \{(\text{graded}) \mathrm{gr}_F(V)\text{-modules}\}.$$

We now introduce a definition not given by Li in [Li04]. Let V be an \mathbb{N} -graded conformal vertex algebra with conformal vector ω , let $(a^i)_{i \in I}$ be a family of homogeneous strong generators of V , and let M be an $h + \mathbb{N}$ -graded (V, ω) -module. This means M is a V -module with L_0^M diagonalizable whose eigenvalues are in the set $h + \mathbb{N}$ for some $h \in \mathbb{C}$. We set $M_\Delta = \ker(L_0^M - \Delta \mathrm{Id}_M)$ for $\Delta \in \mathbb{C}$, so we have $M = \bigoplus_{n \in \mathbb{N}} M_{h+n}$. For $p \in \mathbb{Z}$, we set

$$\begin{aligned} G^p M &= \text{span}\{a_{(-n_1-1)}^{i_1 M} \cdots a_{(-n_s-1)}^{i_s M} u \mid s, n_1, \dots, n_s \in \mathbb{N}, i_1, \dots, i_s \in I, u \in M \text{ homogeneous}, \\ &\quad \Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_s}} + \Delta_u - h \leq p\}. \end{aligned}$$

Proposition 4.4.5. *The filtration $(G^p M)_{p \in \mathbb{Z}}$ satisfies:*

- (i) $G^p M = 0$ for $p < 0$;
- (ii) $G^0 M \subseteq G^1 M \subseteq \dots$;
- (iii) $M_{h+n} \subseteq G^n M$ for $n \in \mathbb{Z}$;
- (iv) $M = \bigcup_{p \in \mathbb{N}} G^p M$;
- (v) $a_{(n)}^M G^q M \subseteq G^{p+q} M$ for $p, q \in \mathbb{Z}$, $a \in G^p V$ and $n \in \mathbb{Z}$;
- (vi) $a_{(n)}^M G^q M \subseteq G^{p+q-1} M$ for $p, q \in \mathbb{Z}$, $a \in G^p V$ and $n \in \mathbb{N}$;
- (vii) $L_0^M(G^p M) \subseteq G^p M$ and $L_{-1}^M(G^p M) \subseteq G^{p+1} M$ for $p \in \mathbb{Z}$.

Proof. The proofs in [Li04] also work here. □

Remark 4.4.6. We do not have the property $L_{-1}^M(G^p M) \subseteq G^p M$ as in the case of vertex algebras.

Let

$$\mathrm{gr}^G(M) = \bigoplus_{p \in \mathbb{N}} G^p M / G^{p-1} M$$

be the associated graded vector space. The vector space $\mathrm{gr}^G(M)$ is a module over $\mathrm{gr}^G(V)$ with operations given as follows. For $p, q \in \mathbb{N}$, $a \in G^p V$ and $u \in G^q M$, we set:

$$\begin{aligned} \alpha^p(a) \alpha_M^q(u) &= \alpha_M^{p+q}(a_{(-1)}^M u), \\ Y_-^M(\alpha^p(a), z) \alpha_M^q(u) &= \sum_{n \in \mathbb{N}} \alpha_M^{p+q-1}(a_{(n)}^M u) z^{-n-1}, \end{aligned}$$

where $\alpha_M^p : G^p M \rightarrow \mathrm{gr}^G(M)$ is the principal symbol map. The filtration $(G^p M)_{p \in \mathbb{Z}}$ is called the standard filtration of M . By Proposition 4.4.8 ahead, the standard filtration does not depend on the choice of the strong generators of V .

By Proposition 4.3.4(vii), we can define an operator $H^M \in \mathrm{End}(\mathrm{gr}^G(M))$ as $H^M(\alpha_M^p(u)) = \alpha_M^p(L_0^M u)$ for $p \in \mathbb{Z}$ and $u \in G^p M$. For $p \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define $G^p M_{h+n} = G^p M \cap M_{h+n}$. Since $L_0^M(G^p M) \subseteq G^p M$ for $p \in \mathbb{Z}$, Lemma 1.5.12 implies that

$$G^p M = \bigoplus_{n \in \mathbb{N}} G^p M_{h+n} \quad \text{for } p \in \mathbb{Z}.$$

For $n \in \mathbb{N}$, we define $\mathrm{gr}^G(M)_{h+n} = \bigoplus_{p \in \mathbb{N}} \alpha_M^p(G^p M_{h+n})$. Then $H^M u = (h+n)u$ for $u \in \mathrm{gr}^G(M)_{h+n}$. The family of subspaces $(\mathrm{gr}^G(M)_{h+n})_{n \in \mathbb{N}}$ satisfies $\mathrm{gr}^G(M) = \bigoplus_{n \in \mathbb{N}} \mathrm{gr}^G(M)_{h+n}$. Therefore, the operator $H^M \in \mathrm{End}(\mathrm{gr}^G(M))$ is diagonalizable with $\mathrm{gr}^G(M)_{h+n} = \ker(H - (h+n) \mathrm{Id}_{\mathrm{gr}^G(M)})$. In fact, more is true.

Theorem 4.4.7. *This diagonalizable operator H^M is a Hamiltonian of $\text{gr}^G(M)$.*

Proof. The proof of Theorem 4.3.3 also works here. \square

We have the natural vector space isomorphisms

$$\alpha_M^p(G^p M_{h+n}) \cong G^p M_{h+n} / G^{p-1} M_{h+n} \quad \text{for } p \in \mathbb{Z} \text{ and } n \in \mathbb{N}$$

and the refined grading

$$\text{gr}^G(M) = \bigoplus_{p,n \in \mathbb{N}} \alpha_M^p(G^p M_{h+n}). \quad (4.4.2)$$

By (4.4.2), when M is ordinary, which means $\dim(M_{h+n}) < \infty$ for $n \in \mathbb{N}$, it is natural to define the refined character of M with respect to the standard filtration as

$$\text{ch}_{\text{gr}^G(M)}(t, q) = \sum_{p,n \in \mathbb{N}} \dim(\alpha_M^p(G^p M_{h+n})) t^p q^{h+n} \in q^h \mathbb{C}[[t, q]].$$

If $f : M_1 \rightarrow M_2$ is a homomorphism of $h + \mathbb{N}$ -graded (V, ω) -modules, then

$$\begin{aligned} \text{gr}^G(f) : \text{gr}^G(M_1) &\rightarrow \text{gr}^G(M_2), \\ \text{gr}^G(f)(\alpha_{M_1}^p(u)) &= \alpha_{M_2}^p(f(u)) \quad \text{for } p \in \mathbb{N} \text{ and } u \in G^p M_1 \end{aligned}$$

defines a homomorphism of $h + \mathbb{N}$ -graded $\text{gr}^G(V)$ -modules. Therefore, we obtain a functor

$$\text{gr}^G : \{h + \mathbb{N}\text{-graded } (V, \omega)\text{-modules}\} \rightarrow \{h + \mathbb{N}\text{-graded } \text{gr}^G(V)\text{-modules}\}.$$

Proposition 4.4.8. *Let (V, ω) be an \mathbb{N} -graded conformal vertex algebra, and let M be an $h + \mathbb{N}$ -graded (V, ω) -module. Then the Li filtration and the standard filtration satisfy*

$$F_p M_{h+n} = G^{n-p} M_{h+n} \quad \text{for } p, n \in \mathbb{N}.$$

An explicit isomorphism $\text{gr}_F(M) \xrightarrow{\sim} \text{gr}^G(M)$ of modules is defined by extending linearly the isomorphisms of vector spaces given by

$$\begin{aligned} \sigma_p^M(F_p M_{h+n}) &\xrightarrow{\sim} \alpha_M^{n-p}(G^{n-p} M_{h+n}), \\ \sigma_p^M(u) &\mapsto \alpha_M^{n-p}(u) \quad \text{for } p, n \in \mathbb{N} \text{ and } u \in F_p M_{h+n}. \end{aligned}$$

Proof. We recall three facts:

- (a) $M_{h+n} \subseteq G^n M$ for $n \in \mathbb{Z}$;
- (b) $\Delta_{a_{(n)}^M} u = \Delta_a + \Delta_u - n - 1$ for homogeneous $a \in V$, $u \in M$ and $n \in \mathbb{Z}$;
- (c) For $p \in \mathbb{Z}_+$ and $n \in \mathbb{N}$,

$$\begin{aligned} F_p M_{h+n} = \text{span}\{a_{(-n_1-1)}^{i_1 M} u \mid i_1 \in I, n_1 \in \mathbb{Z}_+ \text{ and } u \in F_{p-n_1} M_{\Delta_u} \\ \text{satisfy } \Delta_{a^{i_1}} + \Delta_u + n_1 = h + n\}. \end{aligned}$$

First, we prove the inclusion $F_p M_{h+n} \subseteq G^{n-p} M_{h+n}$ for $p, n \in \mathbb{N}$. We do this by induction on $p \in \mathbb{N}$. The base case $p = 0$ is true by property (a) above. We now assume $p \geq 1$ and $F_q M_{h+n} \subseteq G^{n-q} M_{h+n}$ for $q < p$ and $n \in \mathbb{N}$. We pick an element $a_{(-n_1-1)}^{i_1 M} u$ from the spanning set of $F_p M_{h+n}$ in property (c) above, with $i_1 \in I$, $n_1 \in \mathbb{Z}_+$ and $u \in F_{p-n_1} M_{\Delta_u}$. By the induction hypothesis, we know that $u \in G^{\Delta_u - h - p + n_1} M_{\Delta_u}$, and also $a \in V_{\Delta_{a^{i_1}}} \subseteq G^{\Delta_{a^{i_1}}} V$. Therefore, by Proposition 4.4.5(v), $a_{(-n_1-1)}^{i_1 M} u \in G^{\Delta_{a^{i_1}} + \Delta_u - h - p + n_1} M = G^{n-p} M$.

We now prove the inclusion $G^{n-p} M_{h+n} \subseteq F_p M_{h+n}$. We pick an element $a_{(-n_1-1)}^{i_1 M} \cdots a_{(-n_s-1)}^{i_s M} u$ from the spanning set of $G^{n-p} M_{h+n}$, where $u \in M_{\Delta_u}$, $s, n_1, \dots, n_s \in \mathbb{N}$, $i_1, \dots, i_s \in I$ and $\Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_s}} + \Delta_u - h \leq n - p$. By property (b) above, we must have $\Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_s}} + \Delta_u + n_1 + \cdots + n_s = h + n$. Therefore, $p \leq n_1 + \cdots + n_s$, so we get $a_{(-n_1-1)}^{i_1 M} \cdots a_{(-n_s-1)}^{i_s M} u \in F_p M$ straight from the definition of $F_p M$.

We verify that we obtain an isomorphism $\text{gr}_F(M) \xrightarrow{\sim} \text{gr}^G(M)$ directly from the definitions (cf. Proposition 4.3.6). \square

Proposition 4.4.8 translates into an identity of the refined characters of M , as the following proposition shows. This is also valid for vertex algebras because all the definitions are compatible.

Proposition 4.4.9. *Let (V, ω) be an \mathbb{N} -graded conformal vertex algebra, and let M be an $h + \mathbb{N}$ -graded (V, ω) -module. Then:*

- (i) $\text{ch}_M(q) = \text{ch}_{\text{gr}_F(M)}(q) = \text{ch}_{\text{gr}^G(M)}(q) = \text{ch}_{\text{gr}_F(M)}(1, q) = \text{ch}_{\text{gr}^G(M)}(1, q)$;
- (ii) $\text{ch}_{\text{gr}^G(M)}(t^{-1}, tq) = t^h \text{ch}_{\text{gr}_F(M)}(t, q)$.

Proof.

- (i) This is clear from the properties and definitions of the filtrations.
- (ii) This follows from Proposition 4.4.8 and by replacing p by $n - p$ in the following computation

$$\begin{aligned} \text{ch}_{\text{gr}^G(M)}(t^{-1}, tq) &= \sum_{p, n \in \mathbb{N}} \dim(\alpha_M^p(G^p M_{h+n})) t^{-p} (tq)^{h+n} \\ &= \sum_{p, n \in \mathbb{N}} \dim(\alpha_M^p(G^p M_{h+n})) t^{h+n-p} q^{h+n} \\ &= \sum_{p, n \in \mathbb{N}} \dim(\alpha_M^{n-p}(G^{n-p} M_{h+n})) t^{h+p} q^{h+n} \\ &= t^h \text{ch}_{\text{gr}_F(M)}(t, q). \quad \square \end{aligned}$$

Example 4.4.10 ($\text{gr}^G(M(c, h))$). We pick a highest weight (c, h) . By Theorem 1.6.5, the Verma module $M(c, h)$ is a module over Vir^c . From Example 4.3.9, $\text{gr}^G(\text{Vir}^c) \cong \mathbb{C}[L_{-2}, L_{-3}, \dots]$. Similarly, we can prove that $\text{gr}^G(M(c, h))$ is a free $\text{gr}^G(\text{Vir}^c)$ -module

$$\text{gr}^G(M(c, h)) \cong \bigoplus_{k \in \mathbb{N}} \text{gr}^G(\text{Vir}^c) L_{-1}^k.$$

More precisely, the isomorphism is given by

$$\begin{aligned} \text{gr}^G(M(c, h)) &\xrightarrow{\sim} \bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}, L_{-3}, \dots] L_{-1}^k, \\ \alpha_{M(c, h)}^{2s+k} (L_{-n_1-2}^{M(c, h)} \dots L_{-n_s-2}^{M(c, h)} (L_{-1}^{M(c, h)})^k | c, h) &\mapsto L_{-n_1-2} \dots L_{-n_s-2} L_{-1}^k \quad \text{for } s, k, n_1, \dots, n_s \in \mathbb{N}. \end{aligned}$$

Since $\Delta_{L_{-2}|0} = 2$ and $\Delta_{(L_{-1}^{M(c, h)})^k | c, h} = k + h$ for $k \in \mathbb{N}$,

$$\text{ch}_{\text{gr}^G(M(c, h))}(t, q) = \frac{q^h}{(1 - tq) \prod_{k \geq 2} (1 - t^2 q^k)}.$$

Remark 4.4.11. That $+k$ in the isomorphism above is what makes this filtration different from the PBW filtration (see Appendix A), where all L_n for $n \leq -1$ have the same length. On the other hand, with the standard filtration, L_{-1} has length equal to 1, while L_{-2}, L_{-3}, \dots have length equal to 2.

Remark 4.4.12. The definition of the standard filtration $(G^p M)_{p \in \mathbb{Z}}$ apparently depends on h , so we should write it as $(G_h^p M)_{p \in \mathbb{Z}}$. However, since $G_{h-1}^p M = G_h^{p-1} M$ for $p \in \mathbb{Z}$, $\text{gr}_h^G(M)$ and $\text{gr}_{h-1}^G(M)$ are isomorphic and $\text{ch}_{\text{gr}_{h-1}^G(M)}(t, q) = t \text{ch}_{\text{gr}_h^G(M)}(t, q)$. If we further require that $M_h \neq 0$ when $M \neq 0$, then $\text{gr}^G(M)$ and $\text{ch}_{\text{gr}^G(M)}(t, q)$ are well-defined.

Remark 4.4.13. We could have done the theory of standard filtrations of modules for admissible modules over \mathbb{N} -graded vertex algebras (or maybe even over lower truncated \mathbb{Z} -graded vertex algebras, as it is done in [Li04]).

4.5. The Zhu C_2 -algebra and the C_2 -cofiniteness condition. Let V be a vertex algebra. By Lemma 4.3.2, we have

$$C_2V = \text{span}\{a_{(-2)}b \mid a, b \in V\} = F_1V.$$

We define the *Zhu C_2 -algebra of V* by

$$R_V = V/C_2V = F_0V/F_1V \subseteq \text{gr}_F(V).$$

In Zhu's original article [Zhu96] and other works, R_V is denoted by $A(V)$.

The fact that $\text{gr}_F(V)$ is a vertex Poisson algebra implies that R_V is a Poisson algebra with operations given as follows. For $a, b \in V$, we set:

$$\begin{aligned} \sigma_0(a)\sigma_0(b) &= \sigma_0(a_{(-1)}b), \\ \{\sigma_0(a), \sigma_0(b)\} &= \sigma_0(a_{(0)}b). \end{aligned}$$

If V is graded, then, as we explained in §4.3, $\text{gr}_F(V)$ is graded. Thus, R_V is also graded.

We have constructed a functor

$$R : \{(\text{graded}) \text{ vertex algebras}\} \rightarrow \{(\text{graded}) \text{ Poisson algebras}\}.$$

Oftentimes, some condition on R_V implies or is equivalent to some condition on V , as we shall see in this subsection. The vertex algebra V is called *C_2 -cofinite* if R_V is finite dimensional.

Example 4.5.1 (R_{Vir^c} and R_{Vir_c}). We pick $c \in \mathbb{C}$. We consider $\mathbb{C}[L_{-2}]$ as the polynomial algebra in one variable L_{-2} , and we equip it with the trivial Poisson bracket. By Example 4.3.9 and Proposition 4.3.6, we have the following isomorphism of Poisson algebras

$$\begin{aligned} R_{\text{Vir}^c} &\xrightarrow{\sim} \mathbb{C}[L_{-2}], \\ \sigma_0(L_{-2} | 0) &\mapsto L_{-2}. \end{aligned}$$

We now move onto Vir_c . If c is not of the form $c_{p,q}$ for some $p, q \geq 2$ relatively prime integers, then $\text{Vir}_c = \text{Vir}^c$ by Theorem 1.5.16, and we have already solved the problem. Therefore, we assume c is of this form.

We have a natural quotient map

$$\begin{aligned} \pi_0^c : \text{Vir}^c &\twoheadrightarrow \text{Vir}_c, \\ \pi_0^c(a) &= a + U(\text{Vir})\{a_{p,q}\}. \end{aligned}$$

Applying the functor R , we obtain an epimorphism

$$R_{\pi_0^c} : R_{\text{Vir}^c} \twoheadrightarrow R_{\text{Vir}_c}.$$

From the equation $\ker(R_{\pi_0^c}) = \sigma_0(U(\text{Vir})\{a_{p,q}\})$ and Theorem 1.5.16, we obtain

$$\ker(R_{\pi_0^c}) = (\sigma_0(L_{-2}^{(p-1)(q-1)/2} | 0)).$$

In summary,

$$R_{\text{Vir}_{p,q}} \cong \mathbb{C}[L_{-2}] / (L_{-2}^{(p-1)(q-1)/2}).$$

Thus, Vir^c is never C_2 -cofinite, while Vir_c is C_2 -cofinite only when c is of the form $c_{p,q}$ for some $p, q \geq 2$ relatively prime integers.

Let V be a vertex algebra (no condition on V). The *associated variety of V* , denoted by X_V , and the *singular support of V* , denoted by $SS(V)$, are defined by:

$$\begin{aligned} X_V &= \text{Spec}(R_V), \\ SS(V) &= \text{Spec}(\text{gr}_F(V)). \end{aligned}$$

Lemma 4.5.2 ([Li05, Corollary 4.3]). *Let V be a vertex algebra. As a differential algebra, $\text{gr}_F(V)$ is generated by R_V , i.e.,*

$$\text{gr}_F(V) = (R_V)_T.$$

Let V be a (graded) vertex algebra. We have a natural (graded) algebra inclusion $\text{inc} : R_V \hookrightarrow \text{gr}_F(V)$. By the universal property of $\text{inc} : R \hookrightarrow JR_V$ (see Appendix C), there is a (graded) differential algebra homomorphism $\phi_V : JR_V \rightarrow \text{gr}_F(V)$ such that the following diagram commutes

$$\begin{array}{ccc} R_V & \xleftarrow{\text{inc}} & JR_V \\ & \searrow \text{inc} & \downarrow \phi_V \\ & & \text{gr}_F(V) \end{array}$$

Because R_V is a (graded) Poisson algebra, we can equip JR_V with the level 0 vertex Poisson algebra structure. From now on, JR_V will be considered as a (graded) vertex Poisson algebra.

Lemma 4.5.3. *Let V be a (graded) vertex algebra. The (graded) differential algebra homomorphism $\phi_V : JR_V \rightarrow \text{gr}_F(V)$ defined above is surjective and is actually a (graded) vertex Poisson algebra homomorphism. Thus, $SS(V)$ is isomorphic to a closed subscheme of JX_V .*

Proof. The homomorphism ϕ_V is surjective by Lemma 4.5.2. The fact that ϕ_V is a (graded) vertex Poisson algebra homomorphism is explained in [Ara12, Proposition 2.5.1]. \square

When ϕ_V is an isomorphism, we say V is *classically free*.

Example 4.5.4 (JR_{Vir^c} and JR_{Vir_c}). Let $c \in \mathbb{C}$. Then Vir^c is classically free because

$$\text{gr}_F(\text{Vir}^c) \cong \mathbb{C}[L_{-2}, L_{-3}, \dots]$$

by Example 4.3.9 and Proposition 4.3.6 and

$$JR_{\text{Vir}^c} \cong J(\mathbb{C}[L_{-2}]) = \mathbb{C}[L_{-2}, L_{-3}, \dots]$$

by Example 4.5.1 and Appendix C.

We now move onto Vir_c . If c is not of the form $c_{p,q}$ for some $p, q \geq 2$ relatively prime integers, then $\text{Vir}_c = \text{Vir}^c$ by Theorem 1.5.16, and we have already solved the problem. Therefore, we assume c is of this form. Then

$$JR_{\text{Vir}_{p,q}} \cong J(\mathbb{C}[L_{-2}]/(L_{-2}^{(p-1)(q-1)/2})) = \mathbb{C}[L_{-2}, L_{-3}, \dots]/(L_{-2}^{(p-1)(q-1)/2})_{\partial}$$

by Example 4.5.1 and Appendix C.

Example 4.5.5. If $p > q \geq 2$ are relatively prime integers, then $\text{Vir}_{p,q}$ is classically free if and only if $q = 2$ by [VEH21].

Example 4.5.6. It was proven in [AVEH22] that the Ising model $\text{Vir}_{3,4}$ is not classically free. In fact, by [AVEH22, Theorem 2],

$$\ker(\phi_{\text{Vir}_{3,4}}) = (b)_{\partial},$$

where

$$b = L_{-4}L_{-3}L_{-2} + \frac{1}{6}L_{-5}L_{-2}^2,$$

and $(b)_{\partial}$ is the differential ideal generated by b , cf. Appendix C and Example 4.3.9 where ∂ is denoted by T .

In algebra, commutative algebras are often required to be finitely generated. In the theory of \mathbb{N} -graded vertex algebras, the equivalent of this is assuming that V is finitely strongly generated. Fortunately, these two concepts are related, as the following theorem shows.

Theorem 4.5.7. *Let $S \subseteq V$ be a subset of homogeneous elements of an \mathbb{N} -graded vertex algebra V . The following are equivalent:*

- (i) S strongly generates V ;
- (ii) $\sigma_0(S)$ generates R_V as an algebra without the Poisson structure.

In particular, V is finitely strongly generated if and only if R_V is finitely generated.

Proof. We assume that S strongly generates V . Let $\sigma_0(a) \in R_V$ with $a \in V$ arbitrary. We know that a is a linear combination of elements of the form $a_{(-n_1-1)}^1 \cdots a_{(-n_s-1)}^s |0\rangle$, where $s, n_1, \dots, n_s \in \mathbb{N}$ and $a^1, \dots, a^s \in S$. However, $\sigma_0(a_{(-n_1-1)}^1 \cdots a_{(-n_s-1)}^s |0\rangle) = 0$ if any $n_i > 0$, and if $n_1 = \cdots = n_s = 0$, we have $\sigma_0(a_{(-n_1-1)}^1 \cdots a_{(-n_s-1)}^s |0\rangle) = \sigma_0(a_{(-1)}^1 \cdots a_{(-1)}^s |0\rangle) = \sigma_0(a^1) \cdots \sigma_0(a^s) \in \mathbb{C}[\sigma_0(S)]$. Therefore, $\sigma_0(a) \in \mathbb{C}[\sigma_0(S)]$ and $R_V = \mathbb{C}[\sigma_0(S)]$.

We now assume that $R_V = \mathbb{C}[\sigma_0(S)]$. We define the subspace W strongly generated by S by setting

$$W = \text{span}\{a_{(-n_1-1)}^1 \cdots a_{(-n_s-1)}^s |0\rangle \mid s, n_1, \dots, n_s \in \mathbb{N}, a^1, \dots, a^s \in S\}.$$

As the elements of S are homogeneous, the subspace W is H -invariant. By Lemma 1.5.12,

$$W = \sum_{n \in \mathbb{N}} W \cap V_n. \quad (4.5.1)$$

Our objective is to show that $V = W$. By Theorem 1.4.5(i)–(ii) and Corollary 1.4.9, for $m, n \in \mathbb{N}$ and $a, b \in V$, we have

$$\begin{aligned} \sigma_n(T^n(a))\sigma_m(T^m(b)) &= \sigma_n(n!a_{(-n-1)} |0\rangle)\sigma_m(m!b_{(-m-1)} |0\rangle) \\ &= \sigma_{n+m}((n!a_{(-n-1)} |0\rangle)_{(-1)}(m!b_{(-m-1)} |0\rangle)) \\ &= n!m!\sigma_{n+m}(a_{(-n-1)}b_{(-m-1)} |0\rangle). \end{aligned}$$

By Lemma 4.5.2,

$$\text{gr}_F(V) = \mathbb{C} \left[\bigcup_{n \in \mathbb{N}} T^n(R_V) \right] = \mathbb{C} \left[\bigcup_{n \in \mathbb{N}} T^n(\mathbb{C}[\sigma_0(S)]) \right] = \mathbb{C} \left[\bigcup_{n \in \mathbb{N}} T^n(\sigma_0(S)) \right].$$

It is enough to show that for $n \in \mathbb{N}$, $V_n \subseteq W$. Thus, we fix $n_0 \in \mathbb{N}$. By Proposition 4.3.6, there exists $N \in \mathbb{N}$ such that $F_p V_{n_0} = 0$ for $p > N$. We now pick $a \in F_N V_{n_0}$. We know there exist $s, n_1, \dots, n_s \in \mathbb{N}$, $a^1, \dots, a^s \in S$ and $p(x_1, \dots, x_s) = \sum_{v \in \mathbb{N}^s} t_v x_1^{v_1} \cdots x_s^{v_s} \in \mathbb{C}[x_1, \dots, x_s]$, where $t_v \in \mathbb{C}$ for $v \in \mathbb{N}^s$, such that

$$\begin{aligned} \sigma_N(a) &= p(T^{n_1}(\sigma_0(a^1)), \dots, T^{n_s}(\sigma_0(a^s))) \\ &= p(\sigma_{n_1}(T^{n_1}(a^1)), \dots, \sigma_{n_s}(T^{n_s}(a^s))) \\ &= \sum_{v \in \mathbb{N}^s} t_v (n_1!)^{v_1} \cdots (n_s!)^{v_s} \sigma_{v_1 n_1 + \cdots + v_s n_s}((a_{(-n_1-1)}^1)^{v_1} \cdots (a_{(-n_s-1)}^s)^{v_s} |0\rangle). \end{aligned}$$

Therefore,

$$\sigma_N(a) = \sigma_N(b),$$

where

$$b = \sum_{\substack{v \in \mathbb{N}^s \\ v_1 n_1 + \cdots + v_s n_s = N}} t_v (n_1!)^{v_1} \cdots (n_s!)^{v_s} (a_{(-n_1-1)}^1)^{v_1} \cdots (a_{(-n_s-1)}^s)^{v_s} |0\rangle \in W.$$

This means $a - b \in F_{N+1} V$. By (4.5.1), we can write $b = \sum_{n \in \mathbb{N}} b_n$, where $b_n \in W \cap V_n$ for $n \in \mathbb{N}$. Therefore, $a - b_{n_0} \in F_{N+1} V_{n_0} = 0$, and this implies $a \in W$. We concluded that $F_N V_{n_0} \subseteq W$ from $F_{N+1} V_{n_0} = 0 \subseteq W$. In the same way, we can conclude that $F_{N-1} V_{n_0} \subseteq W$ from $F_N V_{n_0} \subseteq W$, and we conclude using induction that $F_0 V_{n_0} = V_{n_0} \subseteq W$, finishing the proof. \square

Remark 4.5.8. For a generalization of Theorem 4.5.7 assuming V is a lower truncated \mathbb{Z} -graded vertex algebra, see [Li05, Theorem 4.8].

Remark 4.5.9. In the proof of [Ara12, Corollary 2.6.2], it is commented that the principal symbol map gives the isomorphism $V \xrightarrow{\sim} \text{gr}_F(V)$. However, there are some difficulties with this because if there were such an isomorphism, it should be natural and valid for every vertex algebra V , given that V is defined for every vertex algebra (see §4.3). We now present an example in which V and $\text{gr}_F(V)$ satisfy:

- (i) $\bigcap_{p \in \mathbb{N}} F_p V = 0$;
- (ii) The vector spaces V and $\text{gr}_F(V)$ have different dimensions;

(iii) R_V is finitely generated, but V is not finitely strongly generated.

We consider the commutative vertex algebra $V = \mathbb{C}[[t]]$ with derivation $T = t^2 \frac{d}{dt}$. We have:

$$\begin{aligned} F_0 V &= \mathbb{C}[[t]], \\ F_p V &= t^{1+p} \mathbb{C}[[t]] \quad \text{for } p \in \mathbb{Z}_+. \end{aligned}$$

From this, we get (i), and we have the following isomorphisms of vector spaces:

$$\begin{aligned} \text{gr}_F(V) &= \bigoplus_{p \in \mathbb{N}} F_p V / F_{p+1} V \\ &= \mathbb{C}[[t]]/t^2 \mathbb{C}[[t]] \oplus t^2 \mathbb{C}[[t]]/t^3 \mathbb{C}[[t]] \oplus t^3 \mathbb{C}[[t]]/t^4 \mathbb{C}[[t]] \oplus \dots \\ &\cong \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C} \oplus \dots, \\ R_V &= V/F_1 V \\ &= \mathbb{C}[[t]]/t^2 \mathbb{C}[[t]] \\ &\cong \mathbb{C}[t]/(t^2). \end{aligned}$$

Thus, $\text{gr}_F(V)$ has countable dimension. However, V has uncountable dimension, and we get (ii).

From the definition of the Zhu C_2 -algebra, we see that R_V is isomorphic to $\mathbb{C}[t]/(t^2)$ as an algebra, and the latter is generated by the single element $t + (t^2)$. However, V cannot be finitely strongly generated because it has uncountable dimension, and we get (iii).

From this example, we see that the assumption of V being \mathbb{N} -graded cannot be removed in Theorem 4.5.7, although we can assume that V is lower truncated \mathbb{Z} -graded.

Theorem 4.5.10. *Let V be a vertex algebra such that R_V is finitely generated. Then the following are equivalent:*

- (i) V is C_2 -cofinite;
- (ii) $\dim(X_V) = 0$;
- (iii) X_V is finite and discrete;
- (iv) $\dim(SS(V)) = 0$.

If V is conical, then $(R_V)_0 = \mathbb{C}\sigma_0(|0\rangle)$ and (i)–(iv) are also equivalent to

- (v) $X_V = \{\text{point}\}$.

Proof.

(i) \iff (ii): By [AK18, Exercise 19.17], R_V is finite dimensional if and only if R_V is Artinian, and by [AK18, Theorem 19.8], R_V is Artinian if and only if $0 = \dim(R_V) = \dim(X_V)$.

(ii) \implies (iii): By [AK18, Theorem 19.8], X_V is finite. Since $\dim(X_V) = 0$, every prime ideal is maximal. Thus, X_V is discrete.

(iii) \implies (ii): Clear.

(ii) \implies (iv): By Theorem C.4, $\dim(JX_V) = 0$. By Lemma 4.5.3, $\dim(SS(V)) = 0$.

(iv) \implies (ii): This follows by considering the projection onto the 0-th component $\text{gr}_F(V) \rightarrow R_V$ which shows that $\dim(R_V) \leq \dim(\text{gr}_F(V))$.

We now assume that V is conical, i.e., $V = \bigoplus_{n \in \mathbb{N}} V_n$ and $V_0 = \mathbb{C}|0\rangle$. Clearly, $R_V = \bigoplus_{n \in \mathbb{N}} (R_V)_n$ and $(R_V)_0 = \mathbb{C}\sigma_0(|0\rangle)$. Since $\Delta_{\Delta_0(a)\Delta_0(b)} = \Delta_{\sigma_0(a)} + \Delta_{\sigma_0(b)}$, $\sum_{n \in \mathbb{Z}_+} (R_V)_n$ is a maximal ideal of R_V . In fact, it is the unique maximal proper ideal of R_V .

(ii) \implies (v): By definition, every prime in R_V is maximal. Since R_V has a unique maximal proper ideal, $X_V = \{\text{point}\}$.

(v) \implies (iii): Clear. □

Thus, C_2 -cofinite vertex algebras can be regarded as a generalization of finite-dimensional finitely generated algebras.

Theorem 4.5.11 ([ABD03, Theorem 4.5]). *Let V be a conical conformal vertex algebra. Then V is regular if and only if V is C_2 -cofinite and rational.*

Example 4.5.12. By Example 1.6.15, Example 4.5.1 and Theorem 4.5.11, $\text{Vir}_{p,q}$ is regular for $p, q \geq 2$ relatively prime integers.

Let V be a vertex algebra, and let M be a module over V . By Lemma 4.4.3, we have

$$C_2M = \text{span}\{a_{(-2)}^M u \mid a \in V, u \in M\} = F_1M.$$

We define the *Zhu C_2 -module of M* by

$$R_M = M/C_2M = F_0M/F_1M \subseteq \text{gr}_F(M).$$

The fact that $\text{gr}_F(M)$ is a module over $\text{gr}_F(V)$ implies that R_M is a module over R_V with operations given as follows. For $a \in V$ and $u \in M$, we set:

$$\begin{aligned} \sigma_0(a)\sigma_0^M(u) &= \sigma_0^M(a_{(-1)}^M u), \\ \{\sigma_0(a), \sigma_0^M(u)\} &= \sigma_0^M(a_{(0)}^M u). \end{aligned}$$

If V is graded and M is a graded V -module, then, as we explained in §4.3, $\text{gr}_F(M)$ is graded $\text{gr}_F(V)$ -module. Thus, R_M is also a graded R_V -module.

We have constructed a functor

$$R : \{(\text{graded}) V\text{-modules}\} \rightarrow \{(\text{graded}) R_V\text{-modules}\}.$$

A V -module M is called *C_2 -cofinite* if R_M is finite dimensional. The condition of C_2 -cofiniteness on modules has several implications, as we will see later.

Example 4.5.13 ($R_{M(c,h)}$ and $R_{L(c,h)}$). We pick a highest weight (c, h) . As in Example 4.5.1, we consider $\mathbb{C}[L_{-2}]$ as the polynomial algebra in one variable L_{-2} , and we equip it with the trivial Poisson bracket. We consider $\bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}]L_{-1}^k$ as a module over the Poisson algebra $\mathbb{C}[L_{-2}]$ with Poisson bracket given by $\{L_{-2}, L_{-1}^k\} = L_{-1}^{k+1}$ for $k \in \mathbb{N}$. By Proposition 4.4.8, Example 4.4.10 and Example 4.5.1, we have the following isomorphism of modules over Poisson algebras

$$\begin{aligned} R_{M(c,h)} &\xrightarrow{\sim} \bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}]L_{-1}^k, \\ \sigma_0^{M(c,h)}(L_{-2}^{M(c,h)}|c, h\rangle) &\mapsto L_{-2}, \\ \sigma_0^{M(c,h)}(L_{-1}^{M(c,h)}|c, h\rangle) &\mapsto L_{-1}. \end{aligned}$$

We have a natural epimorphism

$$\begin{aligned} \pi_{c,h} : M(c, h) &\twoheadrightarrow L(c, h), \\ \pi_{c,h}(u) &= u + J(c, h), \end{aligned}$$

and it satisfies $\ker(\pi_{c,h}) = J(c, h)$. Applying the functor R , we obtain an epimorphism

$$R_{\pi_{c,h}} : R_{M(c,h)} \twoheadrightarrow R_{L(c,h)}.$$

From the equation $\ker(R_{\pi_{c,h}}) = \sigma_0(J(c, h))$, we obtain

$$R_{L(c,h)} \cong \frac{\bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}]L_{-1}^k}{\sigma_0(J(c, h))}.$$

Let V be a vertex algebra such that R_V is finitely generated, and let M be a V -module. The *associated variety of M* , denoted by X_M , and the *singular support of M* , denoted by $SS(M)$, are defined by:

$$\begin{aligned} X_M &= \text{Supp}_{R_M}(R_V), \\ SS(M) &= \text{Supp}_{\text{gr}_F(V)}(\text{gr}_F(M)). \end{aligned}$$

Theorem 4.5.14 ([Li05, Lemma 4.2]). *Let V be a vertex algebra, and let M be a V -module. As a $\text{gr}_F(V)$ -module without the Poisson structure, $\text{gr}_F(M)$ is generated by R_M , i.e.,*

$$\text{gr}_F(M) = \text{gr}_F(V)R_M.$$

Let V be a (graded) vertex algebra, and let M be a (graded) V -module. We have a natural (graded) inclusion $\text{inc} : R_M \hookrightarrow \text{gr}_F(M)$ of (graded) R_V -modules. As we explained in §4.2, we can consider $JR_V \otimes_{R_V} R_M$ as a (graded) JR_V -module. Because we have a (graded) epimorphism of vertex Poisson algebras $\phi_V : JR_V \rightarrow \text{gr}_F(V)$, we can consider the (graded) $\text{gr}_F(V)$ -module $\text{gr}_F(M)$ as a (graded) JR_V -module. By the universal property of $\text{inc} : R_M \hookrightarrow JR_V \otimes_{R_V} R_M$ (see §4.2), there is a (graded) JR_V -module homomorphism $\phi_M : JR_V \otimes_{R_V} R_M \rightarrow \text{gr}_F(M)$ such that the following diagram commutes

$$\begin{array}{ccc} R_M & \xrightarrow{\text{inc}} & JR_V \otimes_{R_V} R_M \\ & \searrow \text{inc} & \downarrow \phi_M \\ & & \text{gr}_F(M) \end{array}$$

Lemma 4.5.15. *The (graded) JR_V -module homomorphism $\phi_M : JR_V \otimes_{R_V} R_M \rightarrow \text{gr}_F(M)$ defined above is surjective.*

Proof. The assertion follows from Lemma 4.5.3 and Theorem 4.5.14. \square

When ϕ_M is an isomorphism, we say the M is *classically free*.

Example 4.5.16 ($JR_{\text{Vir}^c} \otimes_{R_{\text{Vir}^c}} R_{M(c,h)}$). We pick a highest weight (c, h) . The Verma module $M(c, h)$ is always classically free as a Vir^c -module because by Proposition 4.4.8, Example 4.4.10, Example 4.5.1 and Example 4.5.13,

$$\begin{aligned} JR_{\text{Vir}^c} \otimes_{R_{\text{Vir}^c}} R_{M(c,h)} &\cong \mathbb{C}[L_{-2}, L_{-3}, \dots] \otimes_{\mathbb{C}[L_{-2}]} \bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}] L_{-1}^k \\ &\cong \bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}, L_{-3}, \dots] L_{-1}^k \\ &\cong \text{gr}_F(M(c, h)). \end{aligned}$$

Theorem 4.5.17. *Let V be a vertex operator algebra, let M be an admissible V -module, and let $T \subseteq M$ be a subset of homogeneous elements of M . The following are equivalent:*

- (i) M is strongly generated over V by T ;
- (ii) $\sigma_0^M(T)$ generates R_M as a module over R_V without the Poisson structure.

In particular, M is finitely strongly generated over V if and only if R_M is a finitely generated R_V -module.

Proof. We assume that M is strongly generated over V by T . Let $\sigma_0^M(v) \in R_V$ with $v \in M$ arbitrary. We know that v is a linear combination of elements of the form $a_{(-n_1-1)}^{1M} \dots a_{(-n_s-1)}^{sM} u$, where $s, n_1, \dots, n_s \in \mathbb{N}$ and $a^1, \dots, a^s \in V$ and $u \in T$. However, $\sigma_0^M(a_{(-n_1-1)}^{1M} \dots a_{(-n_s-1)}^{sM} u) = 0$ if any $n_i > 0$, and if $n_1 = \dots = n_s = 0$, we have $\sigma_0^M(a_{(-n_1-1)}^{1M} \dots a_{(-n_s-1)}^{sM} u) = \sigma_0^M(a_{(-1)}^{1M} \dots a_{(-1)}^{sM} u) = \sigma_0(a^1) \dots \sigma_0(a^s) \sigma_0^M(u) \in R_V \sigma_0^M(T)$. Therefore, $\sigma_0^M(v) \in R_V \sigma_0^M(T)$ and $R_M = R_V \sigma_0^M(T)$.

We now assume that $R_M = R_V \sigma_0^M(T)$. We define

$$M^0 = \text{span}\{a_{(-1)}^{sM} u \mid a \in V \text{ homogeneous and } u \in T\}.$$

Because any element of V is a sum of a finite number of homogeneous elements, the equation $R_M = R_V \sigma_0^M(T)$ implies that $M = M^0 + C_2 M$. By [Li05, Proposition 4.12],

$$M = \text{span}\{a_{(-n_1-1)}^{1M} \dots a_{(-n_s-1)}^{sM} w \mid s \in \mathbb{Z}_+, n_1 > \dots > n_s \geq 0, a^1, \dots, a^s \in V, w \in M^0\}.$$

By the definition of M^0 , we conclude that M is strongly generated over V by T . \square

Theorem 4.5.18. *Let V be a vertex algebra such that R_V is finitely generated, let M be a V -module, and we assume that R_M is a finitely generated R_V -module. Then the following are equivalent:*

- (i) M is C_2 -cofinite;
- (ii) $\dim(X_M) = 0$;

- (iii) X_M is finite and discrete;
- (iv) $\dim(SS(M)) = 0$.

If V is conical, then (i)–(iv) are also equivalent to

- (v) $X_M = \{\text{point}\}$.

Proof. The proof of Theorem 4.5.10 works with minor modifications. \square

Theorem 4.5.19. *Let V be a C_2 -cofinite conical conformal vertex algebra. Then:*

- (i) Any V -module M satisfies $M = M^{\text{Gen}}$ and (1.6.8);
- (ii) $\text{For} \circ \text{Gen} = \text{Id}_{V\text{-Mod}}$;
- (iii) Both For and Gen are equivalences of categories;
- (iv) If M is a V -module, then M is irreducible as a V -module if and only if M with grading $\bigoplus_{n \in \mathbb{Z}} M^{\text{Gen}}(n)$ is irreducible as an admissible V -module;
- (v) Any irreducible V -module is an irreducible ordinary positive energy representation of V .

Proof.

- (i) The equality $M = M^{\text{Gen}}$ is proved in [ABD03, Proposition 5.7] and (1.6.8) is proved in [ABD03, Lemma 5.6].
- (ii) This is obvious.
- (iii) By Proposition 1.6.11(iv), both For and Gen are essentially surjective functors. Both are fully faithful by definition. Thus, both are equivalences of categories.
- (iv) If M is irreducible as a V -module then it clearly is irreducible as an admissible V -module. We now assume M is irreducible as an admissible V -module. By (i), we can assume that $M = \bigoplus_{n \in \mathbb{Z}} M^{\text{Gen}}(n)$. Let N be a proper submodule of M . Then $L_0^M(N) \subseteq N$. By Lemma 1.5.13(iii), $N = \sum_{\Delta \in \mathbb{C}} N \cap M_{\Delta}^{\text{Gen}}$. This implies that $N = \sum_{n \in \mathbb{Z}} N \cap M^{\text{Gen}}(n)$, i.e., N is a proper graded submodule of M . Therefore, $N = 0$. In conclusion, M is irreducible as a V -module.
- (v) This is [ABD03, Corollary 5.8]. \square

By Theorem 4.5.19, when V is a C_2 -cofinite conical conformal vertex algebra, to find all irreducibles in the category of V -modules, it is enough to find all irreducibles in the category of admissible V -modules. By Example 4.5.1, the Ising model $\text{Vir}_{3,4} = \text{Vir}_{1/2}$ satisfies these conditions.

5. PBW BASIS OF $L(1/2, 1/2)$ AND ITS REFINED CHARACTER

In this section, our objective is to determine a PBW basis of $L(1/2, 1/2)$ explicitly and a formula for the refined character of $\text{gr}^G(L(1/2, 1/2))$ with respect to the standard filtration, as introduced in §4.4. We present a family of series in two variables $f_{a,b,c,d}(t, q)$, where $a, b, c, d \in \mathbb{N}$ are the parameters, which are Nahm sums for the same matrix $(\begin{smallmatrix} 8 & 3 \\ 3 & 2 \end{smallmatrix})$ (cf. [Nah07]) when we set $t = 1$. These series satisfy some recursive relations which will help us to prove series identities. Then, we find a formula for the refined character of $\text{gr}^G(L(1/2, 1/2))$ using these series and the recursive relations they satisfy. The arguments are elementary and combinatorial but lengthy. We recover the series obtained in [AVEH22, Theorem 4] when we set $t = 1$, as expected. In the explicit monomial basis of Theorem 0.4, there are ordinary and exceptional partitions. The ordinary partitions are handled by the theory of $\text{Vir}_{1/2}$ -modules done in §3.3. The exceptional partitions are handled with computer software. As we can see, the arguments are similar to those in [AVEH22], but there are some differences such as the use of the Zhu algebra here but not in [AVEH22], and the fact that we do not have a differential available (see Remark 4.4.2). Finally, we give formal proofs of Theorem 0.3 and Theorem 0.4.

5.1. Conventions and notation. Throughout this section, V will always denote the vertex algebra $\text{Vir}^{1/2}$. From now on, some subscripts or superscripts of modules will be omitted, so for example, α_M^p simplifies to α^p . We set:

$$M = M(1/2, 1/2), L = L(1/2, 1/2), W = J(1/2, 1/2).$$

Then M and L are V -modules and, by definition, $L = M/W$. By Theorem 3.3.6,

$$W = U(\text{Vir})\{u_2, u_3\}, \quad (5.1.1)$$

where the generating singular vectors are:

$$u_2 = (L_{-1}^2 - \frac{4}{3}L_{-2})|1/2\rangle, u_3 = (L_{-1}^3 - 3L_{-2}L_{-1} + \frac{3}{4}L_{-3})|1/2\rangle. \quad (5.1.2)$$

We have a natural epimorphism of V -modules

$$\begin{aligned} \pi : M &\twoheadrightarrow L, \\ \pi(u) &= u + W. \end{aligned}$$

Applying the functor gr^G , we obtain an epimorphism of modules over $\text{gr}^G(V)$

$$\text{gr}^G(\pi) : \text{gr}^G(M) \twoheadrightarrow \text{gr}^G(L),$$

and this produces a natural isomorphism of modules over $\text{gr}^G(V)$

$$\text{gr}^G(M)/K \xrightarrow{\sim} \text{gr}^G(L),$$

where

$$K = \ker(\text{gr}^G(\pi)). \quad (5.1.3)$$

The isomorphism in Example 4.4.10 justifies the following definitions. For a partition $\lambda = [\lambda_1, \dots, \lambda_m, 1, \dots, 1]$ with exactly n ones, we define $u_\lambda \in \bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}, L_{-3}, \dots]L_{-1}^k$ by setting

$$u_\lambda = L_{-\lambda_1} \dots L_{-\lambda_m} L_{-1}^n.$$

We define the *length* of λ as

$$\text{len}(\lambda) = 2m + n$$

and the *weight* of λ as

$$\Delta(\lambda) = \lambda_1 + \dots + \lambda_m + n.$$

Therefore, for a partition λ , we have

$$L_\lambda |1/2\rangle \in G^{\text{len}(\lambda)} M_{1/2+\Delta(\lambda)}.$$

For a partition $\lambda = [\lambda_1, \dots, \lambda_m]$ with $\lambda_m \geq 2$, we define $p_\lambda \in \mathbb{C}[L_{-2}, L_{-3}, \dots]$ by setting

$$p_\lambda = L_{-\lambda_1} \dots L_{-\lambda_m}.$$

From the epimorphism of V -modules $\pi : M \twoheadrightarrow L$, we deduce the following four isomorphisms which are going to be used frequently implicitly.

Proposition 5.1.1. *We have four (conformal) weight-preserving vector space isomorphisms:*

$$\begin{aligned} M &\xrightarrow{\sim} \text{gr}^G(M), \\ L_\lambda |1/2\rangle &\mapsto \alpha^{\text{len}(\lambda)}(L_\lambda |1/2\rangle), \\ \text{gr}^G(M) &\xrightarrow{\sim} \bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}, L_{-3}, \dots]L_{-1}^k, \\ \alpha^{\text{len}(\lambda)}(L_\lambda |1/2\rangle) &\mapsto u_\lambda, \\ L &\xrightarrow{\sim} \text{gr}^G(L), \\ L_\lambda(|1/2\rangle + W) &\mapsto \alpha^{\text{len}(\lambda)}(L_\lambda(|1/2\rangle + W)), \\ \text{gr}^G(L) &\xrightarrow{\sim} \text{gr}^G(M)/K, \\ \alpha^{\text{len}(\lambda)}(L_\lambda(|1/2\rangle + W)) &\mapsto \alpha^{\text{len}(\lambda)}(L_\lambda |1/2\rangle) + K, \end{aligned}$$

where λ is a partition.

In this thesis, we will deal with polynomial algebras written as $\mathbb{C}[L_{-2}, L_{-3}, \dots, L_{-N}]$ and free modules of the form $\bigoplus_{n \leq N} \mathbb{C}[L_{-2}, L_{-3}, \dots, L_{-N}]L_{-1}^n$ for some $N \in \mathbb{N}$. We will always use the degree reverse lexicographic order with $L_{-2} > L_{-3} > \dots > L_{-N}$ and TOP (term over position) with $L_{-1}^0 < L_{-1}^1 < \dots < L_{-1}^N$, see [AL94, §3.5]. Let $u \in \bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}, L_{-3}, \dots, L_{-N}]L_{-1}^k$. We can define the *leading monomial of u* , denoted by $\text{lm}(u)$, as follows. We pick N large enough so that $u \in \bigoplus_{n \leq N} \mathbb{C}[L_{-2}, L_{-3}, \dots, L_{-N}]L_{-1}^n$. Then, we define $\text{lm}(u)$ as the leading monomial in $\bigoplus_{n \leq N} \mathbb{C}[L_{-2}, L_{-3}, \dots, L_{-N}]L_{-1}^n$, which is naturally a subset of $\bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}, L_{-3}, \dots, L_{-N}]L_{-1}^k$. This does not depend on the choice of N .

When working with q -series, the following notation is useful. The q -Pochhammer symbol is $(q)_n = \prod_{j=1}^n (1 - q^j) \in \mathbb{C}[q]$ for $n \in \mathbb{N}$. The recursive proof of the series identities in [AVEH22] suggests the following definition. For $a, b, c, d \in \mathbb{N}$, we define

$$f_{a,b,c,d}(t, q) = \sum_{k_1, k_2 \in \mathbb{N}} t^{4k_1 + 2k_2 + d} \frac{q^{4k_1^2 + 3k_1k_2 + k_2^2 + ak_1 + bk_2 + c}}{(q)_{k_1}(q)_{k_2}} \in \mathbb{C}[[t, q]].$$

The following lemma will be used frequently to verify series identities.

Lemma 5.1.2. *The series $f_{a,b,c,d}(t, q)$ satisfies:*

- (i) $t^m q^n f_{a,b,c,d}(t, q) = f_{a,b,c+n,d+m}(t, q)$ for $m, n \in \mathbb{N}$;
- (ii) $f_{a,b,c,d}(tq^n, q) = f_{a+4n,b+2n,c+dn,d}(t, q)$ for $n \in \frac{1}{2}\mathbb{N}$ and $d \in 2\mathbb{N}$;
- (iii) $f_{a,b,c,d}(t, q) - f_{a+n,b,c,d}(t, q) = \sum_{k=0}^{n-1} f_{a+8+k,b+3,a+c+4+k,d+4}(t, q)$ for $n \in \mathbb{Z}_+$;
- (iv) $f_{a,b,c,d}(t, q) - f_{a,b+n,c,d}(t, q) = \sum_{k=0}^{n-1} f_{a+3,b+2+k,b+c+1+k,d+2}(t, q)$ for $n \in \mathbb{Z}_+$.

Proof.

- (i) Clear.
- (ii) Clear.
- (iii) The key step is replacing k_1 by $k_1 + 1$ in the following computation

$$\begin{aligned} f_{a,b,c,d}(t, q) - f_{a+n,b,c,d}(t, q) &= \sum_{k_1, k_2 \in \mathbb{N}} t^{4k_1 + 2k_2 + d} \frac{q^{4k_1^2 + 3k_1k_2 + k_2^2 + ak_1 + bk_2 + c}}{(q)_{k_1}(q)_{k_2}} (1 - q^{nk_1}) \\ &= \sum_{k_1, k_2 \in \mathbb{N}} t^{4k_1 + 2k_2 + d} \frac{q^{4k_1^2 + 3k_1k_2 + k_2^2 + ak_1 + bk_2 + c}}{(q)_{k_1}(q)_{k_2}} (1 - q^{k_1}) \sum_{k=0}^{n-1} q^{kk_1} \\ &= \sum_{k=0}^{n-1} \sum_{k_1, k_2 \in \mathbb{N}} t^{4k_1 + 2k_2 + d} \frac{q^{4k_1^2 + 3k_1k_2 + k_2^2 + (a+k)k_1 + bk_2 + c}}{(q)_{k_1}(q)_{k_2}} (1 - q^{k_1}) \\ &= \sum_{k=0}^{n-1} \sum_{k_1, k_2 \in \mathbb{N}} t^{4k_1 + 2k_2 + d + 4} \frac{q^{4k_1^2 + 3k_1k_2 + k_2^2 + (a+8+k)k_1 + (b+3)k_2 + a+c+4+k}}{(q)_{k_1}(q)_{k_2}} \\ &= \sum_{k=0}^{n-1} f_{a+8+k,b+3,a+c+4+k,d+4}(t, q). \end{aligned}$$

- (iv) Same trick as (iii) but with k_2 instead of k_1 . □

5.2. A combinatorial argument. A partition $\lambda = [\lambda_1, \dots, \lambda_m]$ contains a partition $\eta = [\eta_1, \dots, \eta_n]$, written as $\eta \subseteq \lambda$, if $m \geq n$ and there is $i \in \mathbb{Z}_+$ such that $1 \leq i \leq m - n + 1$ and $[\lambda_i, \lambda_{i+1}, \dots, \lambda_{i+n-1}] = \eta$.

We define

$$p(t, q) = \sum_{\lambda \in P} t^{\text{len}(\lambda)} q^{\Delta(\lambda)} \in \mathbb{C}[[t, q]],$$

where P is the set of partitions that do not contain any partition in R as defined in §0, i.e.,

$$P = \{\lambda \mid \text{for } \eta \in R, \lambda \not\supseteq \eta\}.$$

We call the last eight partitions of R *exceptional partitions*, and the others involving r are called *ordinary partitions*. For $m, n \in \mathbb{N}$, we set:

$$\begin{aligned} P(n) &= \{\lambda \in P \mid \Delta(\lambda) = n\}, \\ p(q) &= \sum_{n \in \mathbb{N}} |P(n)|q^n \in \mathbb{C}[[q]], \\ P(n, m) &= \{\lambda \in P \mid \text{len}(\lambda) = m \text{ and } \Delta(\lambda) = n\}. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} p(t, q) &= \sum_{m, n \in \mathbb{N}} |P(n, m)|t^m q^n, \\ p(1, q) &= p(q). \end{aligned}$$

We wish to find an expression for $p(t, q)$ as a sum of series $f_{a,b,c,d}(t, q)$ for some tuples (a, b, c, d) .

We now define subsets of P , which will help us in finding an expression for $p(t, q)$, by setting:

$$\begin{aligned} P_{>2} &= \{[\lambda_1, \dots, \lambda_m] \in P \mid \lambda_m > 2 \text{ or } \lambda = \emptyset\}, \\ P_2 &= \{[\lambda_1, \dots, \lambda_m] \in P \mid \lambda_m = 2\}, \end{aligned}$$

and both $p_{>2}(t, q)$ and $P_{>2}(n, m)$ are defined like $p(t, q)$ and $P(n, m)$ were defined. Likewise, we define $P_{>6,5,3}$, $P_{6,5,3}$, $p_{>6,5,3}(t, q)$, $p_{6,5,3}(t, q)$, $P_{>6,5,3}(n, m)$ and $P_{6,5,3}(n, m)$. It turns out that P decomposes as a disjoint union of these smaller objects, and we can find recurrence relations between them to find our desired formula for $p(t, q)$.

Lemma 5.2.1. *The formal power series $p_{>2}(t, q)$ is given by*

$$p_{>2}(t, q) = f_{3,2,0,0}(t, q).$$

Proof. We consider the disjoint union

$$P_{>2} = P_{4,3} \cup P_{6,5,3} \cup P_{>6,5,3} \cup P_{>5,3} \cup P_{4,4} \cup P_{5,4} \cup P_{>5,4} \cup P_{>4},$$

from which we get the formula

$$p_{>2}(t, q) = p_{4,3}(t, q) + p_{6,5,3}(t, q) + p_{>6,5,3}(t, q) + p_{>5,3}(t, q) + p_{4,4}(t, q) + p_{5,4}(t, q) + p_{>5,4}(t, q) + p_{>4}(t, q).$$

These subseries satisfy the following recurrences with initial conditions:

$$\begin{aligned} p_{>4}(t, q) &= p_{4,4}(tq^{1/2}, q) + p_{5,4}(tq^{1/2}, q) + p_{>5,4}(tq^{1/2}, q) + p_{>4}(tq^{1/2}, q), & p_{>4}(0, 0) &= 1, \\ p_{>5,4}(t, q) &= p_{6,5,3}(tq^{1/2}, q) + p_{>6,5,3}(tq^{1/2}, q) + p_{>5,3}(tq^{1/2}, q), & p_{>5,4}(0, 0) &= 0, \\ p_{5,4}(t, q) &= p_{4,3}(tq^{1/2}, q), & p_{5,4}(0, 0) &= 0, \\ p_{4,4}(t, q) &= t^2 q^3 p_{>6,5,3}(tq^{2/2}, q) + t^2 q^3 p_{>5,3}(tq^{2/2}, q), & p_{4,4}(0, 0) &= 0, \\ p_{>5,3}(t, q) &= t^2 q^3 p_{>4}(tq^{1/2}, q), & p_{>5,3}(0, 0) &= 0, \\ p_{>6,5,3}(t, q) &= t^2 q^3 p_{>5,4}(tq^{1/2}, q), & p_{>6,5,3}(0, 0) &= 0, \\ p_{6,5,3}(t, q) &= t^2 q^3 p_{5,4}(tq^{1/2}, q), & p_{6,5,3}(0, 0) &= 0, \\ p_{4,3}(t, q) &= t^2 q^3 p_{4,4}(tq^{1/2}, q) + t^2 q^2 p_{>5,4}(tq^{1/2}, q), & p_{4,3}(0, 0) &= 0. \end{aligned}$$

The solution to these equations is unique if it exists, and we can verify using Lemma 5.1.2 that:

$$\begin{aligned} p_{>4}(t, q) &= f_{6,4,0,0}(t, q), & p_{>5,4}(t, q) &= f_{9,5,4,2}(t, q), \\ p_{5,4}(t, q) &= f_{13,6,9,4}(t, q), & p_{4,4}(t, q) &= f_{12,6,8,4}(t, q), \\ p_{>5,3}(t, q) &= f_{8,5,3,2}(t, q), & p_{>6,5,3}(t, q) &= f_{11,6,8,4}(t, q), \\ p_{6,5,3}(t, q) &= f_{15,7,14,6}(t, q), & p_{4,3}(t, q) &= f_{11,5,7,4}(t, q), \end{aligned}$$

is a solution to these equations. We derive the formula $p_{>2}(t, q) = f_{3,2,0,0}(t, q)$ again from Lemma 5.1.2.

The recurrences follow from the following eight bijections for $m, n \in \mathbb{N}$.

$$P_{>4}(n, 2m) \xrightarrow{\sim} P_{4,4}(n-m, 2m) \cup P_{5,4}(n-m, 2m) \cup P_{>5,4}(n-m, 2m) \cup P_{>4}(n-m, 2m),$$

$$\lambda \mapsto \begin{cases} [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4, 4] & \text{if } [\lambda_{m-1}, \lambda_m] = [5, 5]; \\ [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 5, 4] & \text{if } [\lambda_{m-1}, \lambda_m] = [6, 5]; \\ [\lambda_1 - 1, \dots, \lambda_{m-1} - 1, 4] & \text{if } [\lambda_m] = [5] \text{ and } \lambda_{m-1} > 6; \\ [\lambda_1 - 1, \dots, \lambda_m - 1] & \text{if } \lambda_m > 5, \end{cases}$$

which can be verified directly from the definition of P , as we now show. We note that the operation on λ is always “subtract 1 from each entry of λ ”. For example, a partition $[\lambda_1, \dots, \lambda_m] \in P_{>4}(n, m)$ satisfies exactly one of the following four conditions:

$$[\lambda_{m-1}, \lambda_m] = [5, 5], [\lambda_{m-1}, \lambda_m] = [6, 5], [\lambda_m] = [5] \text{ and } \lambda_{m-1} > 6, \text{ or } \lambda_m > 5.$$

We assume λ satisfies $[\lambda_{m-1}, \lambda_m] = [5, 5]$. Then $[\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4, 4]$ belongs to P because: it cannot contain some ordinary partition (otherwise λ would contain some ordinary partition), and we can verify case by case that it cannot contain some exceptional partition. Actually, we have $[\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4, 4] \in P_{4,4}(n-m, 2m)$, as we easily see, so we get the first part of our bijection. The other parts follow a similar reasoning, and an explicit inverse of this function can be written if desired. We also note that $[5] \mapsto [4]$ and $\emptyset \mapsto \emptyset$ in this bijection. We now verify that this bijection implies our first recurrence. One key step is replacing n by $n-m$ in the following computation

$$\begin{aligned} & p_{4,4}(tq^{1/2}, q) + p_{5,4}(tq^{1/2}, q) + p_{>5,4}(tq^{1/2}, q) + p_{>4}(tq^{1/2}, q) \\ &= \sum_{m,n \in \mathbb{N}} (|P_{4,4}(n, m)| + |P_{5,4}(n, m)| + |P_{>5,4}(n, m)| + |P_{>4}(n, m)|)(tq^{1/2})^m q^n \\ &= \sum_{m,n \in \mathbb{N}} (|P_{4,4}(n, 2m)| + |P_{5,4}(n, 2m)| + |P_{>5,4}(n, 2m)| + |P_{>4}(n, 2m)|)(tq^{1/2})^{2m} q^n \\ &= \sum_{m,n \in \mathbb{N}} (|P_{4,4}(n-m, 2m)| + |P_{5,4}(n-m, 2m)| + |P_{>5,4}(n-m, 2m)| + |P_{>4}(n-m, 2m)|)t^{2m} q^n \\ &= \sum_{m,n \in \mathbb{N}} |P_{>4}(n, 2m)|t^{2m} q^n \\ &= p_{>4}(t, q). \end{aligned}$$

In the next bijections, we do something similar. In some cases, the operation on λ is “subtract 2 from each entry of λ ”, and sometimes we need to change the tail of the result after doing this operation to get something in P . The next two lemmas also employ very similar arguments.

$$P_{>5,4}(n, 2m) \xrightarrow{\sim} P_{6,5,3}(n-m, 2m) \cup P_{>6,5,3}(n-m, 2m) \cup P_{>5,3}(n-m, 2m),$$

$$\lambda \mapsto \begin{cases} [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 6, 5, 3] & \text{if } [\lambda_{m-2}, \lambda_{m-1}, \lambda_m] = [7, 6, 4]; \\ [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 5, 3] & \text{if } [\lambda_{m-1}, \lambda_m] = [6, 4] \text{ and } \lambda_{m-2} > 7; \\ [\lambda_1 - 1, \dots, \lambda_{m-1} - 1, 3] & \text{if } [\lambda_m] = [4] \text{ and } \lambda_{m-1} > 6, \end{cases}$$

$$P_{5,4}(n, 2m) \xrightarrow{\sim} P_{4,3}(n-m, 2m),$$

$$\lambda \mapsto [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4, 3],$$

$$P_{4,4}(n, 2m) \xrightarrow{\sim} P_{>6,5,3}(n-2m-1, 2(m-1)) \cup P_{>5,3}(n-2m-1, 2(m-1)),$$

$$\lambda \mapsto \begin{cases} [\lambda_1 - 2, \dots, \lambda_{m-3} - 2, 5, 3] & \text{if } [\lambda_{m-2}, \lambda_{m-1}, \lambda_m] = [7, 4, 4]; \\ [\lambda_1 - 2, \dots, \lambda_{m-2} - 2, 3] & \text{if } [\lambda_{m-1}, \lambda_m] = [4, 4] \text{ and } \lambda_{m-2} > 7, \end{cases}$$

$$P_{>5,3}(n, 2m) \xrightarrow{\sim} P_{>4}(n - m - 2, 2(m - 1)),$$

$$\lambda \mapsto [\lambda_1 - 1, \dots, \lambda_{m-1} - 1],$$

$$P_{>6,5,3}(n, 2m) \xrightarrow{\sim} P_{>5,4}(n - m - 2, 2(m - 1)),$$

$$\lambda \mapsto [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4],$$

$$P_{6,5,3}(n, 2m) \xrightarrow{\sim} P_{5,4}(n - m - 2, 2(m - 1)),$$

$$\lambda \mapsto [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 5, 4],$$

$$P_{4,3}(n, 2m) \xrightarrow{\sim} P_{4,4}(n - m - 2, 2(m - 1)) \cup P_{>5,4}(n - m - 1, 2(m - 1)),$$

$$\lambda \mapsto \begin{cases} [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 4, 4] & \text{if } [\lambda_{m-2}, \lambda_{m-1}, \lambda_m] = [6, 4, 3]; \\ [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4] & \text{if } [\lambda_{m-1}, \lambda_m] = [4, 3] \text{ and } \lambda_{m-2} > 6. \quad \square \end{cases}$$

Lemma 5.2.2. *The formal power series $p_{>2,1}(t, q)$ is given by*

$$p_{>2,1}(t, q) = f_{5,2,1,1}(t, q).$$

Proof. We consider the disjoint union

$$P_{>2,1} = P_{5,3,1} \cup P_{>5,3,1} \cup P_{5,4,1} \cup P_{>5,4,1} \cup P_{>4,1}.$$

We have bijections:

$$P_{>4,1}(n, 2m + 1) \xrightarrow{\sim} P_{>4}(n - 1, 2m),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_m],$$

$$P_{>5,4,1}(n, 2m + 1) \xrightarrow{\sim} P_{>5,4}(n - 1, 2m),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_m],$$

$$P_{5,4,1}(n, 2m + 1) \xrightarrow{\sim} P_{5,4}(n - 1, 2m),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_m],$$

$$P_{>5,3,1}(n, 2m + 1) \xrightarrow{\sim} P_{>5,3}(n - 1, 2m),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_m],$$

$$P_{5,3,1}(n, 2m + 1) \xrightarrow{\sim} P_{>6,5,3}(n - 1, 2m),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_m],$$

from which, together with the proof of Lemma 5.2.1, we get:

$$p_{>4,1}(t, q) = tqp_{>4}(t, q) = f_{6,4,1,1}(t, q),$$

$$p_{>5,4,1}(t, q) = tqp_{>5,4}(t, q) = f_{9,5,5,3}(t, q),$$

$$p_{5,4,1}(t, q) = tqp_{5,4}(t, q) = f_{13,6,10,5}(t, q),$$

$$p_{>5,3,1}(t, q) = tqp_{>5,3}(t, q) = f_{8,5,4,3}(t, q),$$

$$p_{5,3,1}(t, q) = tqp_{>6,5,3}(t, q) = f_{11,6,9,5}(t, q).$$

The formula for $p_{>2,1}(t, q)$ then follows from Lemma 5.1.2. □

Lemma 5.2.3. *The formal power series $p_{>2,1,1}(t, q)$ is given by*

$$p_{>2,1,1}(t, q) = f_{6,3,2,2}(t, q).$$

Proof. We consider the disjoint union

$$P_{>3,1,1} = P_{4,1,1} \cup P_{>4,1,1}.$$

We have bijections:

$$\begin{aligned} P_{>4,1,1}(n, 2m+2) &\xrightarrow{\sim} P_{>4,1}(n-1, 2m+1), \\ \lambda &\mapsto [\lambda_1, \dots, \lambda_m, 1], \end{aligned}$$

$$\begin{aligned} P_{4,1,1}(n, 2m+2) &\xrightarrow{\sim} P_{>5,4,1}(n-1, 2m+1), \\ \lambda &\mapsto [\lambda_1, \dots, \lambda_m, 1], \end{aligned}$$

from which, together with the proof of Lemma 5.2.2, we get:

$$\begin{aligned} p_{>4,1,1}(t, q) &= tqp_{>4,1}(t, q) = f_{6,4,2,2}(t, q), \\ p_{4,1,1}(t, q) &= tqp_{>5,4,1}(t, q) = f_{9,5,6,4}(t, q). \end{aligned}$$

The formula for $p_{>3,1,1}(t, q)$ then follows from Lemma 5.1.2. \square

Lemma 5.2.4. *The formal power series $p(t, q)$ is given by*

$$p(t, q) = f_{3,2,0,0}(t, q) + f_{5,2,1,1}(t, q) + f_{6,3,2,2}(t, q).$$

Proof. This follows from the disjoint union

$$P = P_{>2} \cup P_{>2,1} \cup P_{>3,1,1}$$

together with Lemma 5.2.1, Lemma 5.2.2 and Lemma 5.2.3. \square

Lemma 5.2.5. *The formal power series $p(q)$ satisfies*

$$\text{ch}_L(q) = q^{1/2}p(q).$$

Proof. This follows from Lemma 5.2.4 by setting $t = 1$ and [AVEH22, Theorem 4] together with Lemma 5.1.2. \square

5.3. Computing leading monomials. We recall that we have defined $V = \text{Vir}^{1/2}$, $M = M(1/2, 1/2)$ and $L = L(1/2, 1/2)$.

Lemma 5.3.1. *Let $\lambda = [\lambda_1, \dots, \lambda_m]$ be a partition with $\lambda_m \geq 2$ or $\lambda = \emptyset$, and we consider M as a module over V . Then*

$$(L_\lambda |0\rangle)_{(-1)} |1/2\rangle = L_\lambda |1/2\rangle + u \quad \text{for some } u \in G^{2m-1}M.$$

Proof. The isomorphism $\text{gr}^G(V) \xrightarrow{\sim} \mathbb{C}[L_{-2}, L_{-3}, \dots]$ maps $L_\lambda |0\rangle$ to p_λ . The isomorphism $\text{gr}^G(M) \xrightarrow{\sim} \bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}, L_{-3}, \dots] L_{-1}^k$ maps $|1/2\rangle$ to L_{-1}^0 and $L_\lambda |1/2\rangle$ to u_λ . The equality $p_\lambda \cdot L_{-1}^0 = u_\lambda$ in $\text{gr}^G(M)$ translates to the equality $(L_\lambda |0\rangle)_{(-1)} |1/2\rangle = L_\lambda |1/2\rangle + u$ in M for some $u \in G^{2m-1}M$. \square

We know $\text{gr}^G(M)$ is a free module over $\text{gr}^G(V)$

$$\text{gr}^G(M) = \bigoplus_{k \in \mathbb{N}} \text{gr}^G(V) L_{-1}^k.$$

For $k \in \mathbb{N}$, we call

$$\iota_k : \text{gr}^G(V) \hookrightarrow \text{gr}^G(M)$$

the insertion of $\text{gr}^G(V)$ into the k -th component of $\text{gr}^G(M)$.

Lemma 5.3.2. *Let $a \in G^p V$ for some $p \in \mathbb{N}$. Then $a_{(-1)} |1/2\rangle \in G^p M$ and*

$$\iota_0(\alpha^p(a)) = \alpha^p(a_{(-1)} |1/2\rangle).$$

Proof. The result follows immediately from Lemma 5.3.1. \square

We have a natural quotient map

$$\begin{aligned}\pi_0 : V &\twoheadrightarrow \text{Vir}_{3,4}, \\ \pi_0(a) &= a + U(\text{Vir})\{a_{3,4}\},\end{aligned}$$

where $a_{3,4}$ is the singular vector of V that generates its maximal proper ideal as in [AVEH22] and (1.5.6). Applying the functor gr^G , we obtain an epimorphism of \mathbb{N} -graded vertex Poisson algebras

$$\text{gr}^G(\pi_0) : \text{gr}^G(V) \twoheadrightarrow \text{gr}^G(\text{Vir}_{3,4}).$$

We set $I = \ker(\text{gr}^G(\pi_0))$, following the notation of [AVEH22], and we recall the definition of K given in (5.1.3).

Lemma 5.3.3. *We have the inclusion*

$$\iota_0(I) \subseteq K.$$

Proof. By Theorem 3.3.4, the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\pi_0} & \text{Vir}_{3,4} \\ & \searrow Y_V^L & \downarrow Y_{\text{Vir}_{3,4}}^L \\ & & \mathcal{F}(L) \end{array}$$

The commutativity of this diagram implies the following statement

$$\text{for } a \in U(\text{Vir})\{a_{3,4}\}, u \in M \text{ and } n \in \mathbb{Z}, a_{(n)}u \in W,$$

where W is defined in (5.1.1). We use this statement in the following simplified form

$$\text{for } a \in U(\text{Vir})\{a_{3,4}\}, a_{(-1)}|1/2\rangle \in W. \quad (5.3.1)$$

We note that

$$I = \sum_{p \in \mathbb{N}} \alpha^p(U(\text{Vir})\{a_{3,4}\} \cap G^p V).$$

Similarly, we have a formula for K

$$K = \sum_{p \in \mathbb{N}} \alpha^p(W \cap G^p M).$$

Let $\alpha^p(a) \in I$ with $a \in U(\text{Vir})\{a_{3,4}\} \cap G^p V$. By Lemma 5.3.2 and (5.3.1), we have $a_{(-1)}|1/2\rangle \in W \cap G^p M$ and also $\iota_0(\alpha^p(a)) = \alpha^p(a_{(-1)}|1/2\rangle) \in K$, finishing the proof. \square

Remark 5.3.4. The proof of Lemma 5.3.3 also works for other values of $c = c_{p,q}$ and $h = h_{m,n}$ as in Theorem 3.3.4.

We need to compute all leading monomials of elements of K . To do this, we need to order the PBW basis of $U(\text{Vir}_{\leq -1}) = \text{span}\{L_\lambda \mid \lambda \text{ is a partition}\}$ first by length, then by degree reverse lexicographic order and finally by position. Formally, for any partitions λ and η , we define

$$L_\lambda \leq L_\eta \text{ if and only if } \begin{cases} \text{len}(\lambda) < \text{len}(\eta) \text{ or} \\ \text{len}(\lambda) = \text{len}(\eta) \text{ and } u_\lambda \leq u_\eta. \end{cases}$$

For $x \in U(\text{Vir}_{\leq -1})$ with $x \neq 0$, we may write

$$x = c_1 L_{\lambda_1} + c_2 L_{\lambda_2} + \cdots + c_r L_{\lambda_r},$$

where for $1 \leq i \leq r$, $0 \neq c_i \in \mathbb{C}$ and $L_{\lambda_1} > L_{\lambda_2} > \cdots > L_{\lambda_r}$. We define the *leading monomial* of x as $\text{lm}(x) = L_{\lambda_1}$. We set $\text{lm}(0) = 0$. Next, we extend the definition of lm from $U(\text{Vir}_{\leq -1})$ to M by considering the isomorphism of vector spaces $U(\text{Vir}_{\leq -1}) \xrightarrow{\sim} M, L_\lambda \mapsto L_\lambda |1/2\rangle$, where λ is a partition. For example:

- (i) We have $\text{lm}(L_{-1}^2 - \frac{3}{4}L_{-2}) = L_{-2}$ because the power product of L_{-1}^2 is 1, the power product of L_{-1}^0 is L_{-2} , $L_{-2} > 1$ by degree considerations, and we are using TOP with $L_{-1}^0 < L_{-1}^1 < L_{-1}^2$. We also have $\text{lm}((L_{-1}^2 - \frac{3}{4}L_{-2}) |1/2\rangle) = L_{-2} |1/2\rangle$ because $\text{len}(L_{-1}^2) = \text{len}(L_{-2}) = 2$.
- (ii) We have $\text{lm}(L_{-1}^3 - 3L_{-2}L_{-1} + \frac{3}{4}L_{-3}) = L_{-2}L_{-1}$ because the power product of L_{-1}^3 is 1, the power product of L_{-1}^1 is L_{-2} , the power product of L_{-1}^0 is L_{-3} , $L_{-2} > L_{-3} > 1$ by definition and degree considerations, and we are using TOP with $L_{-1}^0 < L_{-1}^1 < L_{-1}^2 < L_{-1}^3$. Again, we have $\text{lm}((L_{-1}^3 - 3L_{-2}L_{-1} + \frac{3}{4}L_{-3}) |1/2\rangle) = L_{-2}L_{-1} |1/2\rangle$ because $\text{len}(L_{-1}^3) = \text{len}(L_{-2}L_{-1}) = 3$ and $\text{len}(L_{-3}) = 2$.
- (iii) We have $\text{lm}(L_{-1}^4 - 3L_{-3}L_{-1} - 6L_{-4}) = L_{-3}L_{-1}$ because the power product of L_{-1}^4 is 1, the power product of L_{-1}^1 is L_{-3} , the power product of L_{-1}^0 is L_{-4} , $L_{-3} > L_{-4} > 1$ by definition and degree considerations, and we are using TOP with $L_{-1}^0 < L_{-1}^1 < L_{-1}^2 < L_{-1}^3 < L_{-1}^4$. However, we now have $\text{lm}((L_{-1}^4 - 3L_{-3}L_{-1} - 6L_{-4}) |1/2\rangle) = L_{-3}L_{-1} |1/2\rangle$ because $\text{len}(L_{-1}^4) = 4$, $\text{len}(L_{-3}L_{-1}) = 3$ and $\text{len}(L_{-4}) = 2$. In fact, we have $(L_{-1}^4 - 3L_{-3}L_{-1} - 6L_{-4}) |1/2\rangle \in W$, as we will see in a moment.

Remark 5.3.5. The definition of the order in the PBW basis of $U(\text{Vir}_{\leq -1})$ was made so that for a partition λ and $u \in M$, if $\text{lm}(u) = L_\lambda |1/2\rangle$, then $\text{lm}(\alpha^{\text{len}(\lambda)}(u)) = u_\lambda$.

This order also helps us in computing the exceptional partitions faster, as we do not have to compute Gröbner bases explicitly because it is enough to transform matrices into row reduced echelon form.

A basis of $\text{gr}^G(M)_{1/2+n}$ is given by $\{L_\lambda |1/2\rangle \mid \Delta(\lambda) = n\}$ and has $p(n)$ elements. For $n \in \mathbb{N}$, we compute the matrix A_n with $p(n-2) + p(n-3)$ rows and $p(n)$ columns, which is given by stacking the matrix $A_n^{m_3}$ below the matrix $A_n^{m_2}$. The matrix $A_n^{m_k}$ is given by

$$A_n^{m_k}(i, j) = \text{coefficient of } L_{\lambda_j} |1/2\rangle \text{ in } L_{\lambda_i} u_k,$$

for $1 \leq i \leq p(n-k)$, $1 \leq j \leq p(n)$ and $k = 2, 3$, where $\lambda_1, \dots, \lambda_{p(n)}$ are the partitions of n ordered in such a way that $L_{\lambda_1} > L_{\lambda_2} > \dots > L_{\lambda_{p(n)}}$, and u_2, u_3 are given by (5.1.2).

We now transform A_n into row reduced echelon form, obtaining a matrix A_n^W which has an unknown number of nonzero rows and $p(n)$ columns. For each pivot λ of A_n^W , let u_λ^W be the element of $\text{gr}^G(M)$ corresponding to the row which has $L_\lambda |1/2\rangle$ as pivot. In other words, $u_\lambda^W = L_\lambda |1/2\rangle + (\text{lower order terms})$. We set $u_\lambda^K = \alpha^{\text{len}(\lambda)}(u_\lambda^W)$ for each pivot λ of A_n^W .

Remark 5.3.6. By construction and Remark 5.3.5, for each pivot λ of A_n^W , we have $u_\lambda^W \in W$, $u_\lambda^K \in K$ and $\text{lm}(u_\lambda^K) = u_\lambda$.

For example, when $n = 4$, the partitions are ordered in the following way

$$[[2, 2], [2, 1, 1], [1, 1, 1, 1], [3, 1], [4]],$$

the matrix A_4^W is given by

$$A_4^W = \begin{pmatrix} 1 & 0 & 0 & -\frac{3}{16} & -\frac{15}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & -3 & -6 \end{pmatrix},$$

and the pivots are $[2, 2]$, $[2, 1, 1]$ and $[1, 1, 1, 1]$. Therefore:

$$\begin{aligned} u_{[2,2]}^W &= (L_{[2,2]} - \frac{3}{16}L_{[3,1]} - \frac{15}{8}L_{[4]}) |1/2\rangle, & u_{[2,2]}^K &= L_{-2}L_{-2}, \\ u_{[2,1,1]}^W &= (L_{[2,1,1]} - \frac{1}{4}L_{[3,1]} - \frac{5}{2}L_{[4]}) |1/2\rangle, & u_{[2,1,1]}^K &= L_{-2}L_{-1}^2, \\ u_{[1,1,1,1]}^W &= (L_{[1,1,1,1]} - 3L_{[3,1]} - 6L_{[4]}) |1/2\rangle, & u_{[1,1,1,1]}^K &= L_{-1}^4. \end{aligned}$$

The partitions $[2]$, $[1, 1, 1]$, $[3, 1, 1]$, $[3, 3]$, $[4, 3, 1]$, $[4, 4, 1]$, $[5, 4, 1, 1]$ and $[6, 5, 3, 1]$ are pivots of the matrices $A_2^W, A_3^W, A_5^W, A_6^W, A_8^W, A_9^W, A_{11}^W$ and A_{15}^W respectively, see [Sal23, ising-modules.ipynb]. We set

$$K' = (u_{[2]}^K, u_{[1,1,1]}^K, u_{[3,1,1]}^K, u_{[3,3]}^K, u_{[4,3,1]}^K, u_{[4,4,1]}^K, u_{[5,4,1,1]}^K, u_{[6,5,3,1]}^K, \iota_0(I))_{\text{psn}},$$

where psn denotes the Poisson submodule generated by the given subset. By Lemma 5.3.3, Remark 5.3.6 and the fact that K is a vertex Poisson submodule (not just a submodule), we have

$$K' \subseteq K.$$

Let us consider the Poisson structure of $\text{gr}^G(M)$ as a module over $\text{gr}^G(V)$. We have

$$\begin{aligned} (L_{-2})_{(0)}(u_\lambda) &= \alpha^2(\omega)_{(0)}\alpha^{\text{len}(\lambda)}(L_\lambda | 1/2) \\ &= \alpha^{\text{len}(\lambda)+1}(L_{-1}L_\lambda | 1/2) \\ &= u_{[\lambda,1]}, \end{aligned}$$

where λ is any partition, and $[\lambda, 1]$ denotes the partition λ with a 1 appended at the end.

Remark 5.3.7. While the Poisson structure of $\text{gr}^G(V)$ is trivial (i.e., is zero), the Poisson structure of $\text{gr}^G(M)$ is not. In fact, if $u \in \text{gr}^G(M)$, and λ is a partition, then

$$\text{lm}(u) = u_\lambda \text{ implies } \text{lm}((L_{-2})_{(0)}(u)) = u_{[\lambda,1]}.$$

It is interesting to note that we use the Poisson structure, unlike most results in §4 and [AVEH22].

Let \overline{R} be the set of partitions containing some partition of R , i.e.,

$$\overline{R} = \{\lambda \mid \text{there is some partition } \eta \in R \text{ such that } \lambda \supseteq \eta\}.$$

Lemma 5.3.8. *For $\lambda \in \overline{R}$, there exists $u \in K'$ such that $\text{lm}(u) = u_\lambda$.*

Proof. By the definition of K' and [AVEH22, Proposition 5.1], we know that for $\lambda \in R$, there exists $u \in K'$ such that $\text{lm}(u) = u_\lambda$. We now assume $\lambda \in \overline{R}$, which means there is $\eta \in R$ such that $\lambda \supseteq \eta$. Therefore, λ is obtained from η by adding some integers greater than or equal to two and adding k ones. We pick $v \in K'$ such that $\text{lm}(v) = u_\eta$ and some power product $p_\tau \in \mathbb{C}[L_{-2}, L_{-3}, \dots]$, for some partition τ , such that $\text{lm}((L_{-2})_{(0)}^k(p_\tau v)) = u_\lambda$ (this can be done because of Remark 5.3.7 and the fact that lm is multiplicative). Thus, we set $u = (L_{-2})_{(0)}^k(p_\tau v) \in K'$ to get $\text{lm}(u) = u_\lambda$. For example, if we take $\lambda = [3, 2, 1, 1] \in \overline{R}$, then $\eta = [2] \in R$. In this case, $v = L_{-2} - 3/4L_{-1}^2 \in K'$ is such that $\text{lm}(v) = u_{[2]}$, and we take $u = L_{-3}L_{-2}L_{-1}^2 - 3/4L_{-3}L_{-1}^4 \in K'$. \square

Remark 5.3.9. For any partitions λ and η with the same number of ones, if $\lambda \supseteq \eta$, then $u_\eta \mid u_\lambda$. The converse is not true. For example, $u_{[4,2]} \mid u_{[4,3,2]}$, but $[4, 3, 2] \not\supseteq [4, 2]$. However, if $\eta = [\eta_1, \dots, \eta_m, 1, \dots, 1]$ with $\eta_m \geq 2$ and $\eta_1 - \eta_m \leq 1$, then $\lambda \supseteq \eta$ if and only if $u_\eta \mid u_\lambda$, provided λ and η have the same number of ones.

The following lemma says our set of partitions R is special. If R were $\{[4, 2]\}$, for example, the following lemma would not be true.

Lemma 5.3.10. *There is an alternative description for P , namely*

$$P = \{\lambda \mid \text{for } \eta \in \overline{R}, u_\eta \nmid u_\lambda\}.$$

Proof. We have to prove the following equality

$$\{\lambda \mid \text{for } \eta \in R, \lambda \not\supseteq \eta\} = \{\lambda \mid \text{for } \eta \in \overline{R}, u_\eta \nmid u_\lambda\}.$$

First, we prove the inclusion (\supseteq). We assume λ belongs to the right set, and $\lambda \supseteq \eta$ for some $\eta \in R$. Then λ has k more ones than η for some $k \in \mathbb{N}$. By Remark 5.3.9, $u_{[\eta, 1, \dots, 1]} \mid u_\lambda$, where we added k ones to η , a contradiction. Therefore, $\lambda \not\supseteq \eta$ for $\eta \in R$, which means λ belongs to the left set.

We now prove the inclusion (\subseteq). We assume λ belongs to the left set. Then λ has 0, 1 or 2 ones. We assume λ has 0 ones. It is enough to prove that for $\eta \in R$ with 0 ones, $u_\eta \nmid u_\lambda$. By Remark 5.3.9, if η is equal to $[2]$, $[3, 3]$, $[r, r, r]$, $[r+1, r, r]$ or $[r+1, r+1, r]$ for some $r \geq 3$, then $u_\eta \nmid u_\lambda$. We assume $\eta = [r+2, r+1, r]$ for some $r \geq 3$ and $u_\eta \mid u_\lambda$. Then λ contains $[r+2, r+1, \dots, r+1, r]$, where $r+1$ appears $k \geq 1$ times. If $k \geq 2$, then λ contains $[r+1, r+1, r]$,

which is not possible. If $k = 1$, then λ contains $[r + 2, r + 1, r]$, which is not possible. Therefore, we cannot have $u_\eta \mid u_\lambda$. Continuing this way, we obtain that $u_\eta \nmid u_\lambda$ for $\eta \in R$ with 0 ones. The same argument can be applied when λ has 1 or 2 ones. Therefore, λ belongs to the right set. \square

5.4. Proofs of the main theorems.

Proof of Theorem 0.4. Since we cannot apply Gröbner basis theory directly on the free module $\bigoplus_{k \in \mathbb{N}} \mathbb{C}[L_{-2}, L_{-3}, \dots] L_{-1}^k$, we need to truncate somehow. We have to keep in mind the isomorphisms given in Proposition 5.1.1. For $N \in \mathbb{N}$, we define:

$$\begin{aligned} \text{gr}^G(M)_{\leq 1/2+N} &= \sum_{n \leq N} \text{gr}^G(M)_{1/2+n}, \\ \text{gr}^G(L)_{\leq 1/2+N} &= \sum_{n \leq N} \text{gr}^G(L)_{1/2+n}. \end{aligned}$$

We note that $\text{gr}^G(M)_{\leq 1/2+N}$ is a vector subspace of the free module

$$F_N = \bigoplus_{n \leq N} \mathbb{C}[L_{-2}, L_{-3}, \dots, L_{-N}] L_{-1}^n$$

with base ring $\mathbb{C}[L_{-2}, L_{-3}, \dots, L_{-N}]$ because a basis of $\text{gr}^G(M)_{\leq 1/2+N}$ is given by elements of the form u_λ , with all the elements of λ being less than N , and with λ having at most N ones. We note that $K \cap F_N$ is a submodule of F_N for $N \in \mathbb{N}$.

For $N \in \mathbb{N}$, we have natural vector space isomorphisms

$$\frac{F_N}{K \cap F_N} \supseteq \frac{\text{gr}^G(M)_{\leq 1/2+N}}{K \cap F_N} \xrightarrow{\sim} \frac{\text{gr}^G(M)_{\leq 1/2+N}}{K \cap \text{gr}^G(M)_{\leq 1/2+N}} \xrightarrow{\sim} \text{gr}^G(L)_{\leq 1/2+N}. \quad (5.4.1)$$

Therefore, if we find a basis of each vector space $\text{gr}^G(M)_{\leq 1/2+N}/K \cap F_N \subseteq F_N/K \cap F_N$ such that each basis is contained in the next one when considering the isomorphism (5.4.1), we get a basis of $\text{gr}^G(L)$ by taking the union of these bases because $\bigcup_{N \in \mathbb{N}} \text{gr}^G(L)_{\leq 1/2+N} = \text{gr}^G(L)$.

Let G_N be a Gröbner basis of $K \cap F_N$. We define

$$B_N = \{u_\lambda \mid \Delta(\lambda) \leq N \text{ and for } u \in G_N, \text{lm}(u) \nmid u_\lambda\}.$$

By [AL94, Proposition 3.6.4],

$$\{u_\lambda + K \cap F_N \mid u_\lambda \in B_N\} \quad (5.4.2)$$

is a vector space basis of $\text{gr}^G(M)_{\leq 1/2+N}/K \cap F_N$. Therefore, by Lemma 5.2.5, isomorphism (5.4.1) and the isomorphism between $\text{gr}^G(L)_{1/2+n}$ and $L_{1/2+n}$ of Proposition 5.1.1, we have

$$|B_N| = \sum_{n \leq N} \dim(\text{gr}^G(L)_{1/2+n}) = \sum_{n \leq N} \dim(L_{1/2+n}) = \sum_{n \leq N} |P(n)|.$$

We define

$$\overline{B}_N = \{u_\lambda \mid \Delta(\lambda) \leq N \text{ and for } \eta \in \overline{R}, u_\eta \nmid u_\lambda\}.$$

By Lemma 5.3.10, we also have

$$\overline{B}_N = \{u_\lambda \mid \Delta(\lambda) \leq N \text{ and } \lambda \in P\}.$$

From the definition of $P(n)$, we see that

$$|\overline{B}_N| = \sum_{n \leq N} |P(n)|.$$

On the other hand, $B_N \subseteq \overline{B}_N$. This is because if $u_\lambda \in B_N$ and $\eta \in \overline{R}$, then $u_\eta \nmid u_\lambda$, as we now show. Suppose, for the sake of contradiction, that $u_\eta \mid u_\lambda$. By Lemma 5.3.8, there would exist $u \in K' \cap F_N \subseteq K \cap F_N$ such that $\text{lm}(u) = u_\eta$. However, by the definition of a Gröbner basis, there would then exist $v \in G_N$ such that $\text{lm}(v) \mid \text{lm}(u)$, implying $\text{lm}(v) \mid u_\lambda$, a contradiction. As both B_N and \overline{B}_N are finite sets and $|B_N| = |\overline{B}_N|$, we get $B_N = \overline{B}_N$. We see that the family of bases given by (5.4.2) satisfies the property that each basis is a subset of the next. This observation,

together with the isomorphism between L and $\text{gr}^G(L)$ of Proposition 5.1.1, concludes the proof of Theorem 0.4. \square

Proof of Theorem 0.3. We recall that the basis of Theorem 0.4 also gives a basis of $\text{gr}^G(L)$. For a partition λ , we have

$$\alpha^{\text{len}(\lambda)}(L_\lambda(|1/2\rangle + W)) \in \alpha^{\text{len}(\lambda)}(G^{\text{len}(\lambda)}L_{1/2+\Delta(\lambda)}).$$

Thus, when calculating the refined character of $\text{gr}^G(L)$, $\Delta(\lambda)$ is incremented by $1/2$. Therefore, $\text{ch}_{\text{gr}^G(L)}(t, q) = q^{1/2}p(t, q)$ and together with Lemma 5.2.4, we conclude the proof of Theorem 0.3. \square

Corollary 5.4.1. *We have the equality*

$$K' = K.$$

Proof. This proof is a copy of the proof of Theorem 0.3. We know that $K' \subseteq K$.

Let $N \in \mathbb{N}$, and let G'_N be a Gröbner basis of $K' \cap F_N$ considered as a submodule of F_N . Like in the proof of Theorem 0.3, we set

$$B'_N = \{u_\lambda \mid \Delta(\lambda) \leq N \text{ and for } u \in G'_N, \text{lm}(u) \nmid u_\lambda\}.$$

By [AL94, Proposition 3.6.4], we have

$$|B'_N| = \dim \left(\frac{\text{gr}^G(M)_{\leq 1/2+N}}{K' \cap F_N} \right) = \dim \left(\frac{\text{gr}^G(M)_{\leq 1/2+N}}{K' \cap \text{gr}^G(M)_{\leq 1/2+N}} \right).$$

We can apply the same argument as in the proof of Theorem 0.3 and conclude that $B'_N \subseteq \overline{B_N}$, where $\overline{B_N}$ is the defined in the same way. But $B_N = \overline{B_N}$ and

$$|B_N| = \dim \left(\frac{\text{gr}^G(M)_{\leq 1/2+N}}{K \cap \text{gr}^G(M)_{\leq 1/2+N}} \right).$$

Therefore, we have $|B'_N| \leq |B_N|$, and this implies

$$\dim(K' \cap \text{gr}^G(M)_{\leq 1/2+N}) \geq \dim(K \cap \text{gr}^G(M)_{\leq 1/2+N}) \geq \dim(K' \cap \text{gr}^G(M)_{\leq 1/2+N}).$$

Consequently, $K' \cap \text{gr}^G(M)_{\leq 1/2+N} = K \cap \text{gr}^G(M)_{\leq 1/2+N}$ for $N \in \mathbb{N}$. Taking the union of these subspaces, we get $K' = K$. \square

APPENDIX A. ALMOST COMMUTATIVE ALGEBRAS

Let A be an associative (not necessarily commutative) algebra with unit 1 and filtration $(A^p)_{p \in \mathbb{Z}}$ such that:

- (i) $A^p = 0$ for $p < 0$;
- (ii) $1 \in A^0$;
- (iii) $A^0 \subseteq A^1 \subseteq \dots$;
- (iv) $A^p A^q \subseteq A^{p+q}$ for $p, q \in \mathbb{Z}$.

Let

$$\text{gr}(A) = \bigoplus_{p \in \mathbb{N}} A^p / A^{p-1}$$

be the associated graded vector space. The vector space $\text{gr}(A)$ is an associative algebra with unit and multiplication given as follows. For $p, q \in \mathbb{N}$, $a \in A^p$ and $b \in A^q$, we set

$$\gamma^p(a)\gamma^q(b) = \gamma^{p+q}(ab),$$

where $\gamma^p : A^p \rightarrow \text{gr}(A)$ is the *principal symbol map*, which is the composition of the natural maps $A^p \twoheadrightarrow A^p/A^{p-1}$ and $A^p/A^{p-1} \hookrightarrow \text{gr}(A)$. The unit of $\text{gr}(A)$ is $\gamma^0(1)$.

We say A is *almost commutative* if the filtration $(A^p)_{p \in \mathbb{Z}}$ satisfies the following condition: for $p, q \in \mathbb{Z}$, if $a \in A^p$ and $b \in A^q$, then $ab - ba \in A^{p+q-1}$. If A is almost commutative, then $\text{gr}(A)$ is commutative.

Example A.1 (PBW filtration of $U(\mathfrak{g})$). Let \mathfrak{g} be a Lie algebra. The *tensor algebra of \mathfrak{g}* , given by $T(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n}$, is an associative algebra with unit 1. The *universal enveloping algebra of \mathfrak{g}* , given by $U(\mathfrak{g}) = T(\mathfrak{g})/(xy - yx - [x, y] \mid x, y \in \mathfrak{g})$, is also an associative algebra with 1, but we also have an inclusion $\text{inc} : \mathfrak{g} \hookrightarrow U(\mathfrak{g})$, which is a Lie algebra homomorphism. The PBW filtration of $U(\mathfrak{g})$ is given by

$$U(\mathfrak{g})^p = \text{span}\{x_1 x_2 \dots x_s \mid s \leq p, x_1, \dots, x_s \in \mathfrak{g}\} \quad \text{for } p \in \mathbb{Z}.$$

This filtration clearly satisfies axioms (i)–(iv) above. Furthermore, $U(\mathfrak{g})^1 = \mathfrak{g}$. By [Dix96, Lemma 2.1.5], $U(\mathfrak{g})$ is almost commutative, so $\text{gr}(U(\mathfrak{g}))$ is commutative. The *symmetric algebra of \mathfrak{g}* , given by $S(\mathfrak{g}) = T(\mathfrak{g})/(xy - yx \mid x, y \in \mathfrak{g})$, is a commutative associative algebra with unit 1, and we have an inclusion $\text{inc} : \mathfrak{g} \hookrightarrow S(\mathfrak{g})$ satisfying a universal property. We have two natural inclusions of \mathfrak{g} : $\gamma^1 : \mathfrak{g} \hookrightarrow \text{gr}(U(\mathfrak{g}))$ and $\text{inc} : \mathfrak{g} \hookrightarrow S(\mathfrak{g})$. By the universal property of $\text{inc} : \mathfrak{g} \hookrightarrow S(\mathfrak{g})$, there is a homomorphism of commutative algebras $S(\mathfrak{g}) \rightarrow \text{gr}(U(\mathfrak{g}))$ such that $1 \mapsto 1$, and the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{inc}} & S(\mathfrak{g}) \\ & \searrow \gamma^1 & \downarrow \\ & & \text{gr}(U(\mathfrak{g})) \end{array}$$

By [Dix96, Proposition 2.3.6], the homomorphism $S(\mathfrak{g}) \rightarrow \text{gr}(U(\mathfrak{g}))$ is in fact an isomorphism. If $(x_i)_{i \in I}$ is a basis of \mathfrak{g} , then $S(\mathfrak{g})$ is isomorphic to the polynomial algebra $\mathbb{C}[(x_i)_{i \in I}]$. We have described $\text{gr}(U(\mathfrak{g}))$ explicitly.

APPENDIX B. POISSON ALGEBRAS AND THEIR MODULES

Let A be a commutative associative algebra. A *Hamiltonian operator of A* is a diagonalizable operator $H \in \text{End}(A)$ such that

$$H(ab) = H(a)b + aH(b) \quad \text{for } a, b \in A.$$

Thus, a Hamiltonian of A is just a derivation of A . An algebra with a Hamiltonian is called *graded*.

Let A be a differential commutative associative algebra with derivation ∂ . A *Hamiltonian operator of A* is a Hamiltonian of A as a commutative associative algebra such that

$$[H, \partial] = \partial.$$

It is possible to prove inductively that the last equation implies

$$H\partial^n = n\partial^n + \partial^n H \quad \text{for } n \in \mathbb{N}. \tag{B.0.1}$$

A *Poisson algebra* is an algebra A together with two operations $\bullet \bullet \bullet : A \times A \rightarrow A$ and $\{\bullet, \bullet\} : A \times A \rightarrow A$ satisfying:

- (i) $(A, \bullet \bullet \bullet)$ is a commutative associative algebra;
- (ii) $(A, \{\bullet, \bullet\})$ is a Lie algebra;
- (iii) The operations $\bullet \bullet \bullet$ and $\{\bullet, \bullet\}$ are related by the left Leibniz rule

$$\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\} \quad \text{for } a, b, c \in A.$$

The operation $\{\bullet, \bullet\}$ is called *Poisson bracket*.

Lemma B.1. *A Poisson algebra A also satisfies the right Leibniz rule*

$$\{ab, c\} = a\{b, c\} + b\{a, c\} \quad \text{for } a, b, c \in A.$$

Proof. The result follows from the left Leibniz rule and the anticommutativity of the Poisson bracket. \square

Let A be an associative almost commutative algebra with A with unit 1 satisfying (i)–(iv) of Appendix A. We know that $\text{gr}(A)$ is a commutative associative algebra with unit $\gamma^0(1)$. We now make $\text{gr}(A)$ into a Poisson algebra by defining the Poisson bracket as follows. For $p, q \in \mathbb{N}$, $a \in A^p$ and $b \in A^q$, we set

$$\{\gamma^p(a), \gamma^q(b)\} = \gamma^{p+q-1}(ab - ba).$$

We can verify that this is well-defined and makes $\text{gr}(A)$ into a Poisson algebra.

Example B.2 ($\text{gr}(U(\mathfrak{g}))$). Let \mathfrak{g} be a Lie algebra. We study the case when $A = U(\mathfrak{g})$ is given the PBW filtration. In that case, we see that the Poisson bracket $\{\bullet, \bullet\} : \text{gr}(U(\mathfrak{g})) \times \text{gr}(U(\mathfrak{g})) \rightarrow \text{gr}(U(\mathfrak{g}))$ extends the Lie bracket $[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. We wish to make this construction more explicit. By Example A.1, $\text{gr}(U(\mathfrak{g}))$ is canonically isomorphic to $S(\mathfrak{g})$. Let $(x_i)_{i \in I}$ be a basis of \mathfrak{g} . Then $S(\mathfrak{g})$ is isomorphic to the polynomial algebra $\mathbb{C}[(x_i)_{i \in I}]$. We can write

$$[x_i, x_j] = \sum_{k \in I} c_{i,j}^k x_k \quad \text{for } i, j \in I$$

for some scalars $c_{i,j}^k \in \mathbb{C}$. The left and right Leibniz rules imply that the Poisson bracket is given explicitly by

$$\{f, g\} = \sum_{i,j,k \in I} c_{i,j}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} x_k \quad \text{for } f, g \in \mathbb{C}[(x_i)_{i \in I}].$$

Therefore, this Poisson bracket is the unique Poisson bracket extending the Lie bracket of $\bigoplus_{i \in I} \mathbb{C}x_i \cong \mathfrak{g}$ to a Poisson bracket of $\mathbb{C}[(x_i)_{i \in I}] \cong \text{gr}(U(\mathfrak{g}))$.

Let A be a Poisson algebra. A *Hamiltonian operator* of A is a Hamiltonian H of A as a commutative associative algebra such that

$$H(\{a, b\}) = \{a, H(b)\} + \{H(a), b\} - \{a, b\} \quad \text{for } a, b \in A.$$

Let A be a graded commutative associative algebra with Hamiltonian H , and let M be an A -module. If A has a unit 1, we further assume that $1u = u$ for $u \in M$. A *Hamiltonian operator* of M is a diagonalizable operator $H^M \in \text{End}(M)$ such that

$$H^M(au) = H(a)u + aH^M(u) \quad \text{for } a \in A \text{ and } u \in M.$$

Let A be a Poisson algebra. A *module over A* is an A -module M in the usual associative sense equipped with a bilinear map $\{\bullet, \bullet\} : A \times M \rightarrow M$, which makes M a Lie algebra module over A such that for $a, b \in A$ and $u \in M$:

- (i) (Left Leibniz rule) $\{a, bu\} = \{a, b\}u + b\{a, u\}$;
- (ii) (Right Leibniz rule) $\{ab, u\} = a\{b, u\} + b\{a, u\}$.

Remark B.3. We probably cannot prove the right Leibniz rule from the Left Leibniz rule for modules like in Lemma B.1 because we do not have anticommutativity for modules. Some authors only require the left Leibniz rule for modules, but here we also want the right Leibniz rule (see [Car03]).

Let A be a graded Poisson algebra with Hamiltonian H , and let M be an A -module. A *Hamiltonian operator* of M is a Hamiltonian H^M of M as a module over A as a commutative associative algebra such that

$$H^M(\{a, u\}) = \{a, H^M(u)\} + \{H(a), u\} - \{a, u\} \quad \text{for } a \in A \text{ and } u \in M.$$

APPENDIX C. JET ALGEBRAS AND JET SCHEMES

In this appendix, by an algebra we will mean a commutative associative algebra with unit. Let R be a finitely generated algebra. We now construct a differential algebra JR called the *jet algebra* of R and an algebra inclusion $\text{inc} : R \hookrightarrow JR$ universal with this property, i.e., for a

differential algebra A and an algebra homomorphism $\phi : R \rightarrow A$, there is a unique differential algebra homomorphism $\bar{\phi} : JR \rightarrow A$ such that the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{\text{inc}} & JR \\ & \searrow \phi & \downarrow \bar{\phi} \\ & & A \end{array}$$

Assuming $R = \mathbb{C}[x^1, \dots, x^r]/(f_1, \dots, f_s)$ for some polynomials $f_1, \dots, f_s \in \mathbb{C}[x^1, \dots, x^r]$, the construction is as follows. We introduce new variables $x_{(-i)}^j$ for $i = 1, 2, \dots, j = 1, \dots, r$ and a derivation ∂ of the polynomial algebra $\mathbb{C}[x_{(-i)}^j \mid i = 1, 2, \dots, j = 1, \dots, r]$ by setting

$$\partial x_{(-i)}^j = i x_{(-i-1)}^j \quad \text{for } i = 1, 2, \dots, j = 1, \dots, r.$$

We set (identifying x^j with $x_{(-1)}^j$ when considering f_i in the following equation)

$$\begin{aligned} JR &= \mathbb{C}[x_{(-i)}^j \mid i = 1, 2, \dots, j = 1, \dots, r]/(\partial^j f_i \mid i = 1, \dots, s, j = 0, 1, \dots) \\ &= \mathbb{C}[x_{(-i)}^j \mid i = 1, 2, \dots, j = 1, \dots, r]/(f_1, \dots, f_s)_\partial, \end{aligned}$$

where the subscript ∂ indicates the differential subalgebra generated by the given subset. By our definitions, ∂ factors through a derivation of JR , and we have an algebra inclusion $\text{inc} : R \hookrightarrow JR$, $\text{inc}(x^j + (f_1, \dots, f_s)) = x_{(-1)}^j + (f_1, \dots, f_s)_\partial$ for $j = 1, \dots, r$. The fact that $\text{inc} : R \hookrightarrow JR$ satisfies our desired universal property is explained in [Ara12, §2.3] and [EM09].

Remark C.1. We see that the classes of the original variables x^j generate JR as a differential algebra, i.e.,

$$JR = (x^1 + (f_1, \dots, f_s)_\partial, \dots, x^r + (f_1, \dots, f_s)_\partial)_\partial.$$

If $f : R_1 \rightarrow R_2$ is a homomorphism of finitely generated algebras, then $Jf : JR_1 \rightarrow JR_2$ is defined by requiring that $Jf(\partial_1^n(\text{inc}_1(x))) = \partial_2^n(\text{inc}_2(f(x)))$ for $x \in R_1$ and $n \in \mathbb{N}$ (c.f. Remark C.1). For a finitely generated algebra R and a differential algebra A , we have a natural isomorphism

$$\text{Hom}_{\{\text{differential algebras}\}}(JR, A) \cong \text{Hom}_{\{\text{algebras}\}}(R, A).$$

Remark C.2. It is not true that we have defined a functor J which is left adjoint to the forgetful functor $\{\text{differential algebras}\} \rightarrow \{\text{finitely generated algebras}\}$ because a differential algebra is generally not finitely generated.

But we can easily work in the general case as follows. Let R be an algebra (not necessarily finitely generated), and we consider the polynomial algebra in R variables $\mathbb{C}[(x^j)_{j \in R}]$. We have a natural epimorphism $\pi : \mathbb{C}[(x^j)_{j \in R}] \twoheadrightarrow R$, $\pi(x^j) = j$ for $j \in R$. We can repeat the construction of JR with $\mathbb{C}[(x^j)_{j \in R}]$ in place of $\mathbb{C}[x^1, \dots, x^r]$ and $\ker(\pi)$ in place of (f_1, \dots, f_s) . We have $JR = \mathbb{C}[x_{(-i)}^j \mid i = 1, 2, \dots, j \in R]/\ker(\pi)_\partial$, and the inclusion $\text{inc} : R \rightarrow JR$ is given by $\text{inc}(j) = x_{(-1)}^j + \ker(\pi)_\partial$ for $j \in R$. As in Remark C.1, we have $JR = (\text{inc}(R))_\partial$.

This way, we construct a functor

$$J : \{\text{algebras}\} \rightarrow \{\text{differential algebras}\},$$

which is left adjoint to the forgetful functor $\{\text{differential algebras}\} \rightarrow \{\text{algebras}\}$.

Given an algebra R , it is useful to consider the functor

$$JR \otimes_R \bullet : R\text{-Mod} \rightarrow JR\text{-Mod},$$

where JR is merely considered an algebra. Again, the JR -module M together with the R -module inclusion $\text{inc} : M \hookrightarrow JR \otimes_R M$, $\text{inc}(u) = 1 \otimes u$ satisfy a universal property similar to that of $\text{inc} : R \hookrightarrow JR$, and the functor $JR \otimes_R \bullet$ is left adjoint to the forgetful functor $JR\text{-Mod} \rightarrow R\text{-Mod}$.

Let R be a graded algebra with Hamiltonian H . We can extend uniquely the Hamiltonian H to a Hamiltonian $H^{JR} \in \text{End}(JR)$ because of (B.0.1). Furthermore, if M is a graded R -module with Hamiltonian H^M , we can define a Hamiltonian of $JR \otimes_R M$ by setting

$$H^{JR \otimes_R M} = H^{JR} \otimes \text{Id}_M + \text{Id}_{JR} \otimes H^M.$$

The *jet scheme of an affine scheme of finite type* $X = \text{Spec}(R)$ is defined by

$$JX = \text{Spec}(JR).$$

Remark C.3. It is possible to define the jet scheme JX of a general scheme X of finite type (see [EM09]), but the construction is more elaborate, and we will only need the affine case.

Theorem C.4 ([EM09]). *Let R be a finitely generated algebra. Then $\dim(R) = 0$ if and only if $\dim(JR) = 0$.*

APPENDIX D. AN EXPLICIT DESCRIPTION OF K

In [Sal23, ising-modules.ipynb], I wrote a SageMath program to compute u_λ^K for all partitions λ (if λ is not a pivot, it returns 0). From its output, we obtain:

$$\begin{aligned} u_{[2]}^K &= L_{-2} - \frac{3}{4}L_{-1}^2, & u_{[1,1,1]}^K &= L_{-1}^3, \\ u_{[3,1,1]}^K &= L_{-3}L_{-1}^2, & u_{[3,3]}^K &= L_{-3}L_{-3} + \frac{1}{3}L_{-4}L_{-1}^2, \\ u_{[4,3,1]}^K &= L_{-4}L_{-3}L_{-1}, & u_{[4,4,1]}^K &= L_{-4}L_{-4}L_{-1} + \frac{9}{8}L_{-5}L_{-3}L_{-1}, \\ u_{[5,4,1,1]}^K &= L_{-5}L_{-4}L_{-1}^2, & u_{[6,5,3,1]}^K &= L_{-6}L_{-5}L_{-3}L_{-1}. \end{aligned}$$

However, it turns out that $u_{[5,4,1,1]}^K$ and $u_{[6,5,3,1]}^K$ are redundant, as can be seen in [Sal23, m11-m15.ipynb]. From [AVEH22, Theorem 2], we obtain the following explicit expression for K

$$K = (u_{[2]}^K, u_{[1,1,1]}^K, u_{[3,1,1]}^K, u_{[3,3]}^K, u_{[4,3,1]}^K, u_{[4,4,1]}^K, \iota_0((a, b)\partial))_{\text{psn}},$$

where

$$a = L_{-2}^3, b = L_{-4}L_{-3}L_{-2} + \frac{1}{6}L_{-5}L_{-2}^2.$$

APPENDIX E. THE CASE $L(1/2, 1/16)$

We can do the same analysis we did for $L(1/2, 1/2)$ but with $L(1/2, 1/16)$ instead. The arguments are the same, but the computations are, of course, different. These computations will be shown now.

We recall the definitions of $P^{1/16}$ and $R^{1/16}$ given in §0. For the definition of the series $p^{1/16}(t, q)$, the set $P^{1/16}(n, m)$ and other related notation, see §5.2. Again, we omit superscripts.

The maximal proper subrepresentation of $M(1/2, 1/16)$ is generated by the singular vectors:

$$u_2 = (L_{-2} - \frac{4}{3}L_{-1}^2) |1/16\rangle, u_4 = (L_{-2}L_{-2} - \frac{600}{49}L_{-2}L_{-1}^2 + \frac{144}{49}L_{-1}^4 + \frac{264}{49}L_{-3}L_{-1} - \frac{36}{49}L_{-4}) |1/16\rangle.$$

Lemma E.1. *The formal power series $p_{>2}(t, q)$ is given by*

$$p_{>2}(t, q) = f_{2,2,0,0}(t, q).$$

Sketch of proof. We consider the disjoint union

$$P_{>2} = P_{3,3} \cup P_{6,4,3} \cup P_{>6,4,3} \cup P_{6,5,3} \cup P_{8,7,5,3} \cup P_{>8,7,5,3} \cup P_{>7,5,3} \cup P_{>5,3} \cup P_{4,4} \cup P_{5,4} \cup P_{>5,4} \cup P_{>4}.$$

These subseries satisfy the following recurrences with initial conditions:

$$\begin{aligned}
p_{>4}(t, q) &= p_{4,4}(tq^{1/2}, q) + p_{5,4}(tq^{1/2}, q) + p_{>5,4}(tq^{1/2}, q) + p_{>4}(tq^{1/2}, q), & p_{>4}(0, 0) &= 1, \\
p_{>5,4}(t, q) &= p_{6,5,3}(tq^{1/2}, q) + p_{8,7,5,3}(tq^{1/2}, q) + p_{>8,7,5,3}(tq^{1/2}, q) \\
&\quad + p_{>7,5,3}(tq^{1/2}, q) + p_{>5,3}(tq^{1/2}, q), & p_{>5,4}(0, 0) &= 0, \\
p_{5,4}(t, q) &= p_{6,4,3}(tq^{1/2}, q) + p_{>6,4,3}(tq^{1/2}, q), & p_{5,4}(0, 0) &= 0, \\
p_{4,4}(t, q) &= p_{3,3}(tq^{1/2}, q), & p_{4,4}(0, 0) &= 0, \\
p_{>5,3}(t, q) &= t^2 q^3 p_{>4}(tq^{1/2}, q), & p_{>5,3}(0, 0) &= 0, \\
p_{>7,5,3}(t, q) &= t^2 q^3 p_{>5,3}(tq^{2/2}, q), & p_{>7,5,3}(0, 0) &= 0, \\
p_{>8,7,5,3}(t, q) &= t^2 q^4 p_{>6,4,3}(tq^{2/2}, q), & p_{>8,7,5,3}(0, 0) &= 0, \\
p_{8,7,5,3}(t, q) &= t^2 q^3 p_{6,5,3}(tq^{2/2}, q), & p_{8,7,5,3}(0, 0) &= 0, \\
p_{6,5,3}(t, q) &= t^2 q^3 p_{5,4}(tq^{1/2}, q), & p_{6,5,3}(0, 0) &= 0, \\
p_{>6,4,3}(t, q) &= t^2 q^2 p_{>5,4}(tq^{1/2}, q), & p_{>6,4,3}(0, 0) &= 0, \\
p_{6,4,3}(t, q) &= t^2 q^3 p_{4,4}(tq^{1/2}, q), & p_{6,4,3}(0, 0) &= 0, \\
p_{3,3}(t, q) &= t^2 q^3 p_{>6,4,3}(tq^{1/2}, q) + t^2 q^2 p_{>5,3}(tq^{1/2}, q), & p_{3,3}(0, 0) &= 0.
\end{aligned}$$

The solution to these equations is unique if it exists, and we can verify using Lemma 5.1.2 that:

$$\begin{aligned}
p_{>4}(t, q) &= f_{6,4,0,0}(t, q), & p_{>5,4}(t, q) &= f_{9,5,4,2}(t, q), \\
p_{5,4}(t, q) &= f_{13,6,9,4}(t, q), & p_{4,4}(t, q) &= f_{12,6,8,4}(t, q), \\
p_{>5,3}(t, q) &= f_{8,5,3,2}(t, q), & p_{>7,5,3}(t, q) &= f_{12,7,8,4}(t, q), \\
p_{>8,7,5,3}(t, q) &= f_{15,8,15,6}(t, q), & p_{8,7,5,3}(t, q) &= f_{19,9,23,8}(t, q), \\
p_{6,5,3}(t, q) &= f_{15,7,14,6}(t, q), & p_{>6,4,3}(t, q) &= f_{11,6,7,4}(t, q), \\
p_{6,4,3}(t, q) &= f_{14,7,13,6}(t, q), & p_{3,3}(t, q) &= f_{10,5,6,4}(t, q),
\end{aligned}$$

is a solution to these equations. We derive the formula $p_{>2}(t, q) = f_{2,2,0,0}(t, q)$ again from Lemma 5.1.2.

The recurrences follow from the following twelve bijections for $m, n \in \mathbb{N}$:

$$P_{>4}(n, 2m) \xrightarrow{\sim} P_{4,4}(n-m, 2m) \cup P_{5,4}(n-m, 2m) \cup P_{>5,4}(n-m, 2m) \cup P_{>4}(n-m, 2m),$$

$$\lambda \mapsto \begin{cases} [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4, 4] & \text{if } [\lambda_{m-1}, \lambda_m] = [5, 5]; \\ [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 5, 4] & \text{if } [\lambda_{m-1}, \lambda_m] = [6, 5]; \\ [\lambda_1 - 1, \dots, \lambda_{m-1} - 1, 4] & \text{if } [\lambda_m] = [5] \text{ and } \lambda_{m-1} > 6; \\ [\lambda_1 - 1, \dots, \lambda_m - 1] & \text{if } \lambda_m > 5, \end{cases}$$

$$P_{>5,4}(n, 2m) \xrightarrow{\sim} P_{6,5,3}(n-m, 2m) \cup P_{8,7,5,3}(n-m, 2m) \cup P_{>8,7,5,3}(n-m, 2m)$$

$$\cup P_{>7,5,3}(n-m, 2m) \cup P_{>5,3}(n-m, 2m),$$

$$\lambda \mapsto \begin{cases} [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 6, 5, 3] & \text{if } [\lambda_{m-2}, \lambda_{m-1}, \lambda_m] = [7, 6, 4]; \\ [\lambda_1 - 1, \dots, \lambda_{m-4} - 1, 8, 7, 5, 3] & \text{if } [\lambda_{m-3}, \lambda_{m-2}, \lambda_{m-1}, \lambda_m] = [9, 8, 6, 4]; \\ [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 7, 5, 3] & \text{if } [\lambda_{m-2}, \lambda_{m-1}, \lambda_m] = [8, 6, 4] \text{ and } \lambda_{m-3} > 9; \\ [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 5, 3] & \text{if } [\lambda_{m-1}, \lambda_m] = [6, 4] \text{ and } \lambda_{m-2} > 8; \\ [\lambda_1 - 1, \dots, \lambda_{m-1} - 1, 3] & \text{if } [\lambda_m] = [4] \text{ and } \lambda_{m-1} > 6, \end{cases}$$

$$\begin{aligned}
P_{5,4}(n, 2m) &\xrightarrow{\sim} P_{6,4,3}(n-m, 2m) \cup P_{>6,4,3}(n-m, 2m), \\
\lambda &\mapsto \begin{cases} [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 6, 4, 3] & \text{if } [\lambda_{m-2}, \lambda_{m-1}, \lambda_m] = [7, 5, 4]; \\ [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4, 3] & \text{if } [\lambda_{m-1}, \lambda_m] = [5, 4] \text{ and } \lambda_{m-2} > 7, \end{cases} \\
P_{4,4}(n, 2m) &\xrightarrow{\sim} P_{3,3}(n-m, 2m), \\
\lambda &\mapsto [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 3, 3], \\
P_{>5,3}(n, 2m) &\xrightarrow{\sim} P_{>4}(n-m-2, 2(m-1)), \\
\lambda &\mapsto [\lambda_1 - 1, \dots, \lambda_{m-1} - 1], \\
P_{>7,5,3}(n, 2m) &\xrightarrow{\sim} P_{>5,3}(n-2m-1, 2(m-1)), \\
\lambda &\mapsto [\lambda_1 - 2, \dots, \lambda_{m-1} - 2, 3], \\
P_{>8,7,5,3}(n, 2m) &\xrightarrow{\sim} P_{>6,4,3}(n-2m-2, 2(m-1)), \\
\lambda &\mapsto [\lambda_1 - 2, \dots, \lambda_{m-3} - 2, 4, 3], \\
P_{8,7,5,3}(n, 2m) &\xrightarrow{\sim} P_{6,5,3}(n-2m-1, 2(m-1)), \\
\lambda &\mapsto [\lambda_1 - 2, \dots, \lambda_{m-4} - 2, 6, 5, 3], \\
P_{6,5,3}(n, 2m) &\xrightarrow{\sim} P_{5,4}(n-m-2, 2(m-1)), \\
\lambda &\mapsto [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 5, 4], \\
P_{>6,4,3}(n, 2m) &\xrightarrow{\sim} P_{>5,4}(n-m-1, 2(m-1)), \\
\lambda &\mapsto [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 4], \\
P_{6,4,3}(n, 2m) &\xrightarrow{\sim} P_{4,4}(n-m-2, 2(m-1)), \\
\lambda &\mapsto [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 4, 4],
\end{aligned}$$

$$\begin{aligned}
P_{3,3}(n, 2m) &\xrightarrow{\sim} P_{>6,4,3}(n-m-2, 2(m-1)) \cup P_{>5,3}(n-m-1, 2(m-1)), \\
\lambda &\mapsto \begin{cases} [\lambda_1 - 1, \dots, \lambda_{m-3} - 1, 4, 3] & \text{if } [\lambda_{m-2}, \lambda_{m-1}, \lambda_m] = [6, 3, 3]; \\ [\lambda_1 - 1, \dots, \lambda_{m-2} - 1, 3] & \text{if } [\lambda_{m-1}, \lambda_m] = [3, 3] \text{ and } \lambda_{m-2} > 6. \quad \square \end{cases}
\end{aligned}$$

Lemma E.2. *The formal power series $p_{>2,1}(t, q)$ is given by*

$$p_{>2,1}(t, q) = f_{4,2,1,1}(t, q).$$

Proof. We consider the disjoint union

$$P_{>2,1} = P_{6,5,3,1} \cup P_{8,7,5,3,1} \cup P_{>8,7,5,3,1} \cup P_{>7,5,3,1} \cup P_{>5,3,1} \cup P_{4,4,1} \cup P_{5,4,1} \cup P_{>5,4,1} \cup P_{>4,1}.$$

We have bijections:

$$\begin{aligned}
P_{>4,1}(n, 2m+1) &\xrightarrow{\sim} P_{>4}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m],
\end{aligned}$$

$$\begin{aligned}
P_{>5,4,1}(n, 2m+1) &\xrightarrow{\sim} P_{>5,4}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m],
\end{aligned}$$

$$\begin{aligned}
P_{5,4,1}(n, 2m+1) &\xrightarrow{\sim} P_{5,4}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m],
\end{aligned}$$

$$\begin{aligned}
P_{4,4,1}(n, 2m+1) &\xrightarrow{\sim} P_{4,4}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m], \\
P_{>5,3,1}(n, 2m+1) &\xrightarrow{\sim} P_{>5,3}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m], \\
P_{>7,5,3,1}(n, 2m+1) &\xrightarrow{\sim} P_{>7,5,3}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m], \\
P_{>8,7,5,3,1}(n, 2m+1) &\xrightarrow{\sim} P_{>8,7,5,3}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m], \\
P_{8,7,5,3,1}(n, 2m+1) &\xrightarrow{\sim} P_{8,7,5,3}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m], \\
P_{6,5,3,1}(n, 2m+1) &\xrightarrow{\sim} P_{6,5,3}(n-1, 2m), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m],
\end{aligned}$$

from which, together with the proof of Lemma E.1, we get:

$$\begin{aligned}
p_{>4,1}(t, q) &= tqp_{>4}(t, q) = f_{6,4,1,1}(t, q), \\
p_{>5,4,1}(t, q) &= tqp_{>5,4}(t, q) = f_{9,5,5,3}(t, q), \\
p_{5,4,1}(t, q) &= tqp_{5,4}(t, q) = f_{13,6,10,5}(t, q), \\
p_{4,4,1}(t, q) &= tqp_{4,4}(t, q) = f_{12,6,9,5}(t, q), \\
p_{>5,3,1}(t, q) &= tqp_{>5,3}(t, q) = f_{8,5,4,3}(t, q), \\
p_{>7,5,3,1}(t, q) &= tqp_{>7,5,3}(t, q) = f_{12,7,9,5}(t, q), \\
p_{>8,7,5,3,1}(t, q) &= tqp_{>8,7,5,3}(t, q) = f_{15,8,16,7}(t, q), \\
p_{8,7,5,3,1}(t, q) &= tqp_{8,7,5,3}(t, q) = f_{19,9,24,9}(t, q), \\
p_{6,5,3,1}(t, q) &= tqp_{6,5,3}(t, q) = f_{15,7,15,7}(t, q).
\end{aligned}$$

The formula for $p_{>2,1}(t, q)$ then follows from Lemma 5.1.2. □

Lemma E.3. *The formal power series $p_{>2,1,1}(t, q)$ is given by*

$$p_{>2,1,1}(t, q) = f_{9,4,5,4}(t, q) + f_{5,3,2,2}(t, q).$$

Proof. We consider the disjoint union

$$P_{>2,1,1} = P_{7,5,3,1,1} \cup P_{>7,5,3,1,1} \cup P_{>5,3,1,1} \cup P_{5,4,1,1} \cup P_{>5,4,1,1} \cup P_{>4,1,1}.$$

We have bijections:

$$\begin{aligned}
P_{>4,1,1}(n, 2m+2) &\xrightarrow{\sim} P_{>4,1}(n-1, 2m+1), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m, 1], \\
P_{>5,4,1,1}(n, 2m+2) &\xrightarrow{\sim} P_{>5,4,1}(n-1, 2m+1), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m, 1], \\
P_{5,4,1,1}(n, 2m+2) &\xrightarrow{\sim} P_{5,4,1}(n-1, 2m+1), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_m, 1], \\
P_{>5,3,1,1}(n, 2m+2) &\xrightarrow{\sim} P_{>5,4,1}(n, 2m+1), \\
\lambda &\mapsto [\lambda_1, \dots, \lambda_{m-1}, 4, 1],
\end{aligned}$$

$$P_{>7,5,3,1,1}(n, 2m+2) \xrightarrow{\sim} P_{>7,5,3,1}(n-1, 2m+1),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_m, 1],$$

$$P_{7,5,3,1,1}(n, 2m+2) \xrightarrow{\sim} P_{>8,7,5,3,1}(n-1, 2m+1),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_m, 1],$$

from which, together with the proof of Lemma E.2, we get:

$$\begin{aligned} p_{>4,1,1}(t, q) &= tqp_{>4,1}(t, q) = f_{6,4,2,2}(t, q), \\ p_{>5,4,1,1}(t, q) &= tqp_{>5,4,1}(t, q) = f_{9,5,6,4}(t, q), \\ p_{5,4,1,1}(t, q) &= tqp_{5,4,1}(t, q) = f_{13,6,11,6}(t, q), \\ p_{>5,3,1,1}(t, q) &= tp_{>5,4,1}(t, q) = f_{9,5,5,4}(t, q), \\ p_{>7,5,3,1,1}(t, q) &= tqp_{>7,5,3,1}(t, q) = f_{12,7,10,6}(t, q), \\ p_{7,5,3,1,1}(t, q) &= tqp_{>8,7,5,3,1}(t, q) = f_{15,8,17,8}(t, q). \end{aligned}$$

The formula for $p_{>2,1,1}(t, q)$ then follows from Lemma 5.1.2. □

Lemma E.4. *The formal power series $p_{>3,1,1,1}(t, q)$ is given by*

$$p_{>3,1,1,1}(t, q) = f_{7,3,3,3}(t, q).$$

Proof. We consider the disjoint union

$$P_{>3,1,1,1} = P_{4,1,1,1} \cup P_{>4,1,1,1}.$$

We have bijections:

$$P_{>4,1,1,1}(n, 2m+3) \xrightarrow{\sim} P_{>4,3}(n, 2m+2),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_m, 3],$$

$$P_{4,1,1,1}(n, 2m+3) \xrightarrow{\sim} P_{3,3}(n-1, 2m+2),$$

$$\lambda \mapsto [\lambda_1, \dots, \lambda_{m-1}, 3, 3],$$

from which, together with the proof of Lemma E.1 and Lemma 5.1.2, we get:

$$\begin{aligned} p_{>4,1,1,1}(t, q) &= tp_{>4,3}(t, q) \\ &= t(p_{6,5,3}(t, q) + p_{8,7,5,3}(t, q) + p_{>8,7,5,3}(t, q) + p_{>7,5,3}(t, q) + p_{>5,3}(t, q)) \\ &= tf_{7,4,3,2}(t, q) \\ &= f_{7,4,3,3}(t, q), \\ p_{4,1,1,1}(t, q) &= tqp_{3,3}(t, q) = f_{10,5,7,5}(t, q). \end{aligned}$$

The formula for $p_{>3,1,1,1}(t, q)$ then follows from Lemma 5.1.2. □

Lemma E.5. *The formal power series $p(t, q)$ is given by*

$$p(t, q) = f_{1,1,0,0}(t, q) + f_{4,2,1,1}(t, q) + f_{7,3,3,3}(t, q).$$

Proof. This follows from the disjoint union

$$P = P_{>2} \cup P_{>2,1} \cup P_{>2,1,1} \cup P_{>3,1,1,1}$$

together with Lemma E.1, Lemma E.2, Lemma E.3, Lemma E.4 and Lemma 5.1.2. □

The partitions $[2]$, $[1, 1, 1, 1]$, $[3, 1, 1, 1]$, $[3, 3, 1]$, $[4, 3, 1]$, $[4, 4, 1, 1]$, $[5, 4, 1, 1, 1]$, $[5, 5, 1, 1, 1]$, $[6, 5, 3, 1, 1]$, $[6, 6, 3, 1, 1]$, $[7, 6, 4, 1, 1, 1]$ and $[8, 7, 5, 3, 1, 1]$ are pivots of the matrices A_2^W , A_4^W , A_6^W , A_7^W , A_8^W , A_{10}^W , A_{12}^W , A_{13}^W , A_{16}^W , A_{17}^W , A_{20}^W and A_{25}^W respectively, see [Sal23, ising-modules-1-16.ipynb].

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