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ALBERTO MIZRAHY CAMPOS

COVERING PROCESSES

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Covering Processes

PhD thesis presented to the Graduate Program of Instituto Nacional de Matemática Pura e Aplicada - IMPA, as a partial requirement for obtaining the title of Doctor in Mathematics.

Advisor: Prof. Augusto Quadros Teixeira

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"Não sei, só sei que foi assim..."
João Grilo, character in "Auto da Compadecida".

Abstract

This thesis introduces and studies a model that reveals some properties and behaviors of random covering. More specifically, fix $n > 0$ and consider a discrete one-dimensional torus $\mathbb{Z}/n\mathbb{Z}$ of size n, let $(U_k)_k$ be a sequence of independent uniformly distributed elements of the torus, and let $(R_k)_k$ be a set of independent integer valued lengths with some common distribution R. Define the set of random arcs $(\mathcal{O}_k)_k$ where $\mathcal{O}_k = \{U_k, U_k+1, \cdots, U_k+R_k-1\}$ and define the covering time $\tau_n = \inf\{k : \bigcup_{i=1}^{k} \mathcal{O}_k = \mathbb{Z}/n\mathbb{Z}\}$. The first result consists of calculating the impact of the object's size $\mathcal{O}$ in the covering time τ_n. More precisely, by changing the distribution of the random variable R, we are able to find at least four distinct behaviors of the covering time τ_n; they are the Gumbel regime, the compact support regime, the pre-exponential regime, and the exponential regime. In the second part of the text, we show that a model introduced by Mandelbrot and Shepp in 1972 works as a limit distribution for the discrete covering in the compact support regime. Finally, in the third part of this work, we will talk about the covering in spaces with dimension greater than one, quantifying the size of the vacant region within the subcritical and supercritical phase of the process.

Key-words: Covering Processes, Coupon Collector, Covering Time.

Resumo

Esta tese introduz e estuda um modelo simples que revela algumas propriedades e comportamentos de coberturas aleatórias. Especificamente, fixe $n > 0$ e considere um toro unidimensional discreto $\mathbb{Z}/n\mathbb{Z}$ de tamanho n, seja $(U_k)_k$ um conjunto independente de variáveis aleatória distribuídas uniformemente no toro, e seja $(R_k)_k$ um conjunto de comprimentos independentes com alguma distribuição comum R. Defina o conjunto de arcos aleatórios $(\mathcal{O}_k)_k$ onde $\mathcal{O}_k = \{U_k, U_k + 1, \cdots, U_k + R_k - 1\}$ e defina o tempo de cobertura $\tau_n = \inf\{k : \bigcup_{i=1}^k \mathcal{O}_k = \mathbb{Z}/n\mathbb{Z}\}$. O primeiro resultado consiste em calcular o impacto do tamanho do objeto $\mathcal{O}$ no tempo de cobertura τ_n. Mais precisamente, alterando a distribuição da variável aleatória R, conseguimos encontrar pelo menos quatro comportamentos distintos do tempo de cobertura τ_n; eles são o regime Gumbel, o regime de suporte compacto, o regime pré-exponencial e o regime exponencial. No regime compacto, expomos um modelo introduzido por Mandelbrot e Shepp em 1972 que funciona como um limite de distribuição para a cobertura discreta. Por fim, na terceira parte do trabalho, discutiremos o tempo de cobertura para o modelo em dimensões diferentes de um, quantificando o tamanho do conjunto vacante nos regimes subcrítico e supercrítico do processo.

Palavras chave: Processos de Cobertura, Coletor de Cupons, Tempo de Cobertura.

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Chapter 1

Introduction and motivation

Consider a scenario in which objects of random size fall randomly into some space. Over time, as more objects enter the picture, the space will be eventually covered. This general description shares similarities with several other problems in the literature, being a generalization of the coupon collector problem [13, 2, 10, 25] or the set-covering problems [26, 1]. In terms of applications, this problem appears in a diverse set of questions, such as when one tries to complete a stamp collection [11] or when analyzing the number of genetic genes present in a cell [30].

Problems related to coverings are an old topic in the literature, but they are generally presented as a deterministic question. The first problem on random coverage appears in 1956, by Dvoretzky in [12], where the author asks about necessary and sufficient conditions to cover a circle with probability one using a sequence of arcs with fixed lengths $(\ell_n)_n$ placed uniformly at random. Since then, many other papers appears, ed. see for example the coupon collector [13] from P. Erdős and A. Rényi in 1961, the committee problem [24] in 1968, the solution of Dvoretzky's problem by L.A. Shepp [28] in 1972, and in the same year a continuous model of covering was introduced by B.B. Mandelbrot [22].

To explain how these works influence ours, we need to properly define the covering process. Therefore, we postpone the historical context until the end of this chapter, where the problem and the process are well defined.

The main focus of this thesis is to understand some of the fundamental properties of random coverings. To this end, the text is divided into three chapters that present a partial description of the many different possible behaviors of the process. The first part will consider the main model, that exemplifies how the covering process can change depending on the distribution of the size of the objects. The second part is dedicated to exploring a continuous model that works in some sense as a limiting distribution for the main model introduced here. In the final part, the continuous model is generalized to spaces with arbitrary-dimension, and sharp transitions of the probability to cover the space are analyzed.

The main model- Fix a discrete one-dimensional torus $\mathbb{Z}/n\mathbb{Z}$ of size n represented by the points $\{0, ..., n-1\}$, and consider the set of connected arcs in the torus. Informally speaking, the process can be described as a random stack of arcs, where arcs are chosen to start at random points U and have random lengths R.

To describe this process rigorously, pick two sequences of i.i.d. random variables. The first sequence is going to represent the **radii** $(R_k)_k$, where

$$\mathbb{P}\left(R_1 \geq 1\right) = 1 \text{ and } \mathbb{P}\left(R_1 \geq r\right) = f(r), \text{ for } r \geq 1. \tag{1.1}$$

The second sequence of random variables will be the **positions** $(U_k)_k$, where U_k are independent and distributed as uniform random variables in the discrete torus, i.e. $U_i \sim \text{Unif}(\mathbb{Z}/n\mathbb{Z})$. With these two sequences, denote the $k-th$ object as $\mathcal{O}_k = \{U_k \mod (n), U_k + 1 \mod (n), ..., U_k + R_k - 1 \mod (n)\} \subset \mathbb{Z}/n\mathbb{Z}$, an interval that starts in U_k and has total length R_k. Finally, define the **discrete covering process** as the sequence of random sets $(C_k)_k$, where $C_0 = \emptyset$ and inductively $C_k = C_{k-1} \cup \mathcal{O}_k$, as each time one new objects is revealed covering some part of the space.

With the covering process $(C_k)_k$ defined, the **cover time** of the space $\mathbb{Z}/n\mathbb{Z}$ is defined as

$$\tau_n = \min\{k : C_k = \mathbb{Z}/n\mathbb{Z}\}.$$

The first result of this thesis investigates how the distribution of the radius R can influence the scaling and the limit distribution of τ_n.

The process translates easily to continuous time, resulting in simpler proofs and calculations. To define it, let $\{N(t) : t \geq 0\}$ be a Poisson process with rate 1 on the real line and fix a discrete covering process $(C_k)_k$ with radius distribution $f(r) = \mathbb{P}\left(R \geq r\right)$. Then, define the **continuous time covering process** $(X_t)_{t>0}$ as

$$X_t = C_{N(t)}. \tag{1.2}$$

With this process, define the **continuous cover time** as

$$T_n = \inf\{t : X_t = \mathbb{Z}/n\mathbb{Z}\}.$$

Theorems A, B, C, and D, stated below, expose and describe different phases of the covering process. To illustrate these ideas, see Figure 1.1:

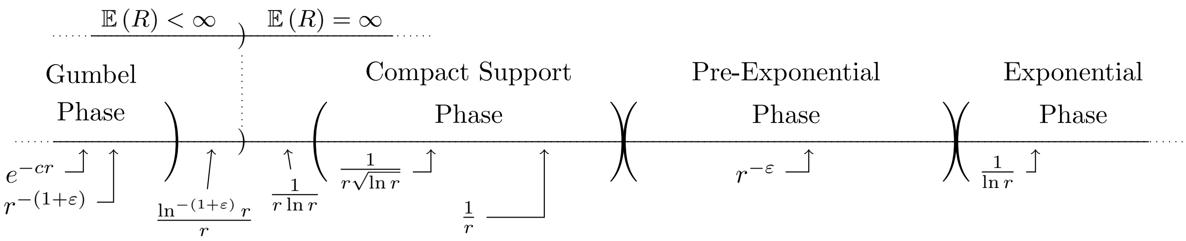

Figure 1.1: A representation of Theorems A to D in a line, with the radius distributions $f(r) = \mathbb{P}(R \geq r)$ disposed in a monotone way. Here, $\varepsilon > 0$ is a positive small number, and $c > 0$ is any constant.

The covering process described above is a flexible model, where similar results using different notation appear scattered in the literature. As far as we can tell, the theorems presented here are new.

Each theorem stated bellow presents a novelty in relation to the existing literature. To contextualize, organize, and structure the statements, a set of references is presented between each of them.

Theorem A (Gumbel Phase)**.** *Assume that* $\mathbb{E}(R^{1+\varepsilon}) < \infty$ *for some* $\varepsilon > 0$*, and set* $\mu = \mathbb{E}(R)$*. Then, as* n *grows,*

$$\frac{\mu}{n} T_n - \ln n \overset{D}{\Longrightarrow} \text{Gumbel}(0, 1), \tag{1.3}$$

where $\mathbb{P}(\text{Gumbel}(0,1) < t) = \exp\{-\exp\{-t\}\}$ *is the Gumbel distribution with parameters* 0 *and* 1*.*

Theorem A deals with the covering problem where the radius R has light tails. Due to the number of available techniques, it is not surprising that this type of covering is the most explored in the literature. For example, [14] establishes the first-order behavior of the cover time, while [14, 4, 32] show convergence in distribution when R is constant.

Remark 1. The article that most resembles the results with Theorem A is [18], where the author considers $\widehat{\tau}_n$, the number of objects needed to cover a continuous circumference

$\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ using random arcs that start at uniform points and have lengths R/n, where $\mathbb{E}\left(R^{1+\varepsilon}\right) < \infty$ for some $\varepsilon > 0$. This covering problem has the following scaling behavior

$$\frac{\mu}{n}\widehat{\tau}_n - \ln\left(\frac{n}{\mu}\right) - \ln\left(\ln\left(\frac{n}{\mu}\right)\right) \overset{D}{\Longrightarrow} \text{Gumbel}(0,1).$$

The above result has a different normalization scale then that of Theorem A. Intuitively speaking, the cover time in the continuum is larger because covering the discrete points of the form $\left\{\frac{i}{n} : i \in \{0, 1, \cdots, n-1\}\right\}$ does not imply that points in between are covered too. Also, it is worth mentioning that the techniques employed in [18] are different from those discussed in here.

A new phenomenon emerges when the radius stops having a first moment; not only does the cover time grow on a different scale, but its limiting distribution presents a compact support.

Theorem B (Compact Support Phase)**.** *Let $b > -1$, and assume that $f(r) = \min\{\frac{\ln^b(r)}{r}, 1\}$. Then, $(f(n)T_n)_n$ is tight. Moreover, for every subsequence $(n_k)_k$ such that*

$$f(n_k)T_{n_k} \overset{D}{\Longrightarrow} Y, \tag{1.4}$$

we have that $Y = Y(f, (n_k)_k)$ is a non-degenerate distribution with compact support.

We can generalize the result of Theorem B for other choices of functions $f(r)$. But due to some technicalities, the general statement for the compact support phase is postponed to Section 2.2.

To state the next result, following the notation in [27], we say that a measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying at infinity with index $p \in \mathbb{R}$ if, for all $t > 0$:

$$\lim_{x\to\infty} \frac{f(xt)}{f(x)} = t^p. \tag{1.5}$$

Let RV_p be the space of all **regularly varying functions at infinity with index** p, and when (1.5) holds, denote $f \in \text{RV}_p$. In particular, when $p = 0$, denote RV_0 as the space of **slowly varying functions**.

When exploring heavier tails for R, limiting distributions with unbounded support are found once again.

Theorem C (Pre-Exponential Phase). *Take $p \in (-1, 0)$, and set $f \in \mathrm{RV}_p$. Then, $(f(n)T_n)_n$ is tight. Moreover, for every subsequence $(n_k)_k$ such that*

$$f(n_k)T_{n_k} \overset{D}{\Longrightarrow} Z, \tag{1.6}$$

we can conclude that $Z = Z(f, (n_k)_k)$ is a distribution that satisfies

$$1 - e^{-z} < \mathbb{P}(Z \geq z) < 1, \text{ for every } z \geq 0. \tag{1.7}$$

In particular, it is not compactly supported.

Remark 2. With $p \in (-1, 0)$, as a standard example of a function in RV_p, one may consider $f(r) = r^p$.

Note that equation (1.7) rules out the possibility that Z is an exponential random variable. This observation highlights the contrast with the next theorem, which focuses on the heaviest tail considered in this thesis and exhibits an exponential distribution as its limit. In this case, the system waits for the arrival of a single interval that covers the entire torus.

Theorem D (Exponential Phase). *Let f be a slowly varying function, then*

$$f(n)T_n \overset{D}{\Longrightarrow} \mathrm{Exp}\{1\}, \tag{1.8}$$

where $\mathbb{P}(\mathrm{Exp}\{1\} < t) = 1 - e^{-t}$ is the exponential distribution with parameter 1.

Remark 3. As examples of functions in RV_0, consider for instance $\frac{1}{\log n}$ or $\frac{1}{\log^b n}$ for some $b > 0$.

Remark 4. The results appearing in Theorems A, B, C, and D, are also valid in the discrete covering process changing only the random variable T_n to τ_n, see Appendix Subsection 5.2.2.

The continuous space model- With Theorems A, B, C and D, a clear picture of the distributions of the covering process begins to appear. However, within the compact support phase of Theorem B, we are able to find a continuous model that works as a limiting distribution.

The particular case ($b = 0$) in Theorem B, when $f(r) = 1/r$, features some self-similarity that reassembles a previously studied covering process. Introduced by B.B. Mandelbrot in [22], and updated in the same year by L.A. Shepp in [28], we define **the**

Mandelbrot-Shepp model at time $\alpha \geq 0$ as a Poisson point process in the **cylinder** $S = \mathbb{S}^1 \times (0, \infty)$ with rate:

$$d\Lambda_\alpha = \alpha \left(dx \otimes \frac{dr}{r^2} \right),$$

where dx, dr are respectively Lebesgue measured in the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and in $(0, \infty)$. More specifically, the probability space $(\Omega, \mathcal{F}, \mathbb{P}_\alpha)$. Where $\Omega = \{\omega = \sum_{i \in I} \delta_{(x_i, r_i)} : (x_i, r_i) \in S$ for all $i \in I$, and $\omega(K) < \infty$ for all $K \subset S$ compact$\}$, $\mathcal{F}$ is the smallest $\sigma-$algebra that makes the evaluation maps $\{\omega(K) : K$ compact in $S\}$ measurable and $\mathbb{P}_\alpha$ measures $\omega(\cdot)$ as a Poisson random variable with intensity $\Lambda_\alpha(\cdot)$, and it value is independent in disjoint sets.

Associated to this point process, the Mandelbrot-Shepp model introduces a covering of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. This coverage will use the points in the Poisson process to define objects. This means that the coverage will have different properties than the discrete coverages treated so far, for example, for each $\alpha > 0$ the Poisson process has infinitely many points with probability one, so our coverage has infinitely many objects. To formally define it, let $\xi = (x, r) \in S$ and consider **the projection function** of ξ as

$$\Pi(\xi) = \begin{cases} [0, 1), & \text{if } r > 1. \\ (x, x + r), & \text{if } r \leq 1 \text{ and } x + r \leq 1. \\ (x, 1) \cup [0, x + r - 1), & \text{if } r \leq 1 \text{ and } x + r > 1. \end{cases} \tag{1.9}$$

Given any configuration $\omega = \sum_{i \in I} \delta_{(x_i, r_i)}$, define now the random sets:

$$\mathcal{C}(\omega) = \bigcup_{i \in I} \Pi((x_i, r_i)), \text{ and } \mathcal{V}(\omega) = \mathbb{S}^1 \setminus \mathcal{C}(\omega),$$

respectively as **the covered set** and **the vacant set** of the Mandelbrot-Shepp model.

For more details on the model, see a complete description of it in Section 3.1. Originally the model was defined by Mandelbrot and Shepp in [22, 28], and have other modern approaches such the works of [19, 3]. The dynamic of the process is informally described analogous to a covering, where by increasing the rate α the cover will receive more objects. To be more precise, for every $\alpha > 0$ denote by $\mathcal{C}_\alpha$ the covering where the Poisson have rate Λ_α, then set $(\mathcal{C}_\alpha)_\alpha$ a process where the Poisson process is coupled in α, thus increasing the rate α, will increase the number of points and also objects. The novelty of Theorem B*, bellow, consists in proving that the discrete process converges (in a specific sense, that will be made precisely latter) to its continuous counterpart $(\mathcal{C}_\alpha)_{\alpha > 0}$.

Theorem B*. *Consider the discrete covering $X_{\alpha n} = C_{N(\alpha n)}$ of $\mathbb{Z}/n\mathbb{Z}$ that uses the random radius distribution $f(r) = \frac{1}{r}$, then:*

$$\frac{T_n}{n} \overset{D}{\Longrightarrow} Y, \tag{1.10}$$

where Y is a non trivial distribution with $\operatorname{supp}\{Y\} = [0,1]$, that satisfies for $\alpha \in (0,1)$:

$$\mathbb{P}\left(\mathcal{C}_\alpha = \mathbb{S}^1\right) = \mathbb{P}\left(Y \leq \alpha\right), \text{ and}$$
$$\frac{\log |\mathbb{T}_n \setminus X_{\alpha n}|}{\log n} \mathbf{1}\{X_{\alpha n} \neq \mathbb{T}_n\} \overset{D}{\Longrightarrow} (1-\alpha)\mathbf{1}\{\mathcal{C}_\alpha \neq \mathbb{S}^1\}.$$

Remark 5. Given any finite collection of points in $\mathbb{S}^1$, the probability of covering these points in the discrete case converges to one. However, the cover time is not a random variable that depends only on a finite set of points, so there is a tightness issue that is central to Theorem B*. Showing that the discrete cover time converges to its continuous analogue is not trivial, as we need to control the random variables beyond just a local perspective. Remark 1 serves as a cautionary tale that the discrete and continuous processes may feature different behaviors.

Covering in arbitrary dimensions- Inspired by the symmetries of the Mandelbrot-Shepp model, we were tempted to study an analogous process in high-dimensional spaces.

For an integer $d \geq 1$, define the $d-$**dimensional torus** as the quotient $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. In $\mathbb{T}^d$, let $\|.\|$ be **the Euclidean distance** of the space, and define the **open ball with center** x **and radius** $r > 0$ as $B(x,r) = \{y \in \mathbb{T}^d : \|x - y\| < r\}$. Consider **the** $d-$**dimensional cylinder** $S^d = \mathbb{T}^d \times (0,\infty)$, and, for each point $\xi \in S^d$, write $\xi = (x,r)$, where x denotes a point in the torus $\mathbb{T}^d$, and $r > 0$ corresponds to a radius. Informally speaking, the cover process will associate each point $(x,r) \in S^d$ to the ball $B(x,r)$ in the covering.

Define **the Mandelbrot-Shepp model in the** d**-dimensional torus** $\mathbb{T}^d$ at time $\alpha \geq 0$ as a Poisson point process in the product space $S^d = \mathbb{T}^d \times (0,\infty)$ with rate:

$$d\Lambda^d_\alpha = \alpha \left(dx \otimes \frac{dr}{r^{d+1}} \right),$$

where dx is the Lebesgue measure of the d-dimensional torus and dr is the Lebesgue measure in $(0,\infty)$.

Moreover, for every point $\xi = (x,r) \in S^d$, we define the d**-dimensional projection** of the point ξ as:

$$\Pi^d(\xi) = B(x,r) \subset \mathbb{T}^d. \tag{1.11}$$

In particular, given any configuration $\omega = \sum_{i\in I} \delta_{(x_i,r_i)}$, construct the random sets:

$$\mathcal{C}^d(\omega) = \bigcup_{i\in I} \Pi^d((x_i, r_i)), \text{ and } \mathcal{V}^d(\omega) = \mathbb{T}^d \setminus \mathcal{C}^d(\omega).$$

Respectively the **covered set** and the **vacant set** of the Mandelbrot-Shepp model in the d-dimensional torus.

Remark 6. For $d = 1$, the projection Π^1 in equation (1.11) does not agree with the definition of projection Π in equation (1.9). Despite having this difference, both models can be related, by changing the parameter α to 2α, see in Subsection 4.2.1 for a proof of this relation.

Given a configuration $\omega = \sum_{i\in I} \delta_{(x_i,r_i)}$ and a real number $z > 0$, define the **truncated configuration**

$$\omega[z] = \sum_{i\in I} \delta_{(x_i,r_i)} \mathbf{1}\{r_i > z\}$$

as the set of points whose radii are larger than z. By using the truncated configurations, we can define **the covering function** and **the truncated covering function**, respectively as:

$$\pi^d(\alpha) = \mathbb{P}_\alpha \left(\mathcal{C}^d(\omega) = \mathbb{T}^d\right) = \mathbb{P}_\alpha \left(\mathcal{V}^d(\omega) = \emptyset\right),$$
$$\pi^d_z(\alpha) = \mathbb{P}_\alpha \left(\mathcal{C}^d(\omega[z]) = \mathbb{T}^d\right) = \mathbb{P}_\alpha \left(\mathcal{V}^d(\omega[z]) = \emptyset\right).$$

In this text, we give properties of the covering probabilities $\pi^d(\alpha)$ and $\pi^d_z(\alpha)$. The results will characterize a critical threshold below which the process may be vacant with positive probability (subcritical), while above it the process is fully covered with probability one (supercritical). Moreover, we are also interested in describing these probabilities on truncated levels.

More precisely, define the **critical threshold**

$$\alpha_c = \inf \left\{ \alpha > 0 : \lim_{z\to 0} \pi^d_z(\alpha) = 1 \right\}, \tag{1.12}$$

that detects when the space is covered almost surely. The existence of this point is not trivial, however one might expect that arbitrary dimensions follows an analogous property described by Theorem B and B*. This definition divides the set of parameters $\alpha \geq 0$ in two sets: The **supercritical** set $\{\alpha < \alpha_c\}$, and the **subcritical** $\{\alpha > \alpha_c\}$ one.

We now introduce an alternative definition that gives a quantitative information about the critical point. Let λ the **Lebesgue measure on** $\mathbb{T}^d$, where for every Borelian $A \subset \mathbb{T}^d$. Then, define the **mean threshold**:

$$\alpha_M = \inf\left\{\alpha > 0 : \lim_{z\to 0} z^{-d}\mathbb{E}_\alpha\left(\lambda(\mathcal{V}^d\left(\omega[z]\right)\right) = 0\right\}, \tag{1.13}$$

detecting whether the truncated space has a significant vacant volume.

In the subcritical regime, $\alpha > \alpha_c$, the space is fully covered with probability one. However, at each truncated level, the probability $\mathbb{P}_\alpha\left(\mathcal{V}\left(\omega[z]\right) \neq \emptyset\right)$ is positive. This probability goes to zero, and we can ask about the rate at which it occurs. Define the **subcritical rate threshold**:

$$\alpha_+ = \inf\left\{\alpha > 0 : \exists\zeta_+ = \zeta_+(\alpha) > 0 \text{ such that } \mathbb{P}_\alpha\left(\mathcal{V}^d\left(\omega[z]\right) \neq \emptyset\right) < z^{\zeta_+}, \forall z > 0\right\}, \tag{1.14}$$

that measures identifies the region where the probability of having vacant sets in the truncated levels decays polynomially to zero. By definition, we have that $\alpha_+ \geq \alpha_c$.

Remark 7. The probability of not covering a fixed point in space can be computed explicitly; see Lemma 15. Moreover, in $\mathcal{V}^d\left(\omega[z]\right)$, this probability decays polynomially to zero in z. Since the probability of not covering the space is bounded by below by the probability of not covering a point, one cannot expect to observe a decay faster than polynomial.

Define the constant $C_d = \pi^{d/2}\Gamma^{-1}\left(\frac{d}{2}+1\right)$ where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the gamma function; observe that the volume of the ball $B(x,r)$ in $\mathbb{R}^d$ is given by $\lambda(B(x,r)) = C_d r^d$. The main result in the supercritical regime is:

Theorem 1. *For every $d \geq 1$, we have $\frac{d}{C_d} = \alpha_M = \alpha_c = \alpha_+$.*

The proof of Theorem 1 is based on a set of propositions, some of which are direct and others are not. The explicit value of α_M is found by direct computation. However, the relationship of the mean threshold α_M with the critical phenomena is not immediate. To relate both thresholds, we define a shrunk configuration that increases the vacant set but does not change the overall intensity of the objects. Using this shrunk configuration, one can overcome the non-enumerable quantity of vacant sets, and use union bound on the possible regions where the vacant belongs, this together with a sharp computation in the shrunk configuration shows that $\alpha_c \leq \alpha_+ \leq \alpha_M$. To finish and show that $\alpha_c = \alpha_M$, we use Billard's Theorem, which gives conditions to have a positive probability of $\mathbb{T}^d$ not being completely covered.

Now, for the supercritical regime, where the process may not be covered with positive probability, define the notion of a well-behaved point. For any point $\alpha \in [0, \alpha_c)$, we say that it is **well-behaved** if:

$$\begin{gathered}\text{There exists } \gamma_1 = \gamma_1(\alpha),\ \gamma_2 = \gamma_2(\alpha),\ \gamma_1 > \gamma_2 > 0 \text{ such that} \\ \lim_{z \to 0} \mathbb{P}_\alpha \left(z^{\gamma_1} < \lambda(\mathcal{V}\left(\omega[z]\right)) < z^{\gamma_2} | \mathcal{V}(\omega) \neq \emptyset \right) = 1.\end{gathered} \tag{1.15}$$

The definition of well-behaved only requires the volume of the vacant set in the truncated levels to behave as a polynomial with probability one in the limit as $z \to 0$, ignoring how fast this probability converges to one.

The main difficulty of the supercritical regime lies in the continuity of the covering function $\pi(\alpha)$. Since at points where the function is not continuous, the vacant set might present a strange property: changing the parameter a little can instantaneously cover the space. Such moments, if exists, will interfere with a homogeneous description of the supercritical phase.

About the supercritical regime, we have the following result.

Theorem 2. *For any $d \geq 1$, and for every $\alpha \in [0, \alpha_c)$ a continuity point of the covering function π, we have that α is well-behaved.*

The proof of Theorem 2 involves forcing a large number of completely empty small regions to be present in the truncated levels. In a way that with high probability, one of the latter survives as a branching process, leading the order of the vacant regions in the limit.

Remark 8. The covering function $\pi(\alpha)$ is clearly monotone, therefore, almost every point $\alpha \in [0, \alpha_c)$ is well-behaved.

In the specific case $d = 1$, we have that:

Theorem 3. *For $d = 1$, $\pi(\alpha)$ is continuous in $\mathbb{R} \setminus \{\alpha_c\}$, and thus any $\alpha \in [0, \alpha_c)$ is well behaved.*

Remark 9. To simplify the statements and discussions, we fixed the objects to be balls $(B(\cdot, r) : r > 0)$. However, the same results can be generalized using other shapes such as simplex or boxes. We only require that the shape has positive volume, be connected and have a well behaved boundary (differentiable or polygonal). The critical point in each case might have a different value, but the equality $\alpha_c = \alpha_M = \alpha_+$ will continue to hold.

Previous works- With the model well defined and the theorems stated, let us state the connections they have with the existing literature. First, the choice of the name of the process (Mandelbrot-Shepp) is due to the works [22] and [28] that were published in the same year, 1972. Both processes were defined in the one-dimensional real line and try to describe the behavior of the vacant set. Furthermore, after 1972, many other approaches and different results were presented.

For example, the Mandelbrot-Shepp process was extensively studied for dimension $d = 1$, not only in the circle but also in the real line; see [19, 3]. Regarding the works in high-dimensional spaces, we can divide the literature into two distinct branches of articles: Those that resolve the issues proposed by Shepp and those that solve the questions proposed by Mandelbrot. In the branch that discusses the questions proposed by Shepp, inspired by Dvoretzky's covering problem, see [12], the focus lies in finding conditions under which a set of objects with fixed volumes $(v_n)_n$ covers the space or not; see [19, 20, 16]. On the other hand, in the Mandelbrot branch, inspired by the book [23], works related to the so-called Mandelbrot percolation appear; see [8, 5].

Parallel to each branch of references, there were many attempts to calculate the Hausdorff and packing dimensions related to the Mandelbrot-Shepp model and other related processes; see [33] for the carpet model and [15] for the set of points covered infinitely many times.

It is also worth noting that the covering model presents similarities with a continuum percolation model. To be precise, for every set $C \subseteq \mathbb{T}^d = [0,1)^d$ and positive real number $\zeta > 0$ define the multiplication ζC as the subset $\left\{p \in \mathbb{R}^d : \frac{p}{\zeta} \in C \subset [0,1)^d\right\}$. Both models can be related as follows: Consider the cover set at the truncated level $1/n$, that is, $\mathcal{C}\left(\omega[\frac{1}{n}]\right) \subset \mathbb{T}^d$, then scale it by n to become a random set $n\mathcal{C}\left(\omega[\frac{1}{n}]\right) \subseteq n\mathbb{T}^d = [0,n)^d$. For every value of $n \geq 1$ and $\alpha > 0$ fixed, the configuration $n\mathcal{C}\left(\omega[\frac{1}{n}]\right) \subseteq n\mathbb{T}^d$ has the same distribution as a continuum percolation model using balls with random radii R, where $\mathbb{P}\left(R > r\right) = r^{-d}$, and with intensity $\alpha > 0$ (the intensity and the radius distribution are fixed for every choice of n). See the continuum percolation model defined in [9]. Note also that the mean volume of the objects in continuum percolation diverges, thus the space is fully covered almost surely, see [31] on the hole space $\mathbb{R}^d$.

Using the relation with continuum percolation, we can make the definition of the mean threshold α_M clearer, see equation (1.13). Note that, for any $\alpha > 0$, in the Mandelbrot-Shepp model, the volume of the vacant set always converges to zero in mean. Definition (1.13) asks when the volume of the process converges to zero faster than the function z^d. This choice of function is not arbitrary, and to see that, we can use the connection with the continuum percolation. In the boxes $n\mathbb{T}^d$ of the continuum percolation, since we already multiply the space by n, a more natural definition of α_M emerges. In this case, α_M will correspond to the mean volume of the vacant set in the box of size n (not any more n^{-d} times the expected volume).

The results of Mandelbrot percolation or the carpet model characterize the probability that there exists a positive measure path in the vacant set. Here, our aim is to provide a similar characterization, but now for coverage phenomena. In essence, this consists of exposing how the coverage process behaves as we vary α and also giving bounds on how well the model can be approximated by the truncated version.

The thesis is divided as follows. The proofs of Theorems A, B, C, and D, are presented in Chapter 2, respectively, in Sections 2.1, 2.2, 2.3, and 2.4. Chapter 3 is dedicated to proving Theorem B* and constructing the continuous model that serves as a limit distribution. In Chapter 4, we study processes in spaces with arbitrary dimension. More precisely, in Section 4.1, we work with the subcritical phase, where the process is covered with probability one, and the proof Theorem 1 that describes this phase. Next, in Section 4.2, we work in the supercritical regime, where the process may not be covered with some positive probability. In Subsection 4.2.1, we proof Theorem 2 which controls the number of vacant points in the limit.In Subsection 4.2.2, we talk about the continuity of the function $\pi(\alpha)$ for $d = 1$. In Chapter 5, Section 5.1 is dedicated to open problems that remain unsolved, and finally, Appendix 5.2 is divided into two subsections; Subsection 5.2.1 presents some lemmas and useful tools used in the proof of theorems, and Subsection 5.2.2 consider the discrete-time case.

Notations- During the thesis, the real numbers are denoted by $\mathbb{R}$, the integers correspond to $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$, the set I is used as an arbitrary index set. Concerning measures, we denote by δ the Dirac measure and by $\lambda(.)$ the Lebesgue measure. We also use the same notation for the Lebesgue measure in the Torus and in the space $\mathbb{R}^d$, but we clarify which spaces are considered in each case.

CHAPTER 2

The different phases of the covering process

This chapter contains the proofs of Theorems A, B, C, and D. Each proof is self-contained and can be read independently.

2.1 Gumbel Phase

This Section is devoted to the proof of Theorem A and is organized as follows: First, some auxiliary lemmas about the behavior of the light tail radius are proved. Then, three propositions are stated and used to prove Theorem A. Finally, we present the proof of these propositions.

Lemma 1. *Let R be a discrete random variable with $\mathbb{P}(R \geq r) = f(r)$. The following statements are equivalent:*

1. $\mathbb{E}\left(R^{p+1}\right) < \infty$ *for some* $p > 1$.
2. *There exists* $\lambda > 0$ *such that* $\lim_{k\to\infty} f(k)k^{1+\lambda} = 0.$ (2.1)
3. *There exists* $\lambda' > 0$ *such that* $\lim_{k\to\infty} f(k)k^{1+\lambda'} \ln k = 0.$ (2.2)

Lemma 1 plays a role in simplifying several proofs in this Section. Its proof is elementary and can be found in Appendix Subsection 5.2.1. Continuing with the lemmas, let $\mu = \mathbb{E}(R)$ and define the auxiliary variable:

$$g_k = \frac{\sum_{i=1}^{k} f(i)}{\mu}. \tag{2.3}$$

About this auxiliary sequence, $g_1 = \mu^{-1}$ due to (1.1) and g_k is a non-decreasing sequence that converges to one. Another important observation is given below.

Lemma 2. *If f satisfies condition (2.1), i.e. exists $\lambda > 0$ such that $\lim_{k\to\infty} f(k)k^{1+\lambda} = 0$. Then*

$$\lim_{n\to\infty} (1 - g_n) \ln n = 0. \tag{2.4}$$

Proof of Lemma 2. Using condition (2.1), for some $\lambda > 0$, exists a $n_0(\lambda)$ and a constant $C = C(\lambda)$ such that for every $n > n_0$:

$$(1 - g_n) \ln n = \frac{\ln n}{\mu} \sum_{i=n+1}^{\infty} f(i) \leq \frac{C \ln n}{n^{\lambda}\mu}.$$

The proof follows by taking the limit. □

Now, fix an arbitrary $\alpha \in (0, 1)$ and define the $\alpha - vacant$ set

$$\mathcal{V}_\alpha = (\mathbb{Z}/n\mathbb{Z}) \setminus X_{\alpha \frac{n \ln n}{\mu}},$$

this is the set of points in $\mathbb{Z}/n\mathbb{Z}$ that have not yet been covered at time $\alpha \frac{n \ln n}{\mu}$.

The first step of the proof of Theorem A consists in showing that $\mathcal{V}_\alpha$ is a set of sparse points that has a polynomial size; see Propositions 1 and 2. The second step relates the cover time of the sparse set $\mathcal{V}_\alpha$ with an time scale of a classical coupon collector. In this relation, we prove that the cover happens one point at a time; see Proposition 3. With this in hand, Theorem A follows by simple observations made in Subsection 2.1.1.

For any $\alpha, \beta \in (0, 1)$, define the event

$$A_\alpha(\beta) = \left\{\text{There exists } x, y \in \mathcal{V}_\alpha \ : \ |x - y| < n^{\beta}\right\}.$$

Proposition 1 (Sparse)**.** *Choose R satisfying $\mathbb{E}(R) = \mu < \infty$. Then, for every choice of $\beta \in (0, 1)$ there exists $\alpha = \alpha(\beta, \mu) < 1$ such that*

$$\lim_{n\to\infty} \mathbb{P}(A_\alpha(\beta)) = 0. \tag{2.5}$$

The proof of this proposition is postponed to Subsection 2.1.2.1 and is based on a union bound argument on the set $A_\alpha(\beta)$.

Proposition 2 (Concentration). *Let R be any distribution satisfying condition (2.4). For all $\alpha < 1$, and for any given $\delta > 0$*

$$\lim_{n\to\infty} \mathbb{P}\left(\left||\mathcal{V}_\alpha| - n^{1-g_n\alpha}\right| \geq \delta n^{1-g_n\alpha}\right) = 0. \tag{2.6}$$

The proof of this proposition is exposed in Subsection 2.1.2.2. It relies on a second moment estimate of the random variable $|\mathcal{V}_\alpha|$.

By Lemma 2 all distributions R with more than one moment satisfies Propositions 1 and 2. However, this condition is stronger than the necessary condition for those two propositions, and there exists random variables without moments bigger than one that satisfies both of them. We choose to use the moment condition, since it simplified some statements, and is needed for the next argument.

To state Proposition 3, let $\beta \in (0, 1)$ and define the family of sets

$$\mathcal{K}_\beta^n = \{K \subset \mathbb{Z}/n\mathbb{Z} : \forall x, y \in K, |x - y| > n^\beta\}.$$

Also, for any set $K \in \mathcal{K}_\beta$ define the time to cover K as $T_K = \inf\{t : K \subset X_t\}$.

Proposition 3 (Covering a sparse set). *Take R satisfying the condition (2.2), i.e., there exists $\lambda > 0$ such that $\lim_{k\to\infty} f(k)k^{1+\lambda} \ln k = 0$. There exists $\beta_0 = \beta_0(\lambda) < 1$, such that, for every $\beta \in (\beta_0, 1)$ and any sequence of sets $\{K(n)\}_n$ that satisfies $\lim_n |K(n)| = \infty$ and $K(n) \in \mathcal{K}_\beta^n$ for every $n > 0$, we uniformly get that:*

$$\frac{\mu}{n} T_{K(n)} - \ln |K(n)| \overset{D}{\Longrightarrow} \text{Gumbel}(0, 1). \tag{2.7}$$

The proof of Proposition 3, presented in Subsection 2.1.2.3, obtains (2.7) by creating a coupling between the covering time of the set $K(n)$ and a time change of the classical Coupon collector problem. We will in fact prove that, with high probability, the covering of $K(n)$ happens one point at a time.

2.1.1 Proof of Theorem A

With no further delay, assuming all the tree propositions above, Theorem A will be proven in this Subsection. The subsequent Subsection 2.1.2 contains the proofs of the propositions.

Proof of Theorem A. The idea of the proof consists in using the hypotheses and Propositions 1 and 2 to find a sparse and large set of vacant points. Next, using Proposition 3 when covering this sparse and large set, the Gumbel distribution will appear.

Taking R such that $\mathbb{E}(R^p) < \infty$ for some $p > 1$, and $\mathbb{E}(R) = \mu$, by Lemma 1 one can find $\lambda > 0$ such that $f(k)k^{1+\lambda}\ln k$ goes to zero, when k goes to infinity. For this fixed $\lambda > 0$, find $\beta_0(\lambda)$ using Proposition 3, and fix any $\beta \in (\beta_0, 1)$ to control how sparse the set needs to be. Now, with β fixed, use Proposition 1 to fix some $\alpha \in (\alpha(\beta,\mu), 1)$.

With the parameters fixed, start using Proposition 1 and 2 to show that the set

$$\Omega_n = \left\{\omega : |\mathcal{V}_\alpha(\omega)| > \ln n, \mathcal{V}_\alpha(\omega) \in \mathcal{K}_\beta^n\right\}$$

has probability converging to one. To exactly compute how big in size the vacant set is, use the full strength of Propositions 2 to get

$$\ln\left(\frac{n^{1-g_n\alpha}}{|\mathcal{V}_\alpha|}\right) \xrightarrow{\mathbb{P}} 0, \tag{2.8}$$

where the right arrow $\mathbb{P}$ indicates convergence in probability.

On the event Ω_n, define $T_\alpha = \inf\{t > 0 : \mathcal{V}_\alpha \subset X_{t+\alpha\frac{n\ln n}{\mu}}\}$ the cover time of the vacant set $\mathcal{V}_\alpha$. In particular, one can observe that

$$\mathbf{1}_{\Omega_n} T_\alpha = \mathbf{1}_{\Omega_n}\left(T_n - \alpha\frac{n\ln n}{\mu}\right), \tag{2.9}$$

where $T_n = \inf\{t : \mathbb{Z}/n\mathbb{Z} \subset X_t\}$. Since it is also the case that $|\mathcal{V}_\alpha|$ diverges on the event Ω_n, apply Proposition 3 to obtain

$$\mathbf{1}_{\Omega_n}\left(\frac{\mu}{n}T_\alpha - \ln|\mathcal{V}_\alpha|\right) \overset{D}{\Longrightarrow} \text{Gumbel}(0,1), \tag{2.10}$$

when $n \to \infty$; Therefore, using equations (2.10) and (2.8):

$$\mathbf{1}_{\Omega_n}\left(\frac{\mu}{n}T_\alpha - (1-\alpha g_n)\ln n\right) \overset{D}{\Longrightarrow} \text{Gumbel}(0,1).$$

Now, by definition (2.9), we get that:

$$\begin{aligned}&\mathbf{1}_{\Omega_n}\left(\frac{\mu}{n}\left(T_n - \alpha\frac{n\ln n}{\mu}\right) - (1-\alpha g_n)\ln n\right)\\ &\quad= \mathbf{1}_{\Omega_n}\left(\frac{\mu}{n}T_n - (1+\alpha(1-g_n))\ln n\right) \overset{D}{\Longrightarrow} \text{Gumbel}(0,1).\end{aligned}$$

Given any sequence of random variables $(Y_n)_n$, and events Ω_n that satisfies: $Y_n\mathbf{1}_{\Omega_n} \overset{d}{\Longrightarrow} Y$ and $\mathbb{P}(\Omega_n) \to 1$ when $n \to \infty$. Then $Y_n \overset{d}{\Longrightarrow} Y$. In our case, using condition (2.4) and the fact that $\mathbb{P}(\Omega_n) \to 1$ when $n \to \infty$, the equation lead to:

$$\frac{\mu}{n}T_n - \ln n \overset{D}{\Longrightarrow} \text{Gumbel}(0,1),$$

proving the Theorem A. □

2.1.2 Proofs of Propositions 1, 2 and 3

This Subsection aims to prove all three propositions. Before doing this, let us start by investigating the probability that a single site remains vacant.

Lemma 3. *On the continuous cover process at time* $\alpha\frac{n\ln n}{\mu}$

$$\mathbb{P}\left(0 \in \mathcal{V}_\alpha\right) = n^{-\alpha g_n}, \text{ and} \tag{2.11}$$

$$\mathbb{P}\left(0, k \in \mathcal{V}_\alpha\right) = n^{-\alpha(g_k + g_{n-k})} \tag{2.12}$$

for any $k \in \{1, ..., n-1\}$.

Proof. Observe that the number of objects covering 0 corresponds to a Poisson random variable. The computation of its rate involves determining the probability that a single object covers the origin. To proceed, we have:

$$\mathbb{P}\left(0 \in \mathcal{O}_1\right) = \sum_{i=1}^{n} \mathbb{P}\left(U_1 = i-1, R \geq i\right) = \sum_{i=1}^{n} \frac{f(i)}{n}.$$

In particular, the rate of the Poisson is given by the product

$$\alpha\frac{n\ln(n)}{\mu}\sum_{i=1}^{n}\frac{f(i)}{n} = \alpha g_n \ln(n).$$

Therefore:

$$\mathbb{P}\left(0 \in \mathcal{V}_\alpha\right) = \exp\{-\alpha g_n \ln n\},$$

proving (2.11).

The proof of the statement of equation (2.12) relies on counting the number of objects that at time $\alpha\frac{n\ln n}{\mu}$ hits 0 or k for some $k \in \{1, ..., n-1\}$. For this, observe that: When the uniform U is between 1 and k, the object need just to hit k; When it is between $k+1$ and n, the object need to hit the origin. So:

$$\mathbb{P}\left(\{0 \in \mathcal{O}_1\} \cup \{k \in \mathcal{O}_1\}\right) = \frac{\sum_{i=1}^{k} f(i)}{n} + \frac{\sum_{i=1}^{n-k} f(i)}{n}.$$

Therefore, following the same steps used to deduce (2.11), we get:

$$\mathbb{P}\left(0, k \in \mathcal{V}_\alpha\right) = \exp\{-\alpha(g_k + g_{n-k})\ln n\},$$

proving (2.12). □

2.1.2.1 Proof of Proposition 1

Proof of Proposition 1. Let $\mu = \mathbb{E}(R)$, then for any choice of $\beta \in (0,1)$, take $\alpha = \alpha(\mu, \beta)$ satisfying:

$$1 > \alpha > \max\left\{\frac{1}{1+\mu^{-1}}, \frac{1+\beta}{2}\right\}.$$

With α fixed, choose $\varepsilon = \varepsilon(\mu, \alpha, \beta)$ such that:

$$0 < \varepsilon < \min\left\{2 - \frac{(1+\beta)}{\alpha}, \frac{2(\alpha(1+\mu^{-1})-1)}{\alpha}\right\}. \tag{2.13}$$

Using a union bound on $A_\alpha(\beta)$, and relation (2.12), the computation leads to:

$$\begin{aligned}\mathbb{P}(A_\alpha(\beta)) &= \mathbb{P}\left(\bigcup_{x,y\in\mathbb{T}_n:\,|x-y|<n^\beta}\{x,y\in\mathcal{V}_\alpha\}\right)\\ &\le n\sum_{k=1}^{\lfloor n^\beta\rfloor}\mathbb{P}(0,k\in\mathcal{V}_\alpha) = \sum_{k=1}^{\lfloor n^\beta\rfloor} n^{1-\alpha(g_k+g_{n-k})}.\end{aligned}$$

With fixed β, and with α and ε chosen accordingly. Since $g_1 = \mu^{-1}$, and $(g_n)_n$ converges monotonously to one, find $k_0 = k_0(\varepsilon)$ such that for every $k > k_0$ we have $g_k > 1 - \varepsilon/2$. Then, for $n > 2k_0$:

$$\begin{aligned}\mathbb{P}(A_\alpha(\beta)) &\le \sum_{k=1}^{k_0} n^{1-\alpha(g_k+g_{n-k})} + \sum_{k=k_0+1}^{\lfloor n^\beta\rfloor} n^{1-\alpha(g_k+g_{n-k})}\\ &\le \sum_{k=1}^{k_0} n^{1-\alpha(g_k+1-\varepsilon/2)} + \sum_{k=1}^{\lfloor n^\beta\rfloor} n^{1-\alpha(2-\varepsilon)}\\ &\le k_0 n^{1-\alpha(1+\mu^{-1}-\varepsilon/2)} + n^{1+\beta-\alpha(2-\varepsilon)}\end{aligned}$$

By the choice of ε and α, it follows that $\mathbb{P}(A_\alpha(\beta))$ decays polynomialy to zero. □

2.1.2.2 Proof of Proposition 2

Proof of Proposition 2. The proof follows by Chebyshev's inequality. Start by using Lemma 3 to get:

$$\begin{aligned}\mathbb{E}(|\mathcal{V}_\alpha|) &= n^{1-\alpha g_n}, \text{ and}\\ \mathbb{E}\left(|\mathcal{V}_\alpha|^2\right) &= \sum_{x\in\mathbb{Z}/n\mathbb{Z}}\mathbb{P}(x\in\mathcal{V}_\alpha) + \sum_{\substack{x,y\in\mathbb{Z}/n\mathbb{Z}\\ x\neq y}}\mathbb{P}(x,y\in\mathcal{V}_\alpha)\\ &= n^{1-\alpha g_n} + 2n\sum_{k=1}^{n/2} n^{-\alpha(g_k+g_{n-k})}.\end{aligned}$$

Now, applying the Chebyshev's inequality it follows that for any $\delta > 0$:

$$\begin{aligned}\mathbb{P}\left(||\mathcal{V}_\alpha| - \mathbb{E}\left(|\mathcal{V}_\alpha|\right)| > \delta n^{1-\alpha g_n}\right) &\leq \frac{\mathbb{E}\left(|\mathcal{V}_\alpha|^2\right) - \mathbb{E}\left(|\mathcal{V}_\alpha|\right)^2}{n^{2(1-\alpha g_n)}\delta^2} \\ &= \frac{n^{1-\alpha g_n} - n^{2(1-\alpha g_n)} + 2n\sum_{k=1}^{n/2} n^{-\alpha(g_k+g_{n-k})}}{n^{2(1-\alpha g_n)}\delta^2} \\ &= \frac{1}{\delta^2 n^{1-\alpha g_n}} + \frac{1}{\delta^2} f(\alpha, n), \qquad (2.14)\end{aligned}$$

where:

$$f(\alpha, n) = \frac{2}{n}\sum_{k=1}^{n/2}(-1 + n^{\alpha(2g_n - g_k - g_{n-k})}).$$

Since $\alpha < 1$, and the distribution f satisfies condition (2.4) the proofs follows directly of the following Lemma. □

Lemma 4. *If condition* (2.4) *is satisfied, that is* $\lim_n(1 - g_n)\ln n = 0$*, and* $\alpha < 1$*, then* $\lim_{n\to\infty} f(\alpha, n) = 0$.

Proof. Fix $\alpha < 1$, $\mu > 1$ and take $\gamma \in \left(0, \frac{1}{2}\right)$, such that $\gamma < 2(1-\alpha(1-\mu^{-1}))$. Since $(g_n)_n$ converges monotonously to one, there exists $k_0 = k_0(\alpha, \gamma)$ such that $1 - g_k \leq \gamma(2\alpha)^{-1}$ for all $k > k_0$, also assume that $n > k_0$. Now divide the function $f(\alpha, n)$ into three parts, so that $f(\alpha, n) = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \frac{2}{n}\sum_{k=1}^{k_0}\left(-1 + n^{\alpha(2g_n - g_k - g_{n-k})}\right), \\ I_2 &= \frac{2}{n}\sum_{k=k_0}^{\lfloor n^\gamma \rfloor}\left(-1 + n^{\alpha(2g_n - g_k - g_{n-k})}\right), \\ I_3 &= \frac{2}{n}\sum_{k=\lceil n^\gamma \rceil}^{n/2}\left(-1 + n^{\alpha(2g_n - g_k - g_{n-k})}\right). \end{aligned}$$

It remains to show that for every $\varepsilon > 0$, there exists a number n_0 such that for every $n > n_0$, then $I_i < \varepsilon$ for every $i \in \{1, 2, 3\}$.

Concerning the term I_1, since k_0 is fixed, it follows that I_1 is a sum of $k_0 + 1$ elements. We have k_0 polynomials in n in the form n^{λ_k} for some λ_k, and one element of the form $k_0 n^{-1}$. Then, taking $n > 2k_0$, observe that:

$$\begin{aligned} \max_{k\leq k_0}\{\lambda_k\} &= \max_{k\leq k_0}\{\alpha(2g_n - g_k - g_{n-k}) - 1\} \\ &\leq \alpha(1 - \mu^{-1} + 1 - g_{n-k_0}) - 1 \\ &\leq \alpha(1 - \mu^{-1}) + \frac{\gamma}{2} - 1 < 0. \end{aligned}$$

Since $\gamma < 2(1-\alpha(1-\mu^{-1}))$, then $\max_{k\leq k_0}\{\lambda_k\}$ is negative. In particular, each term goes to zero in the term I_1, and it is possible to take $n_1 = n_1(\varepsilon, \gamma, \alpha, k_0)$ such that $I_1 < \varepsilon$ for every $n > n_1$.

For the term I_2, find $n_2' = n_2'(\gamma, \alpha)$ such that for every $n > n_2'$ it is true that $g_n - g_{n-k} = (1-g_{n-k}) - (1-g_n) < \gamma(2\alpha)^{-1}$ for every $k \in [k_0, n^\gamma]$. In that way:

$$\begin{aligned}
\frac{2}{n}\sum_{k=k_0}^{\lfloor n^\gamma \rfloor}\left(-1+n^{\alpha(2g_n-g_k-g_{n-k})}\right) &\leq \frac{2}{n}\sum_{k=k_0}^{\lfloor n^\gamma \rfloor}\exp\left\{\ln(n)\left(\alpha(g_n-g_k)+\frac{\gamma}{2}\right)\right\}\\
&\leq 2\exp\left\{\ln(n)\left(\gamma+\frac{\gamma}{2}+\alpha(g_n-g_{k_0})-1\right)\right\}\\
&\leq 2\exp\left\{\ln(n)\left(\frac{3\gamma}{2}+\alpha(1-g_{k_0})-1\right)\right\}\\
&\leq 2\exp\left\{\ln(n)\left(2\gamma-1\right)\right\}.
\end{aligned}$$

Then, by the choice of $\gamma < 1/2$ and k_0, find $n_2 = n_2(\varepsilon, k_0, \gamma) > n_2'$ such that for every $n > n_2$ the value of I_2 satisfies $I_2 < \varepsilon$.

To compute I_3, first use that $(g_n)_n$ is a monotone sequence converging to 1, then for $k \in [n^\gamma, n]$, and n is large enough:

$$4\alpha(g_n - g_k)\ln(n) < \frac{4\alpha}{\gamma}(1-g_{n^\gamma})\ln(n^\gamma). \tag{2.15}$$

Using that $(1+x) \geq e^{x/2}$ when $x < 1$, we can conclude that

$$(1+4\alpha(g_n-g_k)\ln(n))^{1/2\alpha(g_n-g_k)} \geq e^{\ln(n)} = n,$$

when $4\alpha(g_n - g_k)\ln(n) < 1$. Therefore, for big values of n, it is true that:

$$(-1+n^{2\alpha(g_n-g_k)}) \leq 4\alpha(g_n-g_k)\ln(n). \tag{2.16}$$

In particular, by equation (2.16), and since $g_k < g_{n-k}$ for every $k \in (n^\gamma, n/2)$, we get:

$$\begin{aligned}
I_3 &= \frac{2}{n}\sum_{k=\lceil n^\gamma\rceil}^{n/2}\left(-1+n^{\alpha(2g_n-g_k-g_{n-k})}\right)\\
&\leq \frac{2}{n}\sum_{k=\lceil n^\gamma\rceil}^{n/2}\left(-1+n^{2\alpha(g_n-g_k)}\right)\\
&\leq \frac{8\alpha}{n}\sum_{k=\lceil n^\gamma\rceil}^{n/2}(g_n-g_k)\ln n.
\end{aligned}$$

Using Lemma 2 on (2.15), for every fixed γ, take $n_3 = n_3(\varepsilon, k_0, \gamma)$ such that $\frac{16\alpha^2}{\gamma}(1 - g_{n^\gamma})\ln(n^\gamma) < \varepsilon$; more than this, for every $n \geq n_3$:

$$I_3 \leq \frac{8\alpha}{n}\frac{n}{2}\frac{4\alpha}{\gamma}(1 - g_{n^\gamma})\ln(n^\gamma) < \varepsilon.$$

To finish the proof of the lemma take $n_0 = \max\{n_1, n_2, n_3\}$. □

2.1.2.3 Proof of Proposition 3

So far, we did not use the full strength of hypotheses that $\mathbb{E}(R^{1+p}) < \infty$, for some $p > 0$. But here, in Lemma 5, the necessity of this condition will become evident.

Remark 10. Note that the random variable R with distribution $f(r) = \frac{1}{n\ln^3(n)}$ satisfies equation (2.4) from Lemma 2, but does not have any greater moment. In particular, for every $\lambda > 0$, $\lim_{k\to\infty} f(k)k^{1+\lambda}\ln k = \infty$, and R does not satisfy the Lemma 5. Consequentially Proposition 3 is not true for this distribution. Moreover, the covering of the remaining points, using our technique will not have a direct connection to the coupon collector problem.

Lemma 5. *Take R which satisfies the hypothesis in equation (2.2), this is, exists $\lambda > 0$ such that $\lim_{k\to\infty} f(k)k^{1+\lambda}\ln k = 0$. Then there exists a $\beta_0 = \beta_0(\lambda)$, where for every $\beta > \beta_0$, $C > 0$, and for every sequence of sets $(K_n)_n$, with $K_n \in \mathcal{K}_\beta^n$ for every $n \in \mathbb{N}$, we have:*

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k=1}^{N(Cn\ln n)} \{|\mathcal{O}_k \cap K_n| \geq 2\}\right) = 0,$$

where $N(t)$ is the Poisson process in the line with rate 1, used in the definition of the continuous time covering process.

Proof of Lemma 5. The probability that an object intercepts the set K_n in two points or more is bounded by $f(n^\beta) = \mathbb{P}(R > n^\beta)$. Therefore:

$$\mathbb{P}\left(\bigcup_{k=1}^{N(Cn\ln n)} \{|\mathcal{O}_k \cap K_n| \geq 2\}\right) \leq 1 - e^{-Cf(n^\beta)n\ln n}.$$

Now, λ satisfying the condition (2.2), and $m = n^{(1+\lambda)^{-1}}$, it is true that:

$$\lim_{n\to\infty} f(n^{(1+\lambda)^{-1}})n\ln n = \lim_{m\to\infty}(1+\lambda)f(m)m^{1+\lambda}\ln m = 0,$$

In this way, take $\beta > (1+\lambda)^{-1}$ to conclude the proof. □

To finish the proof, we need to understand a simple connection with the Coupon collector. To state it, start by fixing a parameter $p \in (0,1)$, and a set $\{1, ..., K\}$. Define a coupon collector of $\{1, ..., K\}$ with a time change p, in the following way: Consider a Poisson process with rate 1, and for each point in the Poisson process, sample an independent Bernoulli with parameter p. When the Bernoulli is equal to one, with probability p, take one of the possible K points in the space uniformly. When the Bernoulli is equal to zero, with probability $(1-p)$, do nothing. By the thinning argument of Poisson Point process, we can define $(\xi_k)_{k=1}^K$ as a set of independent exponential random variables with rate $\frac{p}{K}$, and, the time need to complete the space as:

$$T_K^\ell = \max_{k=1,\dots,K} \{\xi_k\}.$$

About this process, one may get that:

Lemma 6. *Let $(Y_k)_k$ be a coupon collector of the set $\{1, ..., K\}$ with time change $p \in (0,1]$, that may depend on K. Then, set T_K^ℓ the time needed to take all the coupons, so*

$$\frac{p}{K} T_K^\ell - \ln K \overset{D}{\Longrightarrow} \text{Gumbel}(0,1)$$

uniformly when $|K|$ goes to infinity.

The proof of the Lemma 6 is located in Subsection 5.2.1 in the Appendix. With this result we can conclude Proposition 3.

Proof of Proposition 3. For each object used in the covering $\mathcal{O} = \{U, U+1, \cdots, U+R-1\}$, define the truncated object at height n^β as $\overline{\mathcal{O}} = \{U, U+1, \cdots, U+\min\{R, n^\beta\}-1\}$. With the truncated objects, consider the truncated covering as $\overline{X}_t = \bigcup_{k=1}^{N(t)} \overline{\mathcal{O}}_k$.

Fix any sequence $(K(n))_n$, where $K(n) \in \mathcal{K}_\beta^n$ for every $n > 0$. The set $K(n)$ is composed of disjoint and sparse points of distance at least n^β, therefore, by construction $\overline{X}_t$ behaves like a coupon collector with time change. To compute the time change parameter, notice that the process is defined using a Poisson process and the region that covers each point of $K(n)$ is disjoint and have the same rate (this property is uniform over all sets in $\mathcal{K}_\beta^n$). In particular, the probability that a fixed point 0 is covered in the truncated covering is:

$$\mathbb{P}\left(0 \in \overline{\mathcal{O}}\right) = \frac{1}{n} \sum_{k=1}^{n^\beta} \mathbb{P}\left(R > k\right) = \frac{\mu g_{n^\beta}}{n}.$$

Now, since each point in $K(n)$ have the same probability to be covered in the truncated space, the time change is going to be $\frac{|K(n)|\mu g_{n^\beta}}{n}$. So, define:

$$T^{\ell}_{K(n)} = \inf\left\{t : K(n) \subset \overline{X}_t\right\}.$$

By applying Lemma 6, one gets:

$$\frac{\mu g_{n^\beta}}{n} T^{\ell}_{K(n)} - \ln|K(n)| \xrightarrow{D} \text{Gumbel}(0,1). \tag{2.17}$$

To finish the proof, it remains to replace $T^{\ell}_{K(n)}$ with $T_{K(n)}$, and remove the term g_{n^β} in the equation (2.17).

To prove that T^{ℓ}_{K} is indeed $T_{K(n)}$ with high probability, use Lemma 5 to show that no large object will appear in the time scale needed for the covering. More particular, define the event of having a big object until time t:

$$E_t = \bigcup_{k=1}^{N(t)} \{\mathcal{O}_k \neq \overline{\mathcal{O}}_k\}.$$

Assume $t = 2n\ln n$ to be a suitable value of t. By Lemma 5, the events E_t have probability going to zero, this is:

$$\lim_{n\to\infty} \mathbb{P}\left(E_{2n\ln(n)}\right) = 0.$$

Rest to show that $T^{\ell}_{K(n)}$ is lower than $2n\ln n$ with high probability. For this, consider the following bounds: $|K| < n$ and $g_{n^\beta} < 1$. For big values of n in the limit of equation (2.17), the following holds.

$$\mathbb{P}\left(T^{\ell}_{K(n)} > \frac{2n\log(n)}{\mu}\right) < 1 - \exp\left\{-\frac{1}{n}\right\}. \tag{2.18}$$

Now, lets relate the probability to cover the space with the probability in the truncate covering using the event E_t. Looking to values of t less then $2n\log(n)$, by equation (2.18) with probability converging to one the truncate space is covered. Now, in this frame of time, since the objects in both covered are equal in size, both have the same covering time. Therefore:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(T^{\ell}_{K(n)} = T_{K(n)}\right) &= \lim_{n\to\infty} \mathbb{P}\left(T^{\ell}_{K(n)} = T_{K(n)} | T^{\ell}_{K(n)} < 2n\ln n\right) \\ &\geq \lim_{n\to\infty} 1 - \mathbb{P}\left(E_{2n\ln n}\right) = 1.\end{aligned}$$

Finally, to remove the term g_{n^β} from equation (2.17) using the bound in equation (2.18), we get that $T_{K(n)}$ is of order $n\log(n)$ and therefore by the condition (2.4):

$$\frac{\mu}{n} T_{K(n)} \left(1 - g_{n^\beta}\right) \xrightarrow{\mathbb{P}} 0,$$

concluding that:

$$\frac{\mu}{n} T_{K(n)} - \ln|K(n)| \overset{D}{\Longrightarrow} \text{Gumbel}(0,1).$$

As desired. □

2.2 Compact Support Phase

Unlike Section 2.1, the theorem proved here is a more general version of the theorem stated in the Introduction. The additional conditions make the theorem more general, but less straightforward to understand. Therefore, we have intentionally postponed these conditions until now. Recall that $f(x) \in \mathrm{RV}_{-1}$, if for every $t > 0$ we have that $\lim_{x\to\infty} \frac{f(xt)}{f(x)} = t^{-1}$, or analogous if $f(x) = \frac{L(x)}{x}$ for some slowly varying function $L(x)$.

Theorem 1 (Compact Support Phase)**.** *Let $f \in \mathrm{RV}_{-1}$ that satisfies for all $\beta \in (0,1)$ that:*

$$\limsup_{n\to\infty} \sup_{n^\beta \le x \le n} \frac{xf(x)}{nf(n)} = b(\beta) < \infty, \quad \textit{and} \tag{2.19}$$

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n^\beta} f(i)}{f(n)n\ln n} = d(\beta) < \infty. \tag{2.20}$$

Then, $(f(n)T_n)_n$ is tight. Moreover, for every subsequence $(n_k)_k$ such that

$$f(n_k)T_{n_k} \overset{D}{\Longrightarrow} Y. \tag{2.21}$$

The distribution $Y = Y(f,(n_k)_k)$ is a non-degenerate distribution with compact support.

Remark 11. Hypotheses (2.19) and (2.20) look strange at first glance. To make it tangible to the reader fix $\beta \in (0,1)$, and consider the following examples:

- Take $f(x) = 1/x$. Therefore, the value appearing in condition (2.19) is trivially equal to one. Using the harmonic series, we can calculate the value appearing in condition (2.20) which is equal to β.

- Let $b \in \mathbb{R}$ and take $f(x) = \frac{\ln^b x}{x}$. About condition (2.19), we have that:

$$\sup_{n^\beta \le x \le n} \left\{ \frac{xf(x)}{nf(n)} \right\} = \sup_{\gamma \in (\beta,1)} \left\{ \frac{n^\gamma f(n^\gamma)}{nf(n)} \right\} = \sup_{\gamma \in (\beta,1)} \left\{ \gamma^b \right\}.$$

  For condition (2.20), let C be some constant, then we get that:

$$\begin{aligned} \sum_{i=2}^{n^\beta} f(i) &< \int_2^{n^\beta+1} f(x)dx \\ &= \begin{cases} \frac{\ln^{b+1}(n^\beta)}{b+1} + C, & \text{if } b \neq -1. \\ \ln\ln(n^\beta) + C, & \text{if } b = -1. \end{cases} \end{aligned}$$

  In particular, for all $b > -1$ the value in condition (2.20) if finite, and for all $b \le -1$ the condition (2.20) is not satisfied.

- To see a case where condition (2.19) is not satisfied, let $\gamma \in (0,1)$ and take $f(x) = \frac{\exp\{-\log^\gamma(x)\}}{x}$. We have $f \in \mathrm{RV}_{-1}$, and:

$$\sup_{n^\beta \le x \le n} \frac{xf(x)}{nf(n)} > \frac{\exp\{\log^\gamma(n)\}}{\exp\{\log^\gamma(n^\beta)\}} = \exp\{(1-\beta^\gamma)\log^\gamma(n)\},$$

  that diverges when n grows, for all $\beta \in (0,1)$ fixed.

In order to prove Theorem B, we divided the proof into two subsections. Then, the conclusion follows immediately by applying Prokhorov's Theorem to the sequence $(f(n)T_n)_n$. More precisely, the proof follows the following steps:

1. Subsection 2.2.1 proves that $(f(n)T_n)_n$ is tight, and that any limit in distribution belongs to some compact $[0, a^*]$ with $a^* > 0$.

2. Subsection 2.2.2 will prove that the limit distribution is not degenerate, that is, a probability distribution with support only at a single point. For this, looking to the covering when we place a small number of objects, and controlling the the large and small objects, we can find vacant places with high probability.

To simplify the proof, let us give another description for the continuous covering process that will come in handy. Consider $S = (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{Z}_+$, then define a Poisson Point Process $(\Omega, \mathcal{F}, \mathbb{P})$ on S with rate $\Lambda_\alpha = \alpha\,(\mathrm{Unif}(\mathbb{Z}/n\mathbb{Z}) \otimes dR)$, where dR is the discrete measure associated with the random variable R, and the parameter α will be related with the number of objects placed. Set $\Omega = \{\mathrm{w} : \mathrm{w} = \sum_{i \in I} \delta_{(u_i,r_i)},\ \text{ s.t. } (u_i, r_i) \in$

S for all $i \in I, I < \infty\}$ is the state space, and $\mathcal{F}$ is the smallest $\sigma-$algebra that makes the evaluation measures $\{\mathrm{w}(A) : A \subset S\}$ measurable.

This Poisson process is not artificial; indeed if we place a point $(U_k, R_k) \in S$ for every object $\mathcal{O}_k = \{U_k + \Gamma_{R_k}\}$, then the points placed till time $\alpha > 0$ have the same distribution as a Poisson process with rate Λ_α. To see whether the points imply a covering, define the projection function as

$$\begin{aligned} \Pi : S \to & \mathcal{P}(\mathbb{Z}/n\mathbb{Z}) \\ (u, r) \mapsto & \{u, u+1, ..., u+r\} \in \mathbb{Z}/n\mathbb{Z}. \end{aligned}$$

With the projection Π defined, given any configuration $\mathrm{w} = \sum_{i \in I} \delta_{(u_i, r_i)}$, one can recover the covering process X_α at time α using the configuration w as:

$$X_\alpha = X_\alpha(\mathrm{w}) = \bigcup_{i \in I} \Pi((u_i, r_i)).$$

2.2.1 Compact support

In the Gumbel phase, the typical objects exhibit small sizes in comparison to the torus. Here, however, the presence of objects that are comparable in size to the space itself becomes significant. To control their number and what these big objects cover, we use a Branching Process argument that can be found in the following proposition:

Proposition 4. *Let $f \in \mathrm{RV}_{-1}$. Then there exists $a^* > 0$ such that*

$$\lim_{n \to \infty} \mathbb{P}\left(f(n) T_n \leq a^*\right) = 1.$$

Proof of Proposition 4. The proof of this proposition is based on a comparison between the presence of large objects in the cover with a Branching Process. To properly define the comparison, we first need to define a set of regions in $S = (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{Z}_+$, and a set of intervals on the torus. Then, through the vacant set, a relationship can be constructed between these objects. The essence of the proof lies on the following observation: If there is a vacant interval, and the region of object in S capable of covering it entirely is empty, then we can divide the vacant interval into smaller sets, each set as child of the branching process will have a new independent chance to be covered or not by objects of another non explored region.

Let $\mathcal{T}_4 = (\mathbb{V}_4, \mathbb{E}_4)$ be a rooted tree in which every vertex has four children. Precisely, the set of vertex and edges are respectively:

$$\begin{aligned} \mathbb{V}_4 &= \left\{ v(i,h) \middle| h \geq 0,\ i \in \{0,1,2,3\}^h \right\}, \text{ and} \\ \mathbb{E}_4 &= \left\{ (v(i,h), v(i \times j, h+1)) | v(i,h) \in \mathbb{V}_4,\ j \in \{0,1,2,3\} \right\}. \end{aligned}$$

For any vertex $v(i,h) \in \mathbb{V}_4$ where $i = (i_1, ..., i_h)$, define its order as $|i| = \sum_{j=1}^{h} i_j 4^j$, and set the following regions in $[0,n) \times (0,2n)$:

$$R(i,h) = \left[\frac{n|i|}{4^h}, \frac{n(|i|+1)}{4^h}\right) \times \left[\frac{2n}{4^h}, \frac{2n}{4^{h-1}}\right),$$

For the vertex $v(0,0)$, define the region $R(0,0) = [0,n) \times [2n,\infty)$. Associated with each region $R(i,h)$, define the interval $I(i,h) = \left[\frac{n|i|}{4^h}, \frac{n(|i|+1)}{4^h}\right) \cap \mathbb{Z}$. See Figure 2.1 for a representation of the tree $\mathcal{T}_4$ side by side with the regions.

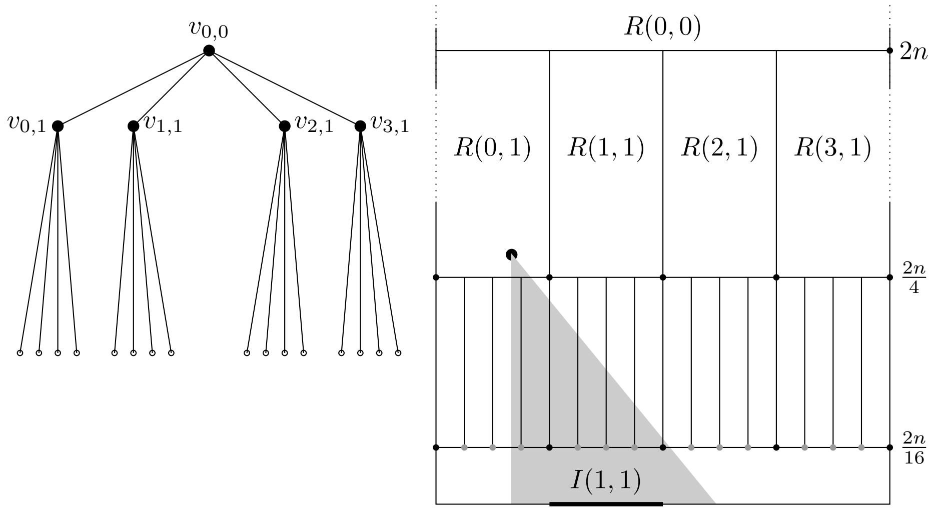

Figure 2.1: A representation side by side of the tree $\mathcal{T}_4$ and the set of intervals $R(i,k)$, on the rectangle $[0,1) \times (0,\infty)$. In the Figure, there exists an object that appears in the region $R(0,1)$, and covers the interval $I(1,1)$ bellor the region $R(1,1)$.

Now, fix a covering process X_α at time $\alpha = \alpha f^{-1}(n)$, and along with a realization of a Poisson point process w with intensity Λ_α. Then, we are going to define a heterogeneous branching process $(Z_h)_h$ in $\mathcal{T}_4$ using the regions $R(i,h)$, where $Z_0 = 1$, and associated with it, we have the vertex $v(0,0)$. For other values of h, define inductively

$$Z_{h+1} = \sum_{i=1}^{Z_h} 4 \cdot \mathbf{1}\{\mathrm{w}(R(\{|v(i)| - 1\} \mod 4^h, h) > 0\},$$

where $\{v(1), ..., v(Z_h)\}$ are the vertex associated to the $h-th$ generation of the branching process. Moreover, define the vertex associated for the next generation as the union of the four children of $v \in \{v(i)\}_{i=1}^{Z_h}$ such that $\{\mathrm{w}(R(\{|v| - 1\} \mod 4^h, h) > 0\}$ happens.

The branching process $(Z_h)_h$, despite having a complex definition, was created to preserve one property: If it dies, it covers the space. In essence, notice two things: The intervals $I(i,h)$ fit inside $I(j,k)$ if and only if $v(i,h)$ is an ancestor of of $v(j,k)$ (this is, $v(j,k)$ belongs to the unique path that connects $v(i,h)$ to the root of the tree); and, if $\{\mathrm{w}(R(i-1 \mod 4^h, h)) > 0\}$ then $I(i,h)$ is completely covered by an object. In particular, if $\{Z_h = 0\}$, each vertex in the tree $v(i,h)$ has a dead parent $v(j,k)$, or analogously, each interval $I(i,h)$ fits within some larger interval $I(j,k)$ such that $\{\mathrm{w}(R(j-1 \mod 4^h, k)) > 0\}$. Therefore, $I(i,h)$ is covered for every i, and the space is fully covered by objects with a radius greater than $2n/4^h$.

Since the process is heterogeneous in probability, we need caution when using classical branching arguments. To understand how the probability changes in each generation, fix a value of $n > 0$. Start by noticing that there is no object $\mathcal{O}$ of size smaller than one in the covering, then the regions $R(i,h)$ with $h > \lfloor \frac{\ln 2n}{\ln 4} \rfloor$ are always empty. About the dependence of the heterogeneous process, notice that by the Poisson construction, since the regions are disjoint, the survival probabilities despite being different are independent. Now, when $h \leq \lfloor \frac{\ln 2n}{\ln 4} \rfloor$, for every fixed region $R(i,h)$ by routine calculation:

$$\mathbb{P}\left(\mathrm{w}(R(0,0)) > 0\right) = 1 - \exp\left\{-\alpha \frac{f(2n)}{f(n)}\right\}$$
$$\mathbb{P}\left(\mathrm{w}(R(i,h)) > 0\right) = 1 - \exp\left\{-\frac{\alpha}{f(n)4^h}\left(f\left(\left\lceil \frac{2n}{4^{h-1}} \right\rceil\right) - f\left(\left\lfloor \frac{2n}{4^h} \right\rfloor\right)\right)\right\}.$$

To understand such values, since $f \in \mathrm{RV}_{-1}$, by the Karamata's representation Theorem, see Proposition 16 item 2, it is true that

$$f(r) = r^{-1}L(r),$$

where $L(r)$ is a slowly varying function. Then, for every fixed $h \geq 0$, by definition (1.5), using the Proposition 16 item 1, we get that:

$$\lim_{n\to\infty} \frac{f(2n)}{f(n)} = \frac{1}{2}, \text{ and} \tag{2.22}$$
$$\lim_{n\to\infty} \frac{1}{4^k f(n)}\left(f\left(\frac{2n}{4^k}\right) - f\left(\frac{2n}{4^{k-1}}\right)\right) = \frac{3}{8}, \text{ and}$$
$$\lim_{n\to\infty} \frac{1}{f(n)4^h}\left(f\left(\left\lceil \frac{2n}{4^{h-1}} \right\rceil\right) - f\left(\left\lfloor \frac{2n}{4^h} \right\rfloor\right)\right) = \frac{3}{8} \tag{2.23}$$

Therefore, the heterogeneous probabilities have a limit for each h fixed using the equations (2.22) and (2.23) that relays just on the fact that $f \in \mathrm{RV}_{-1}$. In particular, taking n to infinity, we get:

$$\lim_{n\to\infty} \mathbb{P}\left(\mathrm{w}(R(0,0)) > 0\right) = 1 - e^{-\frac{\alpha}{2}}, \text{ and}$$
$$\lim_{n\to\infty} \mathbb{P}\left(\mathrm{w}(R(i,h)) > 0\right) = 1 - e^{-3\alpha/8}. \tag{2.24}$$

Throughout the rest of the proof, define $\widehat{Z}_t$ a new homogeneous branching process that lives in the tree $\mathcal{T}_4$, and has all four children in one generation with probability equal to $e^{-\frac{2\alpha}{8}}$, that is, greater than the probability of both limits in the equation (2.24).

About $\widehat{Z}_t$, fix $\alpha^*(\widehat{Z}_t) = 4\ln 2$, and notice that for every $\alpha > \alpha^*$ the branching process $\widehat{Z}_t$ in $\mathcal{T}_4$ dies almost surely.

Now, fix $\alpha > \alpha^*$ and take any $\varepsilon > 0$, set $h_0 = h_0(\varepsilon, \alpha)$ such that

$$\mathbb{P}_\alpha\left(\widehat{Z_h} = 0, \text{ for some } h < h_0\right) > 1 - \varepsilon.$$

Finally, using this fixed value of $h = h(\varepsilon, \alpha)$ and the limits in equation (2.24), find $n_0 = n_0(h)$ such that for every $n > n_0$, the process $\widehat{Z}_h$ dominates the events $\mathbf{1}\{\mathrm{w}(R(i,h)) > 0\}$ with $h < h_0$. In particular, remember that by construction, if the branching Z_h dies, then the space is covered, therefore:

$$\mathbb{P}\left(f(n)T_n \leq \alpha\right) \geq \mathbb{P}\left(\widehat{Z_j}(\alpha) = 0 \text{ for some } j \leq h_0\right) \geq 1 - \varepsilon.$$

Taking the limit when ε goes to zero, one can conclude the theorem, for each $\alpha > \alpha^* = 4\ln 2$. □

2.2.2 Non degenerate distribution

This section is devoted to the proof that $(f(n)T_n)_n$ has a nondegenerate limit. To prove this, it is sufficient to show that for small values of α, we have:

$$0 < \liminf_{n\to\infty} \mathbb{P}\left(f(n)T_n > \alpha\right) \leq \limsup_{n\to\infty} \mathbb{P}\left(f(n)T_n > \alpha\right) < 1. \tag{2.25}$$

Different from Subsection 2.2.1, where a branching process technique shows that the space is covered by objects with size comparable with the space; here, we need a more delicate approach that controls the full range of object sizes at the same time to say that the space is not covered.

Before going into more detail, let us prove the lower bound of equation (2.25). Looking just at the objects greater than the space itself, at time $\alpha/f(n)$, we have that:

$$\liminf_{n\to\infty} \mathbb{P}\left(f(n)T_n > \alpha\right) > \lim_{n\to\infty} \mathbb{P}\left(\mathrm{w}(\{r > n\}) = 0\right) = e^{-\alpha}.$$

In particular, the limit above is non zero.

The proof of the upper bound of equation (2.25) is divided in Propositions 5 and 6. We start using a branching argument very similiar from the proof of Proposition 4, now for small values of α. To define it, we will need a new tree and a new set of regions.

Let $\widehat{\mathcal{T}} = (\widehat{\mathbb{V}}, \widehat{\mathbb{E}}) \subset \mathcal{T}_4$ be a sub-graph where

$$\widehat{\mathbb{V}} = \left\{ v(i,h) \middle| h \geq 0,\ i \in \{1,3\}^h \right\},\ \text{and}$$
$$\widehat{\mathbb{E}} = \{(v(i,h), v(i \times j, h+1)) | v(i,h) \in \mathbb{V},\ j \in \{1,3\}\}.$$

Observe the sub-graph $\widehat{\mathcal{T}}$ of $\mathcal{T}_4$ in the Figure 2.2. This restriction guarantees that any two siblings corresponds to separated intervals, and that distance will help with decoupling them.

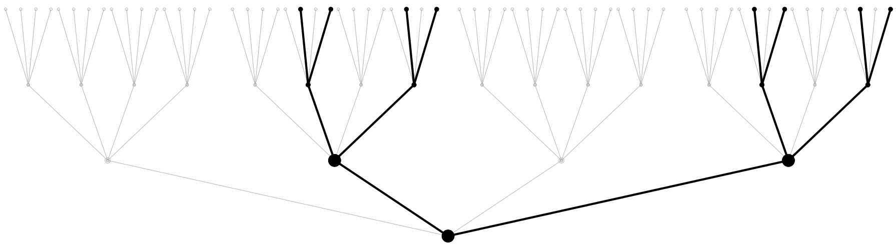

Figure 2.2: The black graph is a representative drawn of the subgraph $\hat{\mathcal{T}}$ inside the graph $\mathcal{T}_4$ drawn as gray.

Using the graph $\widehat{\mathcal{T}}$, for every vertex $\widehat{v}(i,h) \in \widehat{\mathbb{V}}$ with $h > 0$, define the region:

$$\widehat{R}(i,h) = \left[\frac{n|i|}{4^h}, \frac{n((|i|+2)}{4^h} \right) \times \left[\frac{n}{4^{h+1}}, \frac{n}{4^h} \right).$$

For the vertex $v(0,0)$, define $\widehat{R}(0,0) = [0,n) \times \left[\frac{n}{4}, \infty\right)$. And, together with it, we set the intervals $\widehat{I}(i,h) = \left[\frac{n(|i|+1)}{4^h}, \frac{n(|i|+2)}{4^h}\right) \subset \mathbb{Z}/n\mathbb{Z}$. Observe such regions and intervals in Figure 2.3.

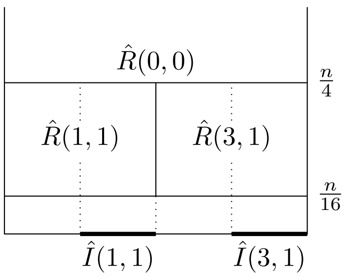

Figure 2.3: A representation of the regions $\hat{R}(i,h)$ and intervals $\hat{I}(i,h)$.

Now fixed any $\alpha > 0$, take the covering process $X_{\alpha/f(n)}$, and the associated configuration w with rate $\Lambda_\alpha/f(n)$. Define a heterogeneous branching process $(Z_h)_h$ in $\widehat{\mathcal{T}_4}$ using the regions $\widehat{R}(i,h)$, where $Z_0 = 1$, and associated with it, we have the vertex $v(0,0)$. For other values of h, define inductively

$$Z_{h+1} = \sum_{i=1}^{Z_h} 2 \cdot \mathbf{1}\{\mathrm{w}(R(|v(i)|, h) = 0\},$$

where $\{v(1), ..., v(Z_h)\}$ are the vertices associated to the $h-th$ generation of the branching process. Moreover, define the vertex associated with the next generation as the union of the two children in $\widehat{\mathcal{T}_4}$ of each $v \in \{v(i)\}_{i=1}^{Z_h}$ such that $\{\mathrm{w}(R(|v|, h) = 0\}$.

To continue, let us explore a relation between surviving in the branching process $(Z_h)_h$ and the covering process. Observe that $I(i,h)$ fits inside $I(j,k)$, if and only if $v(i,h)$ is an ancestor of $v(j,k)$. Also, notice that if $\{\mathrm{w}(\widehat{R}(i,h)) = 0\}$ then $I(i,h)$ does not intersect any object with radius between $n/4^{h+1}$ and $n/4^h$. Therefore, assuming that $\{Z_h > 0\}$, each surviving vertex $v(i,h)$ has a family of survival ancestors (all vertices that belong to the path connecting $v(i,h)$ to the root of the tree). In other words, each interval $I(i,h)$ is not intersected by any object of size greater than $n/4^h$.

Unfortunately, the probability that the process survives in generation h does not behave well. The fact that the tail f belongs to RV_{-1}, controls only objects of size comparable to n, such as $n/4^h$ with h fixed. In particular, to show that we do not cover the space, we need bounds to control objects of arbitrary size, as for example n^β with $\beta \in (0,1)$. It is important to note that for some tails distributions, the rate of the regions $\widehat{R}(i,h)$ can explode as h approaches $\left\lfloor \frac{\ln n}{\ln 4} \right\rfloor$.

To solve the heterogeneity problem, we divide the proof of the upper bound into two steps. First, we will explore the covering of objects with size greater than n^β. Second, we will work with the remaining objects, with sizes smaller than n^β. This division will be informally described by the terms big and small world.

More precisely, for any fix $\beta > 0$, $\alpha > 0$, let $\mathrm{w} = \sum_{i \in I} \delta_{(u_i, r_i)}$ be any configuration of a Poisson Point Process in S with rate $\Lambda_{\alpha/f(n)}$. Define the small and big world to be respectively the processes:

$$X_{\alpha/f(n)}[1, n^\beta) = X_{\alpha/f(n)}[1, n^\beta)(\mathrm{w}) = \bigcup_{i \in I} \Pi((u_i, r_i))\mathbf{1}\{r_i \in [1, n^\beta)\}$$
$$X_{\alpha/f(n)}[n^\beta, \infty) = X_{\alpha/f(n)}[n^\beta, \infty)(\mathrm{w}) = \bigcup_{i \in I} \Pi((u_i, r_i))\mathbf{1}\{r_i \in [n^\beta, \infty)\}$$

Note that the covering process $X_{\alpha/f(n)}$ can be written as $X_{\alpha/f(n)}[1, n^\beta) \cup X_{\alpha/f(n)}[n^\beta, \infty)$. And, by the thinning Poisson Theorem, we can work with these two processes independently.

The proof of the upper bound in relation (2.25) is divided in two steps. Independently, fix $\alpha > 0$ small. The first step reveals the object in the big world, using Proposition 5, it is possible to show that with α small we can find many empty connect regions of size n^{β}. Then, the next step reveals the objects in the small world, and by Proposition 6, we can show by concentration inequality that one of theses vacant regions reveled in the first step have with high probability a empty point. In this way, the probability of covering the space does not approach one, when α is small.

Proposition 5. *Let f be a distribution that satisfies condition (2.19) for some $\beta \in (0,1)$. Now, for all $\varepsilon > 0$, and $\eta \in (0,1)$, we can find $\alpha_0 = \alpha_0(\varepsilon, \beta, \eta)$ such that for all $\alpha < \alpha_0$, with probability greater than $1-\varepsilon$ there exist at least $n^{\frac{\eta}{2}(1-\beta)}$ intervals of size $\lfloor n^{\beta} \rfloor$ with mutual distance at least $\lfloor n^{\beta} \rfloor$ which are not intersected by the process $X_{\alpha/f(n)}[n^{\beta}, \infty)$.*

Proof of Proposition 5. To prove this proposition, we will use condition (2.19) to create a bound on the branching process $\widehat{Z}_h$, for $h < \lceil \ln n^{1-\beta} / \ln 4 \rceil$. Next, we will use the same branching method applied in the proof of Proposition 4, but we will end the calculations using a more specific branching theorem, witch regulates the number of children of the process.

To define the branching technique, take a vertex $\widehat{v}(i,h) \in \widehat{\mathbb{V}}$ and observe that:

$$\mathbb{P}\left(\mathrm{w}(\widehat{R}(0,0)) = 0\right) = \exp\left\{-\alpha \frac{f(n/4)}{f(n)}\right\}$$
$$\mathbb{P}\left(\mathrm{w}(\widehat{R}(i,h)) = 0\right) = \exp\left\{-\frac{2\alpha}{f(n)4^h}\left(f\left(\left\lceil \frac{n}{4^{h+1}} \right\rceil\right) - f\left(\left\lfloor \frac{n}{4^h} \right\rfloor\right)\right)\right\}.$$

In particular, using condition (2.19), we get that:

$$\begin{aligned} \exp\left\{-\frac{2\alpha}{f(n)4^h}\left(f\left(\left\lceil \frac{n}{4^{h+1}} \right\rceil\right) - f\left(\left\lfloor \frac{n}{4^h} \right\rfloor\right)\right)\right\} &> \exp\left\{-\frac{2\alpha f\left(\left\lceil \frac{n}{4^{h+1}} \right\rceil\right)}{f(n)4^h}\right\} \\ &= \exp\left\{-\frac{8\alpha \left\lceil \frac{n}{4^{h+1}} \right\rceil f\left(\left\lceil \frac{n}{4^{h+1}} \right\rceil\right)}{nf(n)}\right\} \\ &> \exp\{-8\alpha b(\beta)\} \end{aligned}$$

Then define $\widehat{Z}_h$ to be a homogeneous Branching process in $\widehat{\mathcal{T}}$ that have two children with probability $e^{-8\alpha b}$. Observe that $\widehat{Z}_h$ dominates the branching process Z_h for every $h \in \left[0, \lceil \ln n^{1-\beta} / \ln 4 \rceil\right)$, so one can couple both process in a way that if $\widehat{Z}_h$ survives, then Z_h also survives.

For every $\varepsilon > 0$ and $\eta \in (0,1)$, there exists $\alpha_0 = \alpha_0(\varepsilon, \eta, \beta)$ such that for all $\alpha < \alpha_0$ we have:

$$\begin{cases} e^{-8\alpha b} > 2^{-1+\sqrt{\eta}} \\ \mathbb{P}\left(\widehat{Z_j} = 0, \text{ for some } j > 0\right) \geq 1 - \varepsilon/2. \end{cases}$$

In particular, with probability close to one the process using just objects with size greater than $\lfloor n^{\beta} \rfloor$ does not cover some intervals of size $\lfloor n^{\beta} \rfloor$. Finally, to count the number of such intervals we can use the concentration of the Branching process in [21], stated for our case as the following Theorem.

Theorem 2. *Let $X_{n,m}$ be independent and equally distributed positive integer random variables with $n > m > 0$. Define the branching process as $Z_n = \sum_{m=1}^{Z_{n-1}} X_{n,m}$. If $\mathbb{E}(X_{1,1}) = \mu > 1$, $\mathrm{Var}(X_{1,1}) = \sigma^2 < \infty$ and $Z_0 = 1$, then there exists a distribution W such that:*

1. $\frac{Z_n}{\mu^n} \to W$, *almost sure.*
2. $\lim_{n\to\infty} \mathbb{E}\left(\left(\frac{Z_n}{\mu^n} - W\right)^2\right) = 0.$
3. $\mathbb{E}(W) = 1$, *and* $\mathrm{Var}(W) = \frac{\sigma^2}{\mu^2 - \mu}$.
4. $\mathbb{P}(W = 0) = q = \mathbb{P}(Z_n = 0$ *for some* $n)$.

Observe that the mean number of decedents in generation $h = \lceil \ln n^{1-\beta} / \ln 4 \rceil$ is equal to $(2e^{-8\alpha b})^h > n^{\frac{\eta}{2}(1-\beta)}$. By Theorem 2, since we are asking for the presence of significantly fewer decedents than the mean, for every $\varepsilon > 0$, there exists n_0 such that for $n > n_0$:

$$\mathbb{P}\left(\widehat{Z}_h > n^{\frac{\eta}{2}(1-\beta)}\right) \geq 1 - \varepsilon.$$

That implies that, the process $\widehat{Z}_h$, that bounds from below the number of intervals of size $\lfloor n^{\beta} \rfloor$ at height $h = \lceil \ln n^{1-\beta} / \ln 4 \rceil$ on the covering $X_t[n^{\beta}, \infty)$, has more than $n^{\frac{\eta}{2}(1-\beta)}$ children with high probability. Since they have a distance between each other greater than $\lfloor n^{\beta} \rfloor$, the proof is finished using the coupling. $\square$

To conclude the proof of the upper bound, let us use McDiarmid's concentration inequality, [17], to give bounds over the small world when α is small.

Proposition 6. *Let f be a distribution that satisfies conditions (2.20) and (2.19) for some $\beta_0 \in (0,1)$. Let $\eta \in (0, 1/2)$, $\beta < \eta$, $\varepsilon > 0$ and fix any set $\mathcal{I}$ formed by n^η disjoint intervals of size n^β that are spaced away from each other by at least n^β. Then there exists $\alpha' = \alpha'(\eta, \beta, \varepsilon)$ such that for all $\alpha < \alpha'$ with probability greater than $1 - \varepsilon$ the covering process $X_{\alpha/f(n)}[1, n^\beta)$ does not cover $\mathcal{I}$.*

Proof of Proposition 6. Start by fixing $\eta \in (0, 1/2)$ and $\beta < \eta$. Fix $\mathcal{I} = \mathcal{I}(\eta, \beta)$ to be an arbitrary set of n^η intervals of size n^β, which are separated by a distance of at least n^β. Let $\alpha > 0$, and consider $\{\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_N\}$ all objects placed in the covering process $X_{\alpha/f(n)}[1, n^\beta)$ that intercept any point in $\mathcal{I}$.

Define a function $F(\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_N)$ that corresponds to the total number of non covered point in the set $\mathcal{I}$ by the objects $\{\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_N\}$. Since the objects have size less than n^β we have that for every $i < N$, it is true that $\Delta_i F \leq n^\beta$, where:

$$\Delta_i F = \sup_{o_1,...,o_i,o'_i,o_{i+1},...,o_N} |F(o_1, ..., o_i, o_{i+1}, ..., o_N) - F(o_1, ..., o'_i, o_{i+1}, ..., o_N)|.$$

Where o_i are possible connected objects of size bounded by n^β.

Notice that the mean of the function F is the average number of missing point at time $\alpha f^{-1}(n)$, so:

$$\begin{aligned}\mathbb{E}\left(F(\mathcal{O}_1, ..., \mathcal{O}_N)\right) = \mu_F = n^{\eta+\beta}\mathbb{P}\left(0 \notin X_t[1, n^\beta)\right) \\ = n^{\eta+\beta} \exp\left\{-\frac{\alpha}{nf(n)}\left(-n^\beta f(n^\beta) + \sum_{i=1}^{n^\beta} f(i)\right)\right\} \\ = n^{\eta+\beta} \exp\left\{\frac{\alpha n^\beta f(n^\beta)}{nf(n)} - \alpha \ln(n) \frac{\sum_{i=1}^{n^\beta} f(i)}{nf(n)\ln n}\right\}.\end{aligned} \tag{2.26}$$

Observe that the total number of objects N that intercepts the set $\mathcal{I}$ is a Poisson random variable with mean equal to λ. Using the fact that the intervals in the set $\mathcal{I}$ are disjoint by a distance of at least n^β, we have that.

$$\begin{aligned}\lambda = n^\eta \frac{\alpha}{f(n)}\left(\sum_{i=1}^{n^\beta} \frac{f(i) - f(n^\beta)}{n} + \sum_{i=1}^{n^\beta} \frac{(1 - f(n^\beta))}{n}\right) \\ = \frac{\alpha n^{\eta+\beta}}{nf(n)}\left((1 - 2f(n^\beta)) + \frac{\sum_{i=1}^{n^\beta} f(i)}{n^\beta}\right)\end{aligned} \tag{2.27}$$

To continue the proof, let $d = d(\beta) \in \mathbb{R}$ from the hypotheses (2.20), δ small, and take constants $C_1, C_2 > 0$ such that $\lambda \leq C_1 n^{\eta+\beta}$ and $\mu_F \geq C_2 n^{\eta+\beta-\alpha(d+\delta)}$. Then, that for every $c > 0$, we have:

$$\mathbb{P}\left(|F - \mu_F| > 2c\mu_F\right) \leq \mathbb{P}\left(|F - \mu_F| > 2c\mu_F, N < \lambda(1 + \lambda^{-1/3})\right) + \mathbb{P}\left(|N - \lambda| \geq \lambda^{2/3}\right).$$

Since, N Poisson distributed with rate λ, and λ diverges as n grows, by the Chebyshev's inequality we have that $\mathbb{P}\left(|N - \lambda| \geq \lambda^{2/3}\right)$ converges to zero. To deal with the other term, notice that since $N < \lambda(1 + \lambda^{-1/3}) < 2C_1 n^{\eta+\beta}$ for big values of n, we get using McDiarmid's inequality that:

$$\mathbb{P}\left(|F - \mu_F| > 2c\mu_F, N < \lambda(1 + \lambda^{-1/3})\right) \leq 2\exp\{-2c^2{\mu_F}^2/\lambda(1 + \lambda^{-1/3})n^{2\beta}\} \leq 2\exp\left\{-\frac{c^2C_2^2}{2C_1^2}n^{\eta-\beta-2\alpha(d+\delta)}\right\}.$$

In particular, choosing δ small enough, for any choice of η and β, for every $\varepsilon > 0$, one can find $\alpha' = \alpha'(\eta, \beta, \delta, \varepsilon)$ such that the probability stays below ε for every $\alpha < \alpha'$, and therefore:

$$\mathbb{P}\left(\{X_t[1, n^\beta)]\}^c \cap \mathcal{I} \neq \emptyset\right) > 1 - \varepsilon,$$

for every choice of set $\mathcal{I}(\beta, \eta)$.

To finish the proof, we just need to show that indeed exists constants $C_1, C_2 > 0$ such that $\lambda \leq C_1 n^{\eta+\beta}$, and $\mu_F \geq C_2 n^{\eta+\beta-\alpha(d+\delta)}$.

To show that there exists $C_1 > 0$ such that $\lambda \leq C_1 n^{\eta+\beta}$, notice that by the hypotheses (2.20), we get for every $n > n_0$:

$$\frac{\sum_{i=1}^{n^\beta} f(i)}{n^\beta n f(n)} < \frac{(d+\delta)\ln n}{n^\beta}.$$

In particular, we get in equation (2.27) that $\lambda \leq C_1 n^{\eta+\beta}$.

To find the constant C_2 and finish the proof in the equation (2.26) we need do two considerations. First find $n_0(\delta)$ such that for every $n > n_0$, by hypotheses (2.20), we get that:

$$\left| d - \frac{\sum_{i=1}^{n^\beta} f(i)}{n f(n) \ln n} \right| < \delta.$$

Then, the we can use condition (2.19) to get that $\frac{n^\beta f(n^\beta)}{n f(n)}$ is bounded in the limit. □

To prove the upper bound on the equation (2.25), we need to combine the Proposition 5 with the Proposition 6. Take any $\eta \in (0, 1)$, then find β small such that $\frac{\eta}{2}(1 - \beta) > \beta$, and that satisfies conditions (2.19) and (2.20). Using Proposition 5, find $\alpha_0(\varepsilon, \beta, \eta)$, such that the probability of surviving $n^{\frac{\eta}{2}(1-\beta)}$ disjoint intervals of size n^{β} is at least $1 - \varepsilon/2$. Next, since $\frac{\eta}{2}(1-\beta) > \beta$, using Proposition 6, one can find a new $\alpha_1(\varepsilon, \beta, \frac{\eta}{2}(1-\beta))$, such that with probability greater than $1-\varepsilon/2$ in the set of surviving intervals of Proposition 5, there exists an empty point. Therefore, the probability of not covering the space when $\alpha < \min\{\alpha_1, \alpha_0\}$ is greater than $1 - \varepsilon$, and that concludes the upper bound.

Also, the same proof allow us to inform that the limit distribution Y has no atom in zero, that is:

Corollary 1. *For every $f \in \mathrm{RV}_{-1}$, which satisfies conditions (2.20) and (2.19). We have:*

$$\lim_{\alpha \to 0} \limsup_{n \to \infty} \mathbb{P}\left(f(n)T_n < \alpha\right) = 0.$$

2.3 Pre-exponential Phase

Although it is possible to apply the same technique used in Section 2.2 to prove Theorem C, we have decided to present an argument that introduces ideas for the next section. The ideas and techniques become more analytical and will need results about slowly varying functions presented in the appendix.

Given any $p \in (-1, 0)$ and $f \in \mathrm{RV}_p$, since p different than -1, then by Karamata's theorem 4 presented in the Appendix, one has that

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} f(i)}{nf(n)} = C_f > 0. \tag{2.28}$$

The rest of the proof will show that for every $\alpha > 0$, we have that:

$$\limsup_{n \to \infty} \mathbb{P}\left(f(n)T_n \leq \alpha\right) \leq 1 - e^{-\alpha C_f}, \text{ and} \tag{2.29}$$

$$\liminf_{n \to \infty} \mathbb{P}\left(f(n)T_n \leq \alpha\right) \geq 1 - e^{-\alpha} + e^{-\alpha}\left(1 - \exp\left\{-\frac{\alpha}{2^{1+p}}\right\}\right)^2. \tag{2.30}$$

The proof of Theorem C follow by just applying Prokhorov's theorem, using the limits (2.29) and (2.30).

The proof of (2.29) starts by observing that if $X_{\alpha/f(n)}$ does not cover the point 0 at time $\alpha/f(n)$, then the space is not covered completely. So:

$$\begin{aligned}\mathbb{P}\left(f(n)T_n > \alpha\right) &\geq \mathbb{P}\left(0 \notin X_{\alpha/f(n)}\right)\\ &= \exp\left\{-\alpha\frac{\sum_{t=1}^{n} f(t)}{nf(n)}\right\}.\end{aligned}$$

By the hypotheses (2.28), we have that

$$\liminf_{n\to\infty} \mathbb{P}\left(f(n)T_n > \alpha\right) \geq \exp\left\{-\alpha C_f\right\}.$$

And, that is the upper bound in implication (2.29).

Now for the lower bound, define the following three disjoint regions in $\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}_+$. Similar to the branching process created in Section 2.2, those three regions correspond to the first children of the argument.

$$\begin{aligned} R_0 &= \{(U,R):\ R \geq n\}, \\ R_1 &= \left\{(U,R) : U \in \left[0, \frac{n}{2}\right), \ R \in \left[\frac{n}{2}, n\right)\right\}, \\ R_2 &= \left\{(U,R) : U \in \left[\frac{n}{2}, n\right), \ R \in \left[\frac{n}{2}, n\right)\right\}. \end{aligned}$$

Notice that, we can cover the space if we hit an object in R_0, but also if R_1 and R_2 are occupied at the same time. Therefore:

$$\mathbb{P}\left(f(n)T_n \leq \alpha\right) \geq \mathbb{P}\left(\mathrm{w}(R_0) > 0\right) + \mathbb{P}\left(\mathrm{w}(R_0) = 0, \mathrm{w}(R_1) > 0, \mathrm{w}(R_2) > 0\right),$$

Now, using that $f \in \mathrm{RV}_p$, for $p \in (0,1)$, we can take the limit of this probabilities, as done in Proposition 4. To get that:

$$\liminf_{n\to\infty} \mathbb{P}\left(f(n)T_n \leq \alpha\right) \geq 1 - e^{-\alpha} + e^{-\alpha}(1 - e^{-\alpha/2^{1+p}})^2.$$

That finishing the proof of (2.30).

2.4 Exponential Phase

The proof of D is direct. We will show that with high probability the covering will occurs exactly when a big object appears. To show this, let $f \in \mathrm{RV}_0$, then apply Karamata's theorem 4 from the appendix to get that

$$\lim_{r\to\infty} \frac{\sum_{i=1}^n f(i)}{nf(n)} = 1. \tag{2.31}$$

Define the region $R_0 = \{(U,R) : R \geq n\}$, and observe that in time $\alpha/f(n)$, the number of objects in R_0, $N(R_0)$, is a Poisson random variable with rate α. In this way, if we do not cover until time $\alpha/f(n)$ then $\{N(R_0) = 0\}$ satisfies:

$$\mathbb{P}\left(T_n > \frac{\alpha}{f(n)}\right) \leq \mathbb{P}\left(N(R_0) = 0\right) = e^{-\alpha}.$$

Now, if the point 0 is not covered, naturally, we do not cover the entire space, so:

$$\begin{aligned}\mathbb{P}\left(f(n)T_n > \alpha\right) &\geq \mathbb{P}\left(0 \notin X_{\alpha/f(n)}\right)\\ &= \exp\left\{-\alpha\frac{\sum_{t=1}^{n} f(t)}{nf(n)}\right\},\end{aligned}$$

that by hypotheses (2.31), we have that:

$$\liminf_{n\to\infty} \mathbb{P}\left(f(n)T_n > \alpha\right) \geq \exp\left\{-\alpha\right\}.$$

And, that concludes the convergence in distribution to the exponential random variable.

CHAPTER 3

The continuous model

This chapter is dedicated to the proof Theorem B*. To remember the reader, the definitions of the continuous model are repeated above, but now with an extra explanation of the model.

3.1 Proof of Theorem B*

In Section 2.2, a representation of the covering process in a cylinder $\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}$ appears in Propositions 4 and 5, such representation makes evident several techniques used to understand the behavior of random coverage. After normalizing by $1/n$, the process in $\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}$ converges into a Poisson process across the continuous cylinder $\mathbb{S}^1\times[0,\infty)$ with a specific rate Λ_α; however, it is important to note that the bare existence of such a limit is not sufficient to guarantee the convergence of the covering phenomena; indeed, the limit may deform or hide small objects essential to the covering in the discrete case, as discussed in Remark 1 for the Gumbel's phase.

This Section introduces and reviews some properties of a continuous covering model. The term continuous is used because we transition from working with a covering of the discrete torus $\mathbb{Z}/n\mathbb{Z}$ to performing a covering of the continuous circle $\mathbb{S}^1$. It is important to clarify that the model and some of the results are not new in the literature; indeed, in 1972 a version of it was introduced by B.B.Mandelbrot in the seminal article [22], and in the same year the model was updated by L.A.Shepp in [29]. Due to these two contributions, we will refer to it as the Mandelbrot-Shepp model.

Both articles introduce the model on the real line, where the process has a regenerative property. In this paper, we focus on the circle; therefore, we need to adapt some of its proofs and definitions. For this reason, this section is divided into two parts. Subsection 3.1.1 aims to describe, define, and give some intuition behind the model on the circle. Subsection 3.1.2 is dedicated to solve the tightness issue of the limit, thus proving Theorem B*, which represents the novel contribution of this work showing that the discrete covering process converges in some sense to the Mandelbrot-Shepp model.

3.1.1 The Mandelbrot-Shepp model

This Subsection defines the Mandelbrot-Shepp model and reviews some standard results about it. Although simple, such properties will be used directly or indirectly to prove Theorem B* in Subsection 3.1.2.

The Mandelbrot-Shepp model is defined as a covering process of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, represented here by the segment $[0,1)$. To define it, consider the cylinder $S = \mathbb{S}^1 \times (0,\infty)$, fix $\alpha \geq 0$, and construct a Poisson point process in S with rate $\Lambda_\alpha = \alpha dx \otimes \frac{dr}{r^2}$. Rigorously, consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}_\alpha)$, where $\Omega = \{\omega = \sum_{i\in I}\delta_{(x_i,y_i)} : (x_i,y_i) \in S\, \forall i \in I, \text{ and } \omega(K) < \infty\, \forall K \subset S \text{ compact}\}$, and $\mathcal{F}$ is the smallest sigma algebra that makes the evaluation maps $\{\omega(K) : K \text{ is compact in } S\}$ measurable.

During the course of this article, configurations with different values of the parameter α will be compared. To simplify, define ω_α as a configuration sampled with the measure Λ_α, in particular, the parameter α is specified in the configuration notation.

To understand how different parameters interact, we should focus our attention on Lemma 7. The proof of the lemma is simple and was omitted from the paper.

Lemma 7. *Fixed $\alpha, \beta > 0$, and consider two independent configurations, that is, $\omega_\alpha = \sum_{i\in I}\delta_{(x_i,y_i)}$ and $\omega_\beta = \sum_{j\in J}\delta_{(x_j,y_j)}$ with intensities Λ_α and Λ_β respectively. Define:*

$$\omega_\alpha \cup \omega_\beta := \sum_{k\in I\cup J}\delta_{(x_k,y_k)}.$$

Then $\omega_\alpha \cup \omega_\beta$ has the same distribution as $\omega_{\alpha+\beta}$ with intensity $\Lambda_{\alpha+\beta}$.

The Lemma 7 allows us to think of the parameter α of the model as a time, and, as time passes, we place more and more objects on the cylinder S. Whenever we work with this type of construction between times α and β, we define the measure $\mathbb{P} = \mathbb{P}_\alpha \otimes \mathbb{P}_\beta$, where each configuration is independent of each other.

With the configurations defined, it is now possible to introduce the covering perspective of the process. Given a point $\xi = (x, y) \in S$, define the **Projection function** of ξ as

$$\Pi(\xi) = \begin{cases} [0,1), & \text{if } y > 1. \\ (x, x+y), & \text{if } y \leq 1 \text{ and } x+y \leq 1. \\ (x,1) \cup [0, x+y-1), & \text{if } y \leq 1 \text{ and } x+y > 1. \end{cases}$$

Given any configuration $\omega = \sum_{i \in I} \delta_{(x_i, y_i)}$ define:

$$\mathcal{C}(\omega) = \bigcup_{i \in I} \Pi((x_i, y_i))$$
$$\mathcal{V}(\omega) = [0,1) \setminus \mathcal{C}(\omega),$$

to be respectively the covered set and the vacant set of the Mandelbrot-Shepp model. Whenever the configuration ω is fixed, or when no confusion arises, we denote by $\mathcal{V}$ and $\mathcal{C}$ those random sets.

Lemma 8. *Given any parameter $\alpha > 0$, and any point $z \in [0,1)$. Then, $\mathbb{P}_\alpha(z \in \mathcal{C}) = 1$.*

Proof of Lemma 8. Let $z \in [0,1)$ and define the region $R_z = \{x \in S : z \in \Pi(x)\}$. Computing the intensity of the Poisson process in R_z, we have:

$$\Lambda_\alpha(R_z) = \int_{R_z} \frac{\alpha}{y^2} dy dx \geq \int_0^z \int_{z-x}^{\infty} \frac{\alpha}{y^2} dy dx = \int_0^z \frac{\alpha}{z-x} dx = \infty.$$

Therefore, for every $\alpha > 0$ the event $\{\omega_\alpha(R_z) > 1\}$ happens almost surely. In particular, z is covered almost surely, concluding the proof. □

Corollary 2. *Given any parameter $\alpha > 0$, and $\mathbb{Q}$ any enumerable set of points in $[0,1)$, we have $\mathbb{P}_\alpha(\mathbb{Q} \subset \mathcal{C}) = 1$.*

As observed by Corollary 2, any enumerable set is almost surely covered by $\mathcal{C}(\omega)$. This observation might lead us to believe that the model is always fully covered. However, we need to be cautious in drawing such conclusions, since the circle $\mathbb{S}^1$ is not countable.

To show that the Mandelbrot-Shepp model presents a non-trivial covering, an argument similar to Proposition 5 can be used, see the following Lemma:

Lemma 9. *Given $\alpha < \frac{\ln(2)}{6}$, then $\mathbb{P}_\alpha(\mathcal{V} \neq \emptyset) > 0$.*

The proof that the model is indeed a non-trivial covering is not new, and can be seen in [22, 29]. In order to make the argument complete, one can find in Appendix 5.2.1 the proof of the Lemma 9 based on the branching process technique used in Proposition 5. Moreover, in Proposition 7 a better bound on the covering probability will be proved based on the Shepp seminal paper [28].

The Mandelbrot-Shepp model is a non-trivial continuous covering process that uses infinitely many objects, whereas the discrete covering process uses only a finite number. The link between these models will be established through a finite truncated version of the Mandelbrot-Shepp model. Consequently, to prove Theorem B*, we must establish two crucial connections: first, the relationship between the Mandelbrot-Shepp model and its truncated version, and second, the connection between the latter and the discrete model.

Given a configuration $\omega = \sum_{i\in I} \delta_{(x_i,y_i)}$ and any real number $z > 0$, define the **truncated configuration at height z** as

$$\omega[z] = \sum_{i\in I} \delta_{(x_i,y_i)} \mathbf{1}\{y_i > z\}. \tag{3.1}$$

In essence, the configuration $\omega[z]$ is given by the points with height greater than z. The next lemma connects such configurations to the un-truncated model.

Lemma 10. *For any parameter $\alpha > 0$, we have*

$$\mathbb{P}_\alpha\left(\mathcal{V} \neq \emptyset\right) = \mathbb{P}_\alpha\left(\bigcap_{n=1}^{\infty}\left\{\mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset\right\}\right).$$

Proof of Lemma 10. Fixed any $n > 0$, note that:

$$\{\mathcal{V} \neq \emptyset\} \subseteq \left\{\mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset\right\}.$$

Therefore:

$$\mathbb{P}_\alpha\left(\mathcal{V} \neq \emptyset\right) \leq \mathbb{P}_\alpha\left(\bigcap_{n=1}^{\infty}\left\{\mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset\right\}\right). \tag{3.2}$$

To prove the opposite inequality, we use a topological argument. Observe that the set of points in $\omega_\alpha\left[\frac{1}{n}\right]$ is almost surely finite. Moreover, since the projection function of any point is an open set, we have that $\mathcal{V}_\alpha\left(\omega\left[\frac{1}{n}\right]\right)$ is the complementary of a finite union of open set, thus it is almost surely closed in $\mathbb{S}^1$, i.e. compact. Finally, consider $m > n$, and notice that $\mathcal{V}\left(\omega\left[\frac{1}{m}\right]\right) \subset \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right)$, in particular, $\bigcap_n \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right)$ is the intersection of nested, compact sets of $\mathbb{S}^1$. Therefore, if all are non empty, there must be a point in the limit, and the space will not be covered at time α. Proving then the equality in (3.2). □

Since the Mandelbrot-Shepp model has a non-trivial covering phenomenon, it is interesting to define the **Cover function** of the the space as

$$\pi(\alpha) = \mathbb{P}_\alpha\left(\mathcal{C} = [0,1)\right).$$

Also, set **the Cover function at height** z as

$$\pi_z(\alpha) = \mathbb{P}_\alpha\left(\mathcal{C}\left(\omega[z]\right) = [0,1)\right).$$

As a consequence of the Lemma 10, the link between the truncated version and the Mandelbrot-Shepp model can be created. Note the following.

Lemma 11. *For every $z > 0$, $\pi_z(\alpha)$ is a continuous function in α and $\lim\limits_{n\to\infty} \pi_{\frac{1}{n}}(\alpha) = \pi(\alpha)$.*

Proof of Lemma 11. To prove that $\pi_z(\alpha)$ is a continuous function for every fixed $z > 0$, define the region $R_z = [0,1) \times [z, \infty)$. Then:

$$\Lambda_\alpha(R_z) = \alpha \int_0^1 \int_z^\infty \frac{dydx}{y^2} = \frac{\alpha}{z}.$$

In particular, for every $\varepsilon > 0$, by Lemma 7, one gets:

$$\begin{aligned} |\pi_z(\alpha+\varepsilon) - \pi_z(\alpha)| &= \mathbb{P}\left(\mathcal{V}\left(\omega_\alpha[z]\right) \neq \emptyset, \mathcal{C}\left(\omega_{\alpha+\varepsilon}[z]\right) = [0,1)\right) \\ &\leq \mathbb{P}_\varepsilon\left(\omega(R_m) > 0\right) = 1 - e^{-\varepsilon/m}. \end{aligned}$$

Implying that for every fixed $z > 0$ the function $\pi_z(\alpha)$ is a right continuous function in α. The proof of left continuity is analogous.

To prove that $\lim_{n\to\infty} \pi_{\frac{1}{n}}(\alpha) = \pi(\alpha)$, it is sufficient to show that for any $\alpha > 0$:

$$\lim_{n\to\infty} \pi(\alpha) - \pi_{\frac{1}{n}}(\alpha) = 0,$$

which is equivalent to

$$\lim_{n\to\infty} \mathbb{P}_\alpha\left(\mathcal{C}(\omega) = [0,1), \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset\right) = 0.$$

To finish the proof, note that by inclusion of the events, the limit is the intersection of them. In particular, it was already shown by Lemma 10, that the limit has zero probability, as we desired. □

Thus, the Mandelbrot-Shepp process has a non-trivial covering function π that can be studied by the approximation through the truncated versions π_z. Furthermore, by approximation, the vacant set $\mathcal{V}$ is also a limit of the vacant sets in $\omega[z]$. Then, to finish the proof, one will need to construct a link between the truncated spaces and the discrete model.

3.1.2 Proof of Theorem B*

The goal of this subsection is to prove Theorem B*. For this, we need to prove two points: The support of the distribution of covering in the Mandelbrot-Shepp model is $[0,1]$, and the limit of the covering probability in the discrete model converges to the function π. Since the proof of these points presents different arguments, we divide this subsection into two. In the first part, Subsection 3.1.2.1, the goal is to show that the function π is not trivial in $[0,1]$, and equal to one above one. In the second part, Subsection 3.1.2.2, the goal will be to find the limit of the vacant sets and the discrete covering probabilities. As a direct consequence of all the propositions presented here, Theorem B* will be derived.

3.1.2.1 Support of the function π

In 1956, Dvoretzky in [12] proposed another problem in the context of covering, which we now introduce. First, fix the space as $\mathbb{S}^1$, the circle with unit length, and fix a decreasing sequence $(\ell_n)_n$. At each time k, one samples a uniform point in the circle and places an arc starting at this point with length ℓ_k. It was shown that if $\sum_n \ell_n = \infty$ each point in $\mathbb{S}^1$ is covered with probability one, but not necessarily $\mathbb{P}(\mathbb{S}^1 \text{ is fully covered}) = 1$. Later, Shepp showed the necessary and sufficient condition in 1972, in [28], described by:

Theorem 3 (Shepp)**.** *Let $0 < \ell_{n+1} \leq \ell_n \leq ... \leq \ell_2 \leq \ell_1 < 1$, $n = 1, 2, ...$, be arcs that are placed independently and uniformly on a circumference $\mathbb{S}^1$ of unit length. The union of these arcs covers $\mathbb{S}^1$ with probability one if and only if*

$$\sum_{n=1}^{\infty} n^{-2} \exp\{\ell_1 + ... + \ell_n\} = \infty.$$

The articles [22, 29] exposed that $\{\alpha = 1\}$ is a threshold for the Mandelbrot-Shepp model in the real line, where conditioning on the origin not being covered, a non-trivial set of vacant objects appears with positive probability. Here in the circle, this value holds the same significance, which makes the result of Proposition 7 not surprising. There are many ways to proceed with the proof of Proposition 7, we choose to use a concentration bound on the Poisson random variables and Theorem 3. This approach establishes an explicit connection between the Dvoretzky problem and our model.

Proposition 7. *For the Mandelbrot-Shepp model $\pi(\alpha) = 1$ for all $\alpha > 1$ and $\pi(\alpha) < 1$ for all $\alpha < 1$.*

Proof of Proposition 7. Theorem 3 does not allow $(\ell_n)_n$ to assume random values, but this problem can be resolved by conditioning. To start, let $\omega = \sum_{i \in I} \delta_{(x_i, y_i)}$ be a configuration, and define the random sequence $(\ell_n)_n$, where:

$$\ell_n = \sup\{r > 0 : \omega\{y \geq r\} = n\}.$$

That is, the size of the $n-th$ biggest object.

Together with the sequence $(\ell_n)_n$, define the regions where objects are expected to belong:

$$R_n^\delta = [0, 1) \times \left[\frac{\delta}{n}, \infty\right).$$

With $\varepsilon > 0$ small and $\delta = 1 - \varepsilon^2$, notice that $\omega_{1+\varepsilon}(R_n^1)$ is a Poisson random variable with rate $(1+\varepsilon)n$, and $\omega_{1-\varepsilon}(R_n^\delta)$ is a Poisson random variable with rate $n/(1+\varepsilon)$. As a application of the Chernoff bound for the Poisson random variable, it is possible to prove the following Lemma; the proof of which is postponed to the Appendix 5.2.1.

Lemma 12. *For any $\varepsilon > 0$, letting $\delta = 1 - \varepsilon^2$, we have that*

$$\sum_{n=1}^{\infty} \mathbb{P}_{1+\varepsilon}\left(\omega(R_n^1) < n\right) < \infty, \text{ and} \tag{3.3}$$

$$\sum_{n=1}^{\infty} \mathbb{P}_{1-\varepsilon}\left(\omega(R_n^\delta) > n\right) < \infty. \tag{3.4}$$

Recall that we need to show two things: First, when $\alpha > 1$ we have $\pi(\alpha) = 1$. Second, when $\alpha < 1$ then $\pi(\alpha) < 1$.

Start by fixing $\alpha > 1$, and observe that:

$$\{\omega(R_n^1) < n\} = \left\{\ell_n < \frac{1}{n}\right\}.$$

Using equation (3.3), together with Borel Cantelli we conclude that:

$$\mathbb{P}\left(\exists n_0, \text{ s.t. } \ell_n > \frac{1}{n} \forall n > n_0\right) = 1. \tag{3.5}$$

Fixed the sequence $(r_n)_n$ where $r_n = 1/n$ when $n > n_0$ and zero otherwise. There exists a constant $C = C(n_0)$ such that:

$$\sum_{n=1}^{\infty} n^{-2} e^{r_1 + \ldots + r_n} = \sum_{n=1}^{\infty} n^{-2} e^{r_1 + \ldots + r_{n_0}} \exp\left\{\sum_{k=n_0}^{n} r_k\right\} \geq \sum_{n=1}^{\infty} n^{-2} \exp\left\{\sum_{k=n_0}^{n} \frac{1}{k}\right\} \geq \sum_{n=1}^{\infty} \frac{C}{n} = \infty.$$

In other words, by Theorem 3, for all $n_0 > 0$, the sequence $(r_n)_n$ with $n > n_0$ covers the circle with probability one. So if $\alpha > 1$, by equation (3.5), one gets the following.

$$\pi(\alpha) = \sum_{k=1}^{\infty} \mathbb{P}\left(\mathcal{C}(\omega) = [0,1), \left\{\ell_n > \frac{1}{n} \forall n > k\right\}\right).$$

And, if instead of placing objects with size ℓ_n we place an smaller object with fixed size $r_n = 1/n$, by Theorem 3 the space is going to be fully covered. So, by coupling one object into another $\pi(\alpha) = 1$ for every $\alpha > 1$ as desired.

Analogously, for any $\varepsilon > 0$ small, and $\alpha = 1 - \varepsilon < 1$ take $\delta = 1 - \varepsilon^2$, and notice that:

$$\{\omega(R_n^\delta) > n\} = \left\{\ell_n > \frac{\delta}{n}\right\}.$$

Using the equation (3.4) together with Borel Cantelli we conclude that there exists just a finite number of regions R_n^δ with more than n objects, thus:

$$\mathbb{P}\left(\exists n_0, \text{ s.t. } \ell_n < \frac{\delta}{n} \forall n > n_0\right) = 1.$$

Fixed a sequence $(r_n)_n$ where $r_n < \frac{\delta}{n}$ for every $n > n_0$, then there exists a constant $c = c(n_0)$ such that:

$$\sum_n n^{-2} e^{r_1 + \ldots + r_n} \leq e^{n_0} + \sum_{n > n_0} n^{-2} \exp\left\{\delta \sum_{k=n_0}^{n} \frac{1}{k}\right\} < e^{n_0} + \sum_n c n^{-1-\varepsilon^2} < \infty.$$

In other words, by Theorem 3 the sequence $(r_n)_n$, where $r_n < \frac{\delta}{n}$ for every $n > n_0$, does not cover the space with probability one. In particular, if $\alpha < 1$, there exists $n_0 > 0$ such that:

$$1 - \pi(\alpha) > \mathbb{P}\left(\mathcal{V}(\omega) \neq \emptyset, \left\{\ell_n < \frac{\delta}{n}, \forall n > n_0\right\}\right) > 0.$$

Where, if instead of placing an object ℓ_n, we place an bigger object of size δ/n. Then by Theorem 3 the space have positive probability to not be covered. So, $\pi(\alpha) < 1$ for every $\alpha < 1$, as desired. □

3.1.2.2 Limits in distribution

In order to prove the limit of the discrete process towards the Mandelbrot-Shepp model, we will create a coupling between its truncated version and the discrete model. Our objective with the coupling is to demonstrate that in the limit, whenever the truncated version covers the space, the discrete model also covers it, and vice verse. This association between the truncated version and the discrete model, together with the limit in Lemma 11, will give us the desired distributional limit.

We utilize the graphical construction of the Mandelbrot-Shepp model to simultaneously construct the discrete covering process. This construction establishes a monotonic coupling between both systems, indicating that covering in the Mandelbrot-Shepp model implies covering in the discrete model. However, it should be noted that in this coupling the converse is not necessarily true: it is possible that while the Mandelbrot-Shepp model is not covered, the associated discrete model is. To address this issue, we need a quantitative connection between the two models.

Given a configuration $\omega = \sum_{i\in I} \delta_{(x_i,y_i)}$ in the cylinder S, we define the process $W^n = W^n(\omega)$ in the torus $\mathbb{Z}/n\mathbb{Z}$ (a covering process with radius distribution $\mathbb{P}(R > r) = 1/r$), as follows: for every $\ell \in \{0, ..., n-1\}$, and $k \in \mathbb{N}$, define the regions in $[0, 1) \times [1/n, \infty)$:

$$\widehat{R}_{\mathcal{O}(\ell,k)} = \left[\frac{\ell}{n}, \frac{\ell+1}{n}\right) \times \left[\frac{k}{n}, \frac{k+1}{n}\right).$$

Using such regions, define the process $W^n = W^n(\omega)$ as:

$$W^n = W^n(\omega) = \bigcup_{\ell=0}^{n-1} \bigcup_{k=1}^{\infty} \mathcal{O}(\ell, k)\mathbf{1}\left\{\omega\left(\widehat{R}_{\mathcal{O}(\ell,k)}\right) > 0\right\},$$

where $\mathcal{O}(\ell, k) = \{\ell, \ell+1, \cdots, \ell+k-1\}$ is an arc.

Notice that since

$$\mathbb{P}\left(\omega\left(R_{\mathcal{O}(\ell,k)}\right) = 0\right) = \exp\left\{-\frac{\alpha}{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right\},$$

the process $W^n(\omega)$ has the same distribution as the continuous covering process with radius distribution $\mathbb{P}(R > r) = 1/r$ at time αn. So, we couple both process in the natural way.

Observe the following problem thought. It is possible that the process W^n covers $\mathbb{Z}/n\mathbb{Z}$, while in the truncated representation $\omega[1/n]$, there exist points in the discrete set $\left\{\frac{\ell}{n}, \ell \in \{0, 1, \cdots, n\}\right\} \subset [0, 1)$ not covered. As a result, providing information on the process W^n relying solely on the configuration $\omega[1/n]$ is a complex task.

Given this challenge, we introduce another discrete auxiliary covering process, which will help us to demonstrate that the covering probability converges to π. Moreover, with this auxiliary covering process, one will be able to relate it back to the process W^n proving then Theorem B*.

Analogously to W^n, define the process $X^n = X^n(\omega)$ in the torus $\mathbb{Z}/n\mathbb{Z}$ (a covering process with radius distribution $\mathbb{P}(R > r) = \log(1 + 1/(r-1))$) as follows: For every

$\ell \in \{0, ..., n-1\}$, and $k \in \mathbb{N}$, define the following regions in $[0,1) \times [1/n, \infty)$:

$$R_{\mathcal{O}(\ell,k)} = \left\{ x \in S : \left\{ \frac{\ell}{n}, \frac{\ell+1}{n}, ..., \frac{\ell+k-1}{n} \right\} \subset \Pi(x) \right\}.$$

As the name suggests, if $\{\omega(R_{\mathcal{O}(\ell,k)}) > 0\}$ the covering X^n will have an object $\mathcal{O}(\ell,k) = \{\ell, \ell+1, ..., \ell+k-1\}$. More precisely, define X^n as:

$$X^n = X^n(\omega) = \bigcup_{\ell=0}^{n-1} \bigcup_{k=1}^{\infty} \mathcal{O}(\ell,k) \mathbf{1} \left\{ \omega \left(R_{\mathcal{O}(\ell,k)} \right) > 0 \right\}. \tag{3.6}$$

Observe in Figure 3.1, the regions related to both processes W^n and X^n.

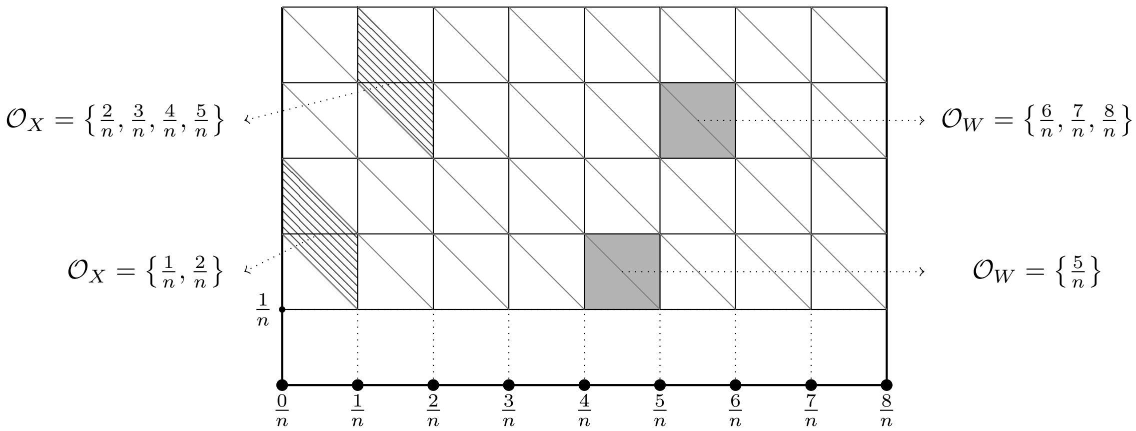

Figure 3.1: In the figure the regions corresponding respectively to the objects in the process W and X are drawn using gray and a pattern of lines respectively. Each square region for W corresponds to an object, and each rhombus region for X corresponds to another object. Notice that given a realization of $\omega[1/n]$, in the coupling looking to the set $P_n = \left\{ \frac{\ell}{n} : \ell \in \{0, 1, \cdots, n\} \right\}$, every point covered by W^n is also covered by X^n.

Concerning X^n, we have that:

$$\mathbb{P}\left(\omega_\alpha \left(R_{\mathcal{O}(\ell,k)}\right) = 0\right) = \exp\left\{ -\frac{\alpha}{n} \left(\log\left(1 + \frac{1}{k-1}\right) - \log\left(1 + \frac{1}{k}\right) \right) \right\}.$$

Thus $X^n(\omega_\alpha)$ has the same distribution as the continuous covering process with radius distribution $\mathbb{P}(R > r) = \log(1 + 1/(r-1))$ at time αn. So, one can couple both process again in the natural way.

The reason why we first work with the process X^n instead of the process W^n lies in a monotonous property presented in the construction. To make it clear, define the set $P_n = \{\ell/n \in [0,1) : \ell \in \{0, 1, ..., n-1\}\}$ and associate it with the torus $\mathbb{Z}/n\mathbb{Z} = \{0, 1, ..., n-1\} = nP_n$. Notice that in the process X^n, any element in the region $R_{\mathcal{O}(\ell,k)}$ covers the points $\{\ell, \cdots, \ell+k-1\}$ in $\mathbb{Z}/n\mathbb{Z}$, and also covers the points $\{\frac{\ell}{n}, \frac{\ell+1}{n}, ..., \frac{\ell+k-1}{n}\} \in P_n$ in the Mandelbrot-Shepp model. Therefore, whenever the torus is covered in the process X^n, the points P_n are also covered, and vice versa. This fact is not true for the process W^n, where it is possible to cover the set P_n in the Mandelbrot-Shepp model, but not cover the set $\mathbb{Z}/n\mathbb{Z}$ using the process W^n.

In the coupling of X^n, observe that it is possible for the truncated configuration $\omega[1/n]$ to cover the set P_n but not necessarily the whole interval $[0,1)$. To address this case, before proving the limit in distribution of the covering, we need to construct a quantitative argument by computing the number of points in the set P_n that are missing in the process X^n, under the condition that the Mandelbrot-Shepp model is not covered; see Proposition 8, where we show that whenever a point is missing in the set $\omega[1/n]$, with high probability there must also be a missing point in the set P_n.

Given any configuration $\omega = \sum_{i\in I} \delta_{(u_i,r_i)}$, define for any $M > 0$ and $z > 0$, the truncated configuration above M and bellow at height z as:

$$\omega[z, M] = \sum_{i\in I} \delta_{(u_i,r_i)}\delta_{\{Z>r_i>z\}}.$$

With this, conditioning that the origin is not covered, we have a small region near the origin with many missing points.

Lemma 13. *Fix $\zeta \in (0,1)$, and $\alpha \in (0,1)$. For any integer $n > 0$, any value of $M > 0$ such that $M+\zeta < 1$, and an arbitrarily $r = r(n) \in \left[0, \frac{1}{n}\right)$, define:*

$$Y(n,\zeta,r,M) = \sum_{k=1}^{\lfloor \zeta n\rfloor} \mathbf{1}\left\{\frac{k}{n} + r \in \mathcal{V}\left(\omega\left[\frac{1}{n}, M\right]\right)\right\},$$

the number of vacant points of the form $\frac{k}{n}+r$ in the open set $(0,\zeta)$, with respect to the truncated measure $\omega\left[\frac{1}{n}, M\right]$. Then, for every positive $\varepsilon > 0$, we get that:

$$\lim_{n\to\infty} \mathbb{P}_\alpha\left(\left|\frac{\ln Y(n,\zeta,r,M)}{\ln n} - (1-\alpha)\right| > \varepsilon \middle| 0 \in \mathcal{V}(\omega)\right) = 0. \tag{3.7}$$

Proof of Lemma 13. The proof is based on the concentration results of the branching process; see Theorem 2. Here, the main strategy is to divide the covering into height scales, and in each scale define two independent processes, a supercritical branching process and an ignition process. The idea consists of proving that many ignitions will occur with high probability, and for each such ignition we can obtain an independent branching processes with positive probability to survive. Whenever the branching process survives, we will be able to concentrate the random variable $Y(n,\zeta,r,M)$ around $n^{1-\alpha}$, the expected number of children of the branching process.

To define the height scales of the problem, with fixed $\alpha>0$ take any $N=N(\alpha,\zeta)$ that satisfies $N>\max\{6,2\zeta^{-1}\}$ and $\frac{1}{2}N^{1-\alpha}\exp\{-\alpha\}>1$. Here, the first condition guarantees a minimal value to start the branching process described in equation (3.10) and proceed with it between scales, and the second condition is used to guarantee a high expected number of children in the construction of the branching processes, see equation (3.9).

With the value of N fixed, divide the cylinders into height scales $H(\ell)=[0,1)\times(N^{-(\ell+1)},N^{-\ell}]$, where $\ell\geq 1$. Also, to simplify the notation, assume that $M=\zeta$ and $n=N^{-\ell_f}$ for some integer $\ell_f\geq 1$. Later in the proof, we can show that this assumption does not affect the results.

To proof the limit in equation (3.7), we are going to show that for any $\delta>0$, and for any $\varepsilon>0$, there exists a value of $n_0=n_0(\alpha,M,\varepsilon,\delta)$ such that for every $n>n_0$:

$$\mathbb{P}_\alpha\left(\left|\frac{\ln Y(n,\zeta,r,M)}{\ln n}-(1-\alpha)\right|>\varepsilon\middle|0\in\mathcal{V}(\omega)\right)<\delta. \tag{3.8}$$

For this, we will also divide the proof in two. The first part is to show that $Y(N^{\ell_f},\zeta,r,M)$ is greater than $N^{\ell_f(1-\alpha-\varepsilon)}$ with high probability. Then using first moment techniques, the second part consists in showing that $Y(N^{\ell_f},\zeta,r,M)$ is smaller than $N^{\ell_f(1-\alpha+\varepsilon)}$ with high probability.

In each scale $(H(\ell))_\ell$, define the following set of intervals used in the construction of the branching process:

$$\mathcal{I}(\ell)=\left\{\left[\frac{2k}{N^\ell}-r,\frac{2k+1}{N^\ell}-r\right):2\leq k\leq\frac{(\zeta+r)N^\ell-1}{2}\right\}.$$

Where $\mathcal{I}(\ell)$ is the set of two by two disjoint intervals that does not exceeds the value of ζ. Also, it does not contain the first two possible intervals (do not starts at zero). Furthermore, in the Mandelbrot-Shepp model, notice that each sequence of fitted vacant intervals $\{(I_\ell)_{\ell=1}^{\ell_f}:I_\ell\in\mathcal{I}(\ell)\}$ contributes to one element in $Y(n,\zeta,r,M)$; therefore, we can bound $Y(n,\zeta,r,M)$ bellow by the number of such sequences. See Figure 3.2 for a representation of the set $H(\ell)$ and the regions $\mathcal{I}(\ell)$.

To count the number of vacant intervals, fix a point $x\in[0,\zeta)$ and $\ell\geq 1$, and define the event:

$$B^\ell(x)=\left\{\left[x-\frac{1}{N^{\ell+1}},x\right)\in\mathcal{V}\left(\omega\left[N^{-(\ell+1)},N^{-\ell}\right]\right)\right\}$$

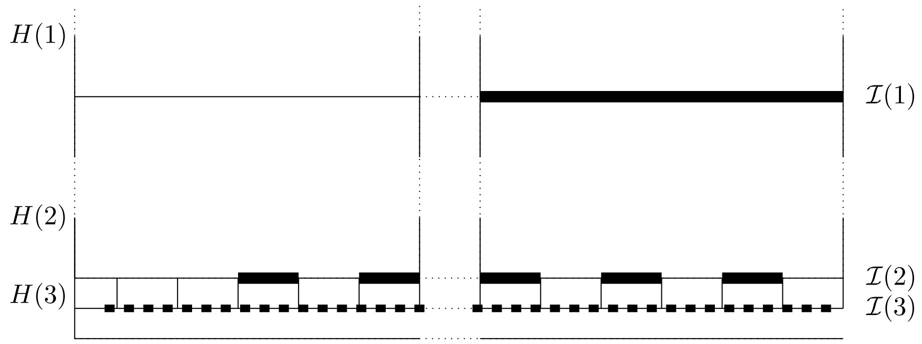

Figure 3.2: The space divided into regions $H(\ell)$, and, in each region $H(\ell)$, the set $\mathcal{I}(\ell)$ of disjoint segments is drawn, allowing for the branching process to survive. Notice that the first rectangle is always deformed by the point r, and the set $\mathcal{I}$ does not start in the beginning.

For any $\ell \geq 1$, fixed any $I \in \mathcal{I}(\ell)$, denote by $I^0 = \inf\{i : i \in I\}$ its first point, and define the random variable:

$$\xi_I^\ell = \sum_{j=1}^{\lfloor N/2 \rfloor} \mathbf{1}\left\{B^\ell\left(I^0 + \frac{2j}{N^{\ell+1}}\right)\right\}.$$

In essence, the number of two by two disjoint empty regions from $I(\ell+1)$ that lies in I, which are not covered using the objects that are in $H(\ell)$.

As the construction suggests, we call ξ_I^ℓ as the number of children in the interval I. Notice that for any distinct $I, J \in \mathcal{I}(\ell)$, the random variables ξ_J^ℓ and ξ_I^ℓ are independent, since the distance between the intervals is greater than the larger object revealed by each event. Moreover, with $\ell + 1 \leq \ell_f$ and for every point not near the origin and not near the end ζ (more precisely, $\frac{1}{N^\ell} \leq x - \frac{1}{N^{\ell+1}} < x < \zeta$), we have the following.

$$\begin{aligned}\mathbb{P}_\alpha\left(B^\ell(x) \middle| 0 \in \mathcal{V}\left(\omega\left[\frac{1}{n}, M\right]\right)\right) &= \exp\left\{\int_{N^{-(\ell+1)}}^{N^{-\ell}} \frac{\alpha}{y}\, dy\right\} \exp\left\{-\int_{N^{-(\ell+1)}}^{N^{-\ell}} \frac{1}{N^{\ell+1}} \frac{\alpha}{y^2}\, dy\right\} \\ &= N^{-\alpha} \exp\left\{-\alpha\left(1 - \frac{1}{N}\right)\right\}.\end{aligned}$$

Witch implies, that the expected number of children is greater than:

$$\mathbb{E}_\alpha\left(\xi_I^\ell\right) \geq \left\lfloor \frac{N}{2} - 1 \right\rfloor N^{-\alpha} \exp\left\{-\alpha\left(1 - \frac{1}{N}\right)\right\} > N^{1-\alpha} \frac{\exp\{-\alpha\}}{2} > 1. \tag{3.9}$$

Finally, given any $I \in \mathcal{I}(\ell)$, define the branching process $(Z_i^I)_i$ as follows: Let $Z_0^I = 1$ and $\Gamma_0 = I$. Then, inductively, in generation $(i-1) - th$ given any set of intervals

$\Gamma_i = \{J_k : J_k \in \mathcal{I}(\ell + i - 1)\}_k$, define

$$Z_i^I = \sum_{J \in \Gamma_i} \xi_J^{\ell+i}, \text{ and} \tag{3.10}$$
$$\Gamma_{i+1} = \left\{L \in I(\ell+i) : L \subset J \in \Gamma_i \text{ and } L \subset \mathcal{V}\left(\omega\left[N^{-(\ell+j)}, N^{-(\ell+j-1)}\right]\right)\right\}.$$

The mean number of children in each generation is greater than one by equation (3.9). So, we have a supercritical branching that can survive indefinitely with positive probability. Since we do not exactly know the distribution of the number of children in each generation, we need to perform an indirect calculation. By Theorem 2, for every positive $\varepsilon > 0$ and any fixed height $\ell_0 > 1$, there exists a probability $\theta = \theta(\alpha, N) > 0$ to survive at the limit when n goes to infinity. By symmetries of the problem, this limit probability does not depend on the height of the first interval $I \in \mathcal{I}(\ell_0)$, since the distribution of the number of children of the process does not change between heights. In particular, for any $\varepsilon > 0$, there exists $\ell_1 = \ell_1(\theta, \varepsilon, \alpha, N)$ such that whenever $\ell_f > \ell_1 + \ell_0$, we get for every $I \in \mathcal{I}(\ell_0)$ that:

$$\mathbb{P}\left(\frac{Z^I_{N^{\ell_f}}}{N^{(\ell_f-\ell_0)(1-\alpha)}} \geq N^{-(\ell_f-\ell_0)\varepsilon}\right) \geq \frac{\theta}{2}. \tag{3.11}$$

Later in the proof , we will use this equation and the value of ℓ_1 to give a positive bound on the probability to survive in the last scale with many points in $Y(n, \zeta, r, M)$ using independent trials that belongs to different initial heights.

With the branching well defined, it is time to construct the ignition process as a sequence of events $(E_\ell)_{\ell=1}^{\ell_2}$ for some $\ell_2 \in \{1, \cdots, \ell_f\}$. Such sequence of events is not independent and will be determined by a set of objects near the origin. The goal of the process is to guarantee the existence of seeds, where each seed gives birth to many new independent branching processes capable of surviving.

For each $\ell \geq 1$, define the region $R(\ell) = [0, N^{-(\ell-1)}] \times (N^{-\ell}, M)$. Then, set the event E_ℓ as:

$$E_\ell = \{\omega(R(\ell)) = \emptyset\}. \tag{3.12}$$

Despite being a sequence of dependent events, it satisfies the following property that guarantees many occurrences when we have a large number of scales to look.

Lemma 14. *For every $\delta > 0$, $\alpha > 0$, $M > 0$, and $N > 0$, define the ignition event as equation (3.12). For every $J > 0$, there exists a fixed integer $\ell_2 = \ell_2(\delta, J, \alpha, N, M) < \infty$*

such that whenever $\ell_f > \ell_2$, we get that:

$$\mathbb{P}_\alpha\left(\sum_{\ell=1}^{\ell_2}\mathbf{1}\{E_\ell\} > J \middle| 0 \in \mathcal{V}\left(\omega\left[\frac{1}{N^{\ell_f}}, M\right]\right)\right) > 1-\frac{\delta}{2}.$$

The proof of Lemma 14 is postponed until the end of this proof. Moreover, to give an idea of the proof, we use the fact that the distribution of the closest object to the origin in each scale has a similar law and does not have mass at the origin.

Observe that, since $N > 6$, whenever an ignition event E_ℓ occurs, we can say that the interval $\left[\frac{4}{N^{\ell+1}} - r, \frac{5}{N^{\ell+1}} - r\right) \in \mathcal{I}(\ell+1)$ is completely empty at height $\omega[N^{-(\ell+1)}]$. This allows a branching process to start there. Moreover, such an exploration of the branching process will be independent of the upcoming ignition events because of their mutual distance.

Therefore, we can guarantee a large vacant set in the limit with high probability. With fixed $\varepsilon > 0$, the idea consists of fixing some $J = J(\theta, \delta)$ large such that $\left(1-\frac{\theta}{2}\right)^J < \frac{\delta}{2(1-\delta)}$, then using the ignition Lemma 14, we can find a minimal height ℓ_2 such that with high probability there are J ignitions until height ℓ_2. In particular, if there are less than J ignitions, we will assume that the vacant set is small, but if there are more than J, each of them will give an independent chance to survive with a large set in the branching, thus using the minimal height ℓ_1 from equation (3.11), and taking $\ell_f > \ell_2 + \ell_1$, we get that:

$$\mathbb{P}\left(\frac{\ln Y(N^{\ell_f}, \zeta.r, M)}{\ln N^{\ell_f}} \le 1-\alpha-\varepsilon \middle| 0 \in \mathcal{V}\left(\omega\left[\frac{1}{N^{\ell_f}}, M\right]\right)\right) \le \frac{\delta}{2} + \left(1-\frac{\delta}{2}\right)\left(1-\frac{\theta}{2}\right)^J < \delta.$$

To prove that Y cannot be greater than $N^{1-\alpha+\varepsilon}$, we can use the Markov inequality. Computing its first moment, there exists a $c = c(n, r)$ such that:

$$\mathbb{E}_\alpha\left(Y(n, \zeta, r, M)\right) = \sum_{k=1}^{n}\mathbb{P}\left(\frac{k}{n} + r \in \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right)\right) \le cn^{1-\alpha}.$$

So, for every $\varepsilon > 0$ and $\delta > 0$, there exits $\ell_3 = \ell_3(\alpha, N, \varepsilon, \delta)$, such that for every $\ell_f > \ell_3$, we get that:

$$\mathbb{P}_\alpha\left(\left|\frac{\ln Y(N^{\ell_f}, \zeta.r, M)}{\ln N^{\ell_f}} - (1-\alpha)\right| \le \varepsilon \middle| 0 \in \mathcal{V}\left(\omega\left[\frac{1}{N^{\ell_f}}, M\right]\right)\right) < \delta. \tag{3.13}$$

Now the proof is almost over, but we need to do a few small considerations. The first one is about the conditional event, and the second one is about the choice of M and n made early in the proof.

We can exchange the conditional event in equation (3.13) for the event $\{0 \in \mathcal{V}(\omega)\}$, for this, notice that $Y(n,\zeta,r,M)$ is a random variable that depends only on the configuration $\omega\left[\frac{1}{n},M\right]$. In particular, by independence of the Poisson random variable, such event is independent from objects smaller than $1/n$ or bigger than M, so for every $y_1 \leq 1/n$, and $y_2 \geq M$, we get that:

$$\begin{aligned}&\mathbb{P}\left(\left|\frac{\ln Y(n,\zeta.r,M)}{\ln n}-(1-\alpha)\right|\leq\varepsilon\middle|0\in\mathcal{V}\left(\omega\left[\frac{1}{n},M\right]\right)\right)\\&=\mathbb{P}\left(\left|\frac{\ln Y(n,\zeta.r,M)}{\ln n}-(1-\alpha)\right|\leq\varepsilon\middle|0\in\mathcal{V}\left(\omega\left[y_1,y_2\right]\right)\right).\end{aligned}$$

Therefore, taking $y_1 \to 0$, and $y_2 \to \infty$, we can conclude that both sequence have the same limit.

The last step to finish the proof is to show that the result can be obtained regardless of the choice of $n = N^{\ell_f}$ and M. To show that the same limit holds for any value of n, observe that on scales $N^{-\ell}$, by construction, we can guarantee completely vacant regions with probability $e^{-\alpha N}$, thus for values of n between $N^{\ell+1}$ and N^{ℓ}, we have by law of large numbers that $Y(n,\zeta,r,M)$ is greater than $e^{-\alpha N}$ times $Y(N^{-\ell},\zeta,r,M)$, giving the desired concentration. For the choice of M, notice that the limit occurs due to the existence of some ignition events defined by Lemma 14; therefore, for any choice of M, we can always look for ignitions smaller than M to guarantee the same survival rate. In this proof, we just choose $M+\zeta<1$, to avoid interference with the fact that $\{0 \in \mathcal{V}\}$. With these considerations, we finish the proof. □

For simplicity, the following proof inherits all the previous definitions.

Proof of Lemma 14. The proof of this lemma consists of a dynamical construction that explores the Mandelbrot-Shepp set from top to bottom trying to find ignition events. Such construction induces a renewal process in the scales, and since such renewal is formed by random variables with well-behaved moments, the lemma is a direct consequence of the weak law of large numbers.

We start the construction with fixed values of $\alpha>0$, $\delta>0$, $\zeta>0$, $M>0$ and $N>0$. In the construction, we will define the first region to have all the irregularities of the problem, so that the next ones are simpler and recursive. Here, we track three main information in each step: a region, a height, and the closest point to the $y-$axis within this region. Define a scale $s_0 = \inf\{\ell \geq 1 : N^{-(\ell-1)} < \min\{\zeta, M\}\}$, and set:

$$h_0 = N^{-s_0}.$$
$$A_0 = [0,\zeta] \times [h_0, M].$$
$$d_0 = \sup\{d : \omega(A_0 \cap [0,d] \times (0,\infty)) = 0\}.$$

In words, we can describe our procedure as follows: Given a fixed region, look for the closest point to the $y-$axis and find its distance d. If d is large, it might be the case that you found an ignition. However, if d is small, the object can influence the ignition event on other scales, so we must use d to find the next scale not influenced by the objects discovered so far. Then, in the next undisturbed region, we repeat this procedure until we find many ignitions and prove the lemma.

With A_0 fixed, the random variable d_0 is an exponential random variable with some fixed positive rate, so it is not zero with probability one. In this proof, we are going to count the number of ignition events that occur just below N^{-s_0}, and show that by looking to deeper scales, we can find as many as we want. Since $\{d_0 > 0\}$ with probability one, we can define the triplet (A_1, h_1, d_1) inductively. To be specific, with a fixed triplet $(A_{k-1}, h_{k-1}, d_{k-1})$ where $\{d_{k-1} > 0\}$, we will define a height $s_k = \inf\{\ell \geq 1 : N^{-(\ell-1)} < d_{k-1}\}$, then set:

$$h_k = N^{-s_k}.$$
$$A_k = [0, N^{-(s_k-1)}] \times [h_k, h_{k-1}).$$
$$d_k = \sup\{d : \omega(A_k \cap [0,d] \times (0,\infty)) = 0\}.$$

Moreover, we have that d_k is an exponential random variable with rate $\alpha\left(h_k^{-1} - h_{k-1}^{-1}\right)$, so it is positive with probability one.

By the continuity of the exponential random variable, it is clear that the process constructed above can be repeated infinitely many times. However, this is not sufficient to prove the lemma. To finish, we need to control the distance between different scales in the sequence and compute the probability of having an ignition in any fixed step.

Define $Z_k = s_{k+1} - s_k$ the distance between the scales and then notice that for every positive integer $x > 0$, the event $\{Z_k \geq x\}$ occurs if and only if $\{d_k < N^{-(s_k+x-1)}\}$ also happens. Thus, we have:

$$\begin{aligned}\mathbb{P}(Z_k \geq x) &= \mathbb{P}\left(d_k \leq N^{-(s_k+x-1)}\right)\\ &= 1 - \exp\{-\alpha N^{-(s_k+x-1)}(h_k^{-1} - h_{k-1}^{-1})\}\\ &\leq 1 - \exp\{-\alpha N^{-x+1}\}.\end{aligned}$$

In particular, the scale distance between different elements in the dynamical construction is a random variable with a light tail that has all the moments.

As a direct consequence of this, for every $\delta > 0$ and $J_0 > 0$, one can find $K_0(J_0, N, \delta)$ such that for every $n > K_0$:

$$\mathbb{P}\left(\sum_{i=1}^{n} Z_k > J_0\right) > 1 - \frac{\delta}{2}.$$

Now, in each of the J_0 steps, there might be an ignition happening. To compute the probability of an ignition on the scale s_k, note that if $\{N^{-(s_k-1)} < d_{k-1}\}$, then no object greater than h_{k-1} can influence the presence of the ignition. Furthermore, in each step, we only need to check if $\{d_k > Nh_k\}$ occurs. In particular, we have that:

$$\mathbb{P}\left(d_k > Nh_k\right) = \exp\{-\alpha N h_k \left(h_k^{-1} - h_{k-1}^{-1}\right)\} \geq e^{-\alpha N}.$$

Where, for $N > 0$ fixed is a constant probability bound.

Therefore, after the scale $s_0 = s_0(N, M, \zeta)$, since J_0 can be arbitrarily large, for every $J > 0$ we can find a $K = K(\delta.J, \alpha, N, M, \zeta)$ such that for every $\ell > K$, we get:

$$\mathbb{P}\left(\sum_{\ell=1}^{K} \mathbf{1}\{E_\ell\} > J \middle| 0 \in \mathcal{V}\left(\omega\left[\frac{1}{N^\ell}, M\right]\right)\right) > 1 - \frac{\delta}{2}.$$

Finishing the proof as desired.

□

The process X^n We are going to adapt the result of the Lemma 13 to the circle. In particular, exclusively for the process X^n, remember that if the truncated Mandelbrot-Shepp process covers the space, then X^n also does. Now, we are going to prove that whenever the Mandelbrot-Shepp model is not covered, we can find many points in the form $\left\{\frac{k}{n} : k \in \{0, 1, \cdots, n-1\}\right\}$ that are missing with high probability, thus forcing the process X^n to be vacant as well.

Proposition 8. *In the Mandelbrot-Shepp model, let $n \in \mathbb{N}$, and set:*

$$Z_n = \sum_{k=1}^{n} \mathbf{1}\left\{\frac{k}{n} \in \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right)\right\}$$

the number of missing points of the form $\left\{\frac{k}{n} : k \in \{0, 1, \cdots, n-1\}\right\}$ in the circle $\mathbb{S}^1$, when we try to cover using just the truncated objects $\omega[1/n]$. Then, we have that for $\alpha \in (0, 1)$:

$$\frac{\log Z_n}{\log n} \mathbf{1}\{\mathcal{V}(\omega) \neq \emptyset\} \overset{D}{\Longrightarrow} (1-\alpha)\mathbf{1}\{\mathcal{V}(\omega) \neq \emptyset\}. \tag{3.14}$$

Proof of Proposition 8. This proof has two steps. The first is to show that Z_n cannot be significantly greater than $n^{1-\alpha}$. The second step focuses on finding a rectangular region $\{L < x < F\} \times \{0 < y < E\}$ with two properties; The first property is that the region itself has not yet been explored, and the second property is that the point L is not covered. Therefore, by Lemma 13, we will have about $n^{1-\alpha}$ empty points in this unexplored region, pushing the number of missing points to $n^{1-\alpha}$.

Observe that $\pi(\alpha) = 1$, then the equation (3.14) is trivially satisfied. Therefore, without loss of generality, fix $\alpha \in (0, 1]$ so that $\pi(\alpha) < 1$ for the rest of the proof.

In order to show that Z_n cannot have a greater quantity of missing points, using first moment, we get that:

$$\mathbb{E}\left(Z_n\right) = n\mathbb{P}\left(0 \in \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right)\right) = e^{-\alpha}n^{1-\alpha}.$$

Therefore, by Markov's inequality, for every $\varepsilon > 0$:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ln Z_n}{\ln n} > 1 - \alpha + \varepsilon, \mathcal{V}(\omega) \neq \emptyset\right) = 0.$$

Finishing one side of the proof.

For the other side, we are going to show that for fixed $\delta > 0$, and for any $\varepsilon > 0$, we have that:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ln Z_n}{\ln n} < 1 - \alpha - \varepsilon, \mathcal{V}(\omega) \neq \emptyset\right) \leq \delta. \tag{3.15}$$

In particular, since δ is arbitrary, we get that:

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{\ln Z_n}{\ln n} - (1 - \alpha)\right| > \varepsilon, \mathcal{V}(\omega) \neq \emptyset\right) = 0.$$

Concluding the proof.

To obtain the bound (3.15), we will approximate the event $\{\mathcal{V}(\omega) = \emptyset\}$. Start by defining the first non covered point of the space as:

$$L = \inf\left\{x \in [0, 1) : x \in \mathcal{V}\left(\omega\right)\right\}.$$

Also, set $L = 1$ if the space is completely covered.

The definitions of the next two random variables: E and F are more complex. In words, $E(L)$ will look at a set of fitted regions in which a large object may appear. The position of the x axis of this object with respect to the point L is defined as $F(L, E(L))$. To illustrate, see in Figure 3.3 a construction of these random variables. Moreover, the main importance of the random variable is that, given a triplet $\{(L, E, F) = (x, s, t)\}$, it will be possible to find a rectangular region that starts in x, ends in t and has height s, that has not yet been explored, and with two properties: the point x is not covered, and the objects that appear in this region cover points not yet discovered. More precisely, define:

$$E(x) = \sup\{s \in (0, 1-x) : \omega\left([x, 1-(x+s)) \times [s, \infty)\right) = 0\},$$
$$F(x, s) = \sup\{t \in (x, 1-(x+s)) : \omega\left([x, t) \times [s, \infty)\right) = 0\}.$$

Also set $E(x) = 0$ when the regions $[x, 1] \times (0, \infty)$ is empty.

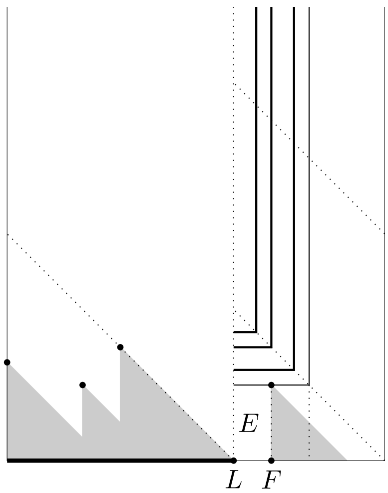

Figure 3.3: A representation of the random variables L,E and F. Given the point L, the space can be divided in three regions; one of them is empty, since L is not covered, the other regions have the possibility of covering the interval $[0, L)$, the third region is formed by objects not yet explored. The event E looks to a region formed by the fitted rectangles, until they find a success draw as a point (In the image the point is given by $(L + F, E)$). The distance of this point to L is given by F. The gray regions are the parts covered by object revealed in this construction

The main point of introducing the triplet $(L, E(L).F(L, E(L))$ is:

$$\mathbb{P}_\alpha\left(\mathcal{V}(\omega) \neq \emptyset\right) = \mathbb{P}_\alpha\left(L < 1, E(L) > 0, F(L, E(L)) > 0\right)$$
$$= \mathbb{P}_\alpha\left(\bigcup_{N=1}^{\infty}\left\{L < 1 - \frac{1}{N}, E(L) > \frac{1}{2N}, F(L, E(L)) > \frac{1}{2N}\right\}\right). \quad (3.16)$$

To prove equality (3.16), observe that the event $\{\mathcal{V}(\omega) \neq \emptyset\}$ is equal to the event $\{L < 1, E(L) \geq 0, F(L, E(L)) \geq 0\}$. Since the events $\{E(L) = 0\}$ and $\{F(L, E(L)) = 0\}$

have zero probability, we conclude the equality (3.16). Here. we just ask for $\{E(L) > 1/2N, F(L, E(L)) > 1/2N\}$ so when L is close to $1 - \frac{1}{N}$ the set of configurations that satisfy the event is not empty.

To simplify the notation, define:

$$G(N) = \left\{ L < 1 - \frac{1}{N}, E(L) > \frac{1}{2N}, F(L, E(L)) > \frac{1}{2N} \right\}.$$

Then, for every $\delta > 0$, since we have a set of increasing events, we get that exists a $N_0(\delta)$ such that for every $N > N_0$, and for every $n \geq 0$, that:

$$\left| \mathbb{P}\left(\frac{\ln Z_n}{\ln n} < 1 - \alpha - \varepsilon, \mathcal{V}(\omega) \neq \emptyset \right) - \mathbb{P}\left(\frac{\ln Z_n}{\ln n} < 1 - \alpha - \varepsilon, G(N) \right) \right| \leq \frac{\delta}{2}.$$

Now, with a fixed value of N, whenever $G(N)$ occurs, we will be able to give a lower bound to the quantity of $\frac{\ln Z_n}{\ln n}$. For this we are going to give a sequence of stochastic dominated random variables that will start in the random variable Z_n given that the event $G(N)$ happens and ends in the random variable Y from the Lemma 13 given that the origin is empty.

For this, given $\omega = \sum_{i \in I} \delta_{(x_i, r_i)}$ and $t > 0$, define the shift:

$$\phi_t(\omega) = \sum_{i \in I} \delta_{(t + x_i, r_i)},$$

where the sum is made in the circle. Also, for any $L \in [0, 1)$, and $n > 0$ define the quantity:

$$r_n(L) = \inf\{r : n(L - r) \in \mathbb{N}\}.$$

Now, assume that $G(N)$ occurs in some configuration ω, then we will have a triplet $(L, E(L), F(L, E(L)))$, and with the triplet we can say that:

1. $L \in \mathcal{V}(\omega)$.
2. $\omega((L, F) \times (E, 1)) = 0$.
3. The distribution of the objects in the region $\left(L, L + \frac{1}{2N}\right) \times \left(0, \frac{1}{2N}\right)$ is a Poisson random variable with rate Λ_α, i.e. does not change.

Here, the facts 1. and 2. are a trivial consequence of the definition of the triplet (L, E, F), and the claim 3. is true since the event $G(N)(\omega)$ given the random variable L, is completely determined by objects outside the region $\left(L, L + \frac{1}{2N}\right) \times \left(0, \frac{1}{2N}\right)$.

The first stochastic bound in Z_n appears now, condition on the measure $G(N)$, since $\left\{F > \frac{1}{2N}\right\}$ occurs, by counting just the missing points in a smaller region, we have:

$$Z_n \succeq \sum_{k=0}^{n-1} \mathbf{1}\left\{\frac{k}{n} \in \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right)\right\}\mathbf{1}\left\{L \leq \frac{k}{n} < L + \frac{1}{2N}\right\},$$

in the measure given the event $G(N)$. Here, $X \succeq Y$, if X stochastically dominates Y.

Using the shift of the configuration ω to send L to 0, we can get that:

$$Z_n(\omega) \succeq \sum_{k=0}^{n-1} \mathbf{1}\left\{\frac{k}{n} - r_n(L) \in \mathcal{V}\left(\phi_{-L}\left(\omega\left[\frac{1}{n}\right]\right)\right)\right\}\mathbf{1}\left\{0 \leq \frac{k}{n} - r(L) < \frac{1}{2N}\right\}.$$

Now, we are close to relating Z_n with the Lemma 13. The last thing to overcome is the restriction in the height $E > 0$. For this, considering the condition 2., if we allow new independent objects to appear in $\left(L, L + \frac{1}{2N}\right) \times (E, 1)$, we can cover more objects and thus diminish the value of Z_n even more. Then, we have:

$$Z_n \succeq Y\left(n, \frac{1}{2N}, r_n(L), 1\right).$$

where Y count the number of missing points of the form $\left\{\frac{k}{n} : k \in \{0, 1, \cdots n-1\}\right\}$ in the interval $\left(0, \frac{1}{2N}\right)$, and allow objects with length smaller than 1 to appear, objects that for the random variable $Z_n(\omega)$ given $G(N)$ is already known to not exists. Moreover, by Lemma 13, we concludes that exists a $n_0 > 0$ such that for every $n > n_0$ we get:

$$\mathbb{P}\left(\frac{\ln Z_n}{\ln n} < 1 - \alpha - \varepsilon, G(N)\right) \leq \frac{\delta}{2}.$$

In particular, for every $n > n_0$:

$$\mathbb{P}\left(\frac{\ln Z_n}{\ln n} < 1 - \alpha - \varepsilon, \mathcal{V}(\omega) \neq \emptyset\right) < \delta.$$

As desired. □

Corollary 3. *Let, $n \in \mathbb{N}$, and set:*

$$Z_n = \sum_{k=1}^{n} \mathbf{1}\left\{\frac{k}{n} \in \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right)\right\}.$$

the number of missing points of the form k/n in the cylinder S, when we try to cover using just the truncated objects $\omega[1/n]$. We have that, for $\alpha \in (0,1)$:

$$\frac{\log Z_n}{\log n}\mathbf{1}\{\mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset\} \overset{D}{\Longrightarrow} (1-\alpha)\mathbf{1}\{\mathcal{V}(\omega) \neq \emptyset\}. \tag{3.17}$$

Proof of Corollary 3. Using that the random variable $\frac{\log Z_n}{\log n}$ is bounded, and $\{\mathcal{V}(\omega) \neq \emptyset\} \subseteq \{\mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset\}$. Furthermore, by Lemma 10, we have $\mathbf{1}\{\mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset\} \overset{D}{\Longrightarrow} \mathbf{1}\{\mathcal{V}(\omega) \neq \emptyset\}$. Then, the proof follows directly that equation (3.17) is the same as equation (3.14), plus some term that goes to zero in probability. □

With the Corollary 3, we can proof Theorem B*, for the process X^n with radius distribution $\log(1+1/(r-1))$ almost directly.

Proposition 9. *Consider $X^n(\omega)$ defined in equation 3.6, then we have that:*

$$\lim_{n\to\infty} \mathbb{P}_\alpha\left(X^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) = \pi(\alpha).$$

Moreover, we have that:

$$\frac{\log|\mathbb{Z}/n\mathbb{Z} \setminus X^n(\omega_\alpha)|}{\log n}\mathbf{1}\{X^n(\omega_\alpha) \neq \mathbb{Z}/n\mathbb{Z}_n\} \overset{D}{\Longrightarrow} (1-\alpha)\mathbf{1}\{\mathcal{V} \neq \emptyset\}. \tag{3.18}$$

Proof of Proposition 9. By the inclusion created by the coupling:

$$\liminf_{n\to\infty} \mathbb{P}\left(X^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) \geq \liminf_{n\to\infty} \pi_{\frac{1}{n}}(\alpha) = \pi(\alpha),$$

Now, with the set $P_n = \left\{\frac{\ell}{n} \in [0,1) : p \in \{0,1,2,...,n-1\right\}$, observe that:

$$\begin{aligned}\mathbb{P}_\alpha\left(X^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) &= \mathbb{P}_\alpha\left(\mathcal{C}\left(\omega\left[\frac{1}{n}\right]\right) = [0,1)\right) + \mathbb{P}_\alpha\left(\mathbb{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset, P_n \subset \mathcal{C}\left(\omega\left[\frac{1}{n}\right]\right)\right)\\ &= \pi_{\frac{1}{n}}(\alpha) + \mathbb{P}\left(Z_n = 0, \mathcal{V}\left(\omega\left[\frac{1}{n}\right]\right) \neq \emptyset\right).\end{aligned}$$

In particular, by Lemma 11, we can take the limit, and get that:

$$\lim_{n\to\infty} \mathbb{P}\left(X^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) = \pi(\alpha).$$

as desired. To finish the proof apply directly the Corollary 3.17, since $|\mathbb{Z}/n\mathbb{Z} \setminus X^n(\omega_\alpha))|$ is equal to the random variable Z_n. □

The proof of Theorem B* is almost over, to complete the proof we need to connect the random process X^n with the process W^n.

The process W^n Unlike X^n, the process W^n does not have a direct connection to the Mandelbrot-Shepp model. By construction, it is entirely plausible that the process W^n is fully covered while the Mandelbrot-Shepp model is not, or, in another direction, it is possible that the Mandelbrot-Shepp model covers the set P_n while W^n is not completely covered. Despite the complications, coverage X^n can be related to coverage W^n in a simple way done in Proposition 9, completing the proof of Theorem B* .

Observe that one side of the relation between X^n with W^n is trivial. Using Figure 3.1, comparing the radius distribution in the coupling, one can conclude that the objects in W^n have a radius of the same size or just one unit smaller than the radius of the objects in X^n. In particular, one may get that:

$$\mathbb{P}_\alpha \left(W^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) \leq \mathbb{P}_\alpha \left(X^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right). \tag{3.19}$$

To finish the proof now, we need to prove that the covering using the law X^n, or analogously, the covering of the Mandelbrot-Shepp model, has the following high probability property: If the space is covered, then it is also covered by changing the sizes of all objects by one unit.

Proposition 10. *The sequence of probabilities:*

$$\left(\mathbb{P}_\alpha \left(W^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) - \mathbb{P}_\alpha \left(X^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right)\right)_n$$

is Cauchy converging to zero. In particular:

$$\lim_{n\to\infty} \mathbb{P}_\alpha \left(W^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) = \pi(\alpha).$$

Moreover, we have that:

$$\frac{\log |\mathbb{Z}/n\mathbb{Z} \setminus W^n(\omega_\alpha)|}{\log n} \mathbf{1}\{W^n(\omega_\alpha) \neq \mathbb{Z}/n\mathbb{Z}_n\} \overset{D}{\Longrightarrow} (1-\alpha)\mathbf{1}\{\mathcal{V} \neq \emptyset\}.$$

Proof. The crucial part of the proof lies on noticing that: In both constructions, as n grows, the set of regions R and $\widehat{R}$ for the objects becomes thinner. And, compared to the process $X^n(\omega)$, the objects in $W^n(\omega)$ are equal in size or have one unit smaller.

Consider an arbitrary configuration ω that covers the space. By Lemma 10, there exists some value of n for which $\omega[1/n]$ also covers the space; this implies that we can extract a covering with a finite number of objects. Now, considering a larger value of m (where $m > n$), we observe that $X^m(\omega)$ continues to cover the space. However, this time it accomplishes this using only the objects present in $\omega[1/n]$. In particular, as we focus solely on these larger objects, the objects in $W^m(\omega)$ progressively approach and align with the objects in $\omega[1/n]$ as m goes to infinity. As a consequence, we can reasonably expect that $W^m(\omega)$ covers the space as well for large values of m.

To get a reasonable limit on how large m should be, we need to understand the properties of the configurations that cover the space at height n. For each configuration

$\omega = \sum_{i\in I} \delta_{(x_i,y_i)}$, define the shift operation:

$$\omega * h = \sum_{i\in I} \delta_{(x_i,hy_i)}.$$

Define also the random variable G that measures how much one can shift down a configuration while still covering it, this is:

$$G = \sup\{\eta \in (0,1) : \mathcal{C}(\omega * (1-\eta)) = [0,1)\}$$

and set $G = 0$ if the space is not covered.

Informally, the random variable G is a measure of how stable coverage is in the Mandelbrot-Shepp model. Notice that for each $\alpha > 0$, we have that $\omega_\alpha * (1-\eta) \sim \omega_{\alpha(1-\eta)}$, thus shrink the space implies changing its space rate. Then, when the space is covered in some truncated level $\omega[1/n]$. Evaluating $\mathcal{C}(\omega[1/n])$, the space is covered almost surely by a finite union of open intervals, each interval having at least two intersections: one at the beginning of the interval and the other at the end. Analyzing the configuration $\mathcal{C}(\omega[1/n] * (1-\eta))$, each object shrinks and, in particular, the size of the intersections also diminish. The value of G will correspond to how much we can shrink the objects' size and still see covering; or analogously, is an evaluation of how much one can change the rate and still see covering.

Notice that, by Lemma 10, the event $\{G = 0\}$ happens, if and only if, we do not cover the space. Thus:

$$\mathbb{P}\left(G > 0 | \mathcal{V} = \emptyset\right) = 1.$$

Therefore, since the events are increasing, we get that $\mathbb{P}\left(G > \eta\right)$ converges to $\mathbb{P}\left(\mathcal{V} = \emptyset\right)$ when η goes to zero. Thus, for fixed value of $\alpha > 0$, and every $\varepsilon > 0$, exists $\eta = \eta(\alpha)$ such that covering and having a positive value of G is close in probability, this is:

$$|\pi(\alpha) - \mathbb{P}_\alpha\left(G > 2\eta\right)| < \varepsilon/2. \tag{3.20}$$

In equation (3.20), with $P_n = \{k/n : k \in \{0,1,\cdots,n-1\}\}$, the probability of covering P_n is $\mathbb{P}\left(X^n(\omega_\alpha = \mathbb{Z}/n\mathbb{Z}\right)$ and this probability converges to $\pi(\alpha)$ by Proposition 9. To approach $\mathbb{P}_\alpha\left(G > 2\eta\right)$, notice that if for every n, we have that when $\{P_n \subset \mathcal{C}\left(\omega_\alpha * (1-\eta)\left[\frac{1}{n}\right]\right)\}$ does not happen, then $\{\mathcal{V}\left(\omega_\alpha * (1-\eta)[1/n]\right) \neq \emptyset\}$, and by Lemma 10, we get $\{\mathcal{V}\left(\omega_\alpha * (1-\eta)\right) \neq \emptyset\}$ happens. Moreover, since $\{G \geq 2\eta\}$ happens if and only if $\omega_\alpha * (1-\eta')$ is covered for every $\eta' < \eta$. Then, we get that, asking for a diminish of η (instead of 2η, to avoid a topology problem in the boundary of the objects), we get:

$$\lim_{n\to\infty} \mathbb{P}\left(P_n \subseteq \mathcal{C}\left(\omega_\alpha * (1-\eta)\left[\frac{1}{n}\right]\right)\middle| G > 2\eta\right) = 1.$$

Applying the limits in equation (3.20), we have:

$$\limsup_n \left|\mathbb{P}\left(X^n(\omega_\alpha) = \mathbb{Z}/n\mathbb{Z}\right) - \mathbb{P}\left(P_n \subseteq \mathcal{C}\left(\omega_\alpha * (1-\eta)\left[\frac{1}{n}\right]\right), G > 2\eta\right)\right| < \varepsilon.$$

In particular, whenever $X^n(\omega_\alpha)$ covers the space and n is large, we can diminish the size of the objects, i.e. $X_{\alpha n}(\omega * (1-\eta)) = \mathcal{C}\left(\omega_\alpha * (1-\eta)\left[\frac{1}{n}\right]\right)$, and still cover with high probability. Since this property starts at some value of n, and will be also true for bigger values. Then, comparing with the covering W^n, and using equation (3.19) we get that, when n is large enough, both models cover the space with high probability. More precisely, we have that for n sufficient large:

$$\begin{aligned}\mathbb{P}_\alpha\left(X^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) \geq \mathbb{P}_\alpha\left(W^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) &\geq \mathbb{P}\left(X^n(\omega_\alpha * (1-\eta)) = \mathbb{Z}/n\mathbb{Z}\right) \\ &\geq \mathbb{P}\left(P_n \subseteq \mathcal{C}\left(\omega_\alpha * (1-\eta)\left[\frac{1}{n}\right]\right), G > 2\eta\right),\end{aligned}$$

In particular, we get for every $\varepsilon > 0$ that:

$$\limsup_n \left|\mathbb{P}_\alpha\left(W^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right) - \mathbb{P}_\alpha\left(X^n(\omega) = \mathbb{Z}/n\mathbb{Z}\right)\right| < \varepsilon.$$

Concluding that the sequence is Cauchy.

To finish the proposition, we need to show the limit of distribution of the number of missing points in the covering W^n, for this just observe that whenever W^n does not cover the torus in the limit when $n \to \infty$, neither does X^n. In particular, by construction, since the vacant set of W^n is bounded by a constant times the size of the vacant set of X^n (the object has the same size, or one unit less). We get that both satisfy the same limit in distribution, since the constant does not change the limit. □

CHAPTER 4

Covering in spaces with arbitrary dimensions

This chapter is dedicated to the study of the coverage of the Mandelbrot-Shepp model when the space has dimension greater than one. The main model can also present interesting results in other dimensions, but we chose to focus the study on the Mandelbrot-Shepp process, as it proved to be in dimension one a continuous model with a critical threshold in relation of the covering probability.

To simplify the calculations, fix $d \in \mathbb{N}$, and if we do not say otherwise, the dimension of the problem is fixed as d. Therefore, we can simplify the text by redefining some random variables presented in Chapter 1. As an abuse of notation, define the vacant set $\mathcal{V} = \mathcal{V}^d$ and the cover set $\mathcal{C} = \mathcal{C}^d$, also for the covering functions, set $\pi(\alpha) = \pi^d(\alpha)$ and $\pi_z(\alpha) = \pi_z^d(\alpha)$.

Whenever $\alpha > 0$ is fixed and there is no confusion, we denote ω_α as an independent Poisson configuration with rate Λ_α. In this way, when it is clear, we omit the parameter α from $\mathbb{P}_\alpha$, placing the parameter α in the event. To simplify even more, define for every $z > 0$ the vacant set:

$$\mathcal{V}_\alpha[z] = \mathcal{V}(\omega_\alpha[z]).$$

Also, when $z = 0$, set $\mathcal{V}_\alpha = \mathcal{V}(\omega_\alpha)$. Let $\lambda(.)$ be the Lebesgue measure on the torus, thus, set:

$$\lambda(\mathcal{V}_\alpha[z]) = \int_{\mathcal{V}_\alpha[z]} dx,$$

to be the volume of the vacant set at time $\alpha > 0$, in the truncated level $z > 0$.

For this chapter, it is also interesting to remember the definitions:

$$\begin{aligned}
\alpha_c &= \inf\left\{\alpha > 0 : \lim_{z\to 0} \pi_z(\alpha) = 1\right\},\\
\alpha_M &= \inf\left\{\alpha > 0 : \lim_{z\to 0} z^{-d}\mathbb{E}\left(\lambda(\mathcal{V}_\alpha[z])\right) = 0\right\},\\
\alpha_+ &= \inf\left\{\alpha > 0 : \forall z > 0, \exists \zeta_+ = \zeta_+(\alpha) > 0 \text{ such that } \mathbb{P}\left(\mathcal{V}_\alpha[z] \neq \emptyset\right) < z^{\zeta_+}\right\},\\
\alpha'_- &= \sup\left\{\alpha > 0 : \begin{array}{l} \exists \gamma_1 = \gamma_1(\alpha),\ \gamma_2 = \gamma_2(\alpha),\ \gamma_1 > \gamma_2 > 0 \text{ such that} \\ \lim_{z\to 0} \mathbb{P}\left(z^{\gamma_1} < \lambda(\mathcal{V}_\alpha[z]) < z^{\gamma_2} | \mathcal{V}(\omega_\alpha) = \emptyset\right) = 1 \end{array}\right\}.
\end{aligned}$$

And remember that for $\alpha > \alpha_c$ we are in the subcritical regime, and for $\alpha < \alpha_c$ we are in the supercritical regime.

4.1 The subcritical phase

When the parameter α is subcritical, the Mandelbrot-Shepp process is covered with probability one. However, in the existing literature, not much is known about the probability of coverage of the truncated levels. Here, using first and second moments, we prove that the thresholds α_M,α_c and α_+ are equal; giving a good description of the subcritical regime.

In essence, the proof will be divided into two main points. The first step is to show that for every $\alpha > \alpha_M$, the probability of not covering the space decays polynomially fast to zero, that is, $\alpha_c \leq \alpha_M$. The next point of the proof consists of showing that for every $\alpha < \alpha_M$, the probability of not covering the space is positive. In this way, we will have $\alpha_c = \alpha_M$ and the probability of not covering the space decays polynomially fast to zero for every subcritical parameter.

In our proof, we explicitly compute the value of α_M. Since this calculation will repeat later for a different kind of configuration, we are going to simplify the text by introducing it now and doing the calculation only once.

More precisely, for any $d \geq 1$, given a configuration $\omega_\alpha = \sum_{i\in I} \delta_{(x_i,r_i)} \subset S^d$, a height $z > 0$, and a value $h \in [0, 1)$. Define the truncated configuration at z shrunk by h as:

$$\omega_\alpha^{(h)}[z] = \sum_{i\in I} \delta_{(x_i,r_i-hz)}\mathbf{1}\{r_i > z\},$$

where each radius in $\omega[z]$ is shrunk by hz. Also define the truncated vacant set at height z shrunk by h at time α as $\mathcal{V}_\alpha^{(h)}[z] = \mathcal{V}(\omega_\alpha^{(h)}[z])$. Since the objects presented in the covering of $\omega_\alpha^{(h)}[z]$ is smaller than those of $\omega_\alpha[z]$, we have that $\mathcal{V}_\alpha[z] \subseteq \mathcal{V}_\alpha^{(h)}[z]$.

Remark 12. This construction decreases the size of the objects but numerically preserves the same rate of the covering probabilities. Later, in Section 4.2, another construction is presented, where the objects are diminished or enhanced and the rate of the probabilities changes. The advantage of this definition is based on the Lemma 15, where we can bound its probability preserving the order.

Recall the constant $C_d = \pi^{d/2}\Gamma^{-1}\left(\frac{d}{2}+1\right)$ (where in $\mathbb{R}^d$, we have $\lambda(B(x,r)) = C_d r^d$). And, compute directly:

Lemma 15. *For any $\alpha > 0$ and fixed $h \in [0,1)$, exists constants $C = C(\alpha, d, h)$, $c = c(\alpha, d, h)$ such that:*

$$Cz^{\alpha C_d} \geq \mathbb{P}\left(0 \in \mathcal{V}_\alpha^{(h)}[z]\right) \geq \mathbb{P}\left(0 \in \mathcal{V}_\alpha[z]\right) \geq cz^{\alpha C_d}, \text{ for all } z < \frac{1}{4(1+h)}.$$

Proof of Lemma 15. For each event on the form $\{0 \in \mathcal{V}_\alpha^{(h)}[z]\}$, there will be two regions of relevance, one that is completely empty when $\{0 \in \mathcal{V}_\alpha^{(h)}[z]\}$ happens, and another that, when empty, implies the occurrence of the event $\{0 \in \mathcal{V}_\alpha^{(h)}[z]\}$. The proof consists of calculating the rate of such regions and showing that this implies the limits of this Lemma.

Start by fixing $h = 0$ (no decrease at all). In this case, to find the upper bound, define the region:

$$A_1 = \left\{(x,r) \in S : \; z \leq r < \frac{1}{4} \text{ and } 0 \in B(x,r)\right\}.$$

The rate of A_1 is given by:

$$\Lambda_\alpha(A_1) = \alpha \int_z^{1/4} \frac{C_d r^d}{r^{d+1}} dz = -\alpha C_d \ln z - \alpha C_d \ln 4.$$

Since $\{0 \in \mathcal{V}_\alpha[z]\}$ implies that A_1 is empty, then:

$$\mathbb{P}\left(0 \in \mathcal{V}_\alpha[z]\right) \leq 4^{\alpha C_d} z^{\alpha C_d}.$$

For the lower bound with $h = 0$, define:

$$A_2 = A_1 \cup \left\{(x,r) \in S : r > \frac{1}{4}\right\}.$$

Where we are considering that every object with radius bigger than 1/4 hits the origin. In particular, we get:

$$\Lambda_\alpha(A_2) = \alpha \int_z^{1/4} \frac{C_d r^d}{r^{d+1}} dz + \alpha \int_{1/4}^{\infty} \frac{dr}{r^{d+1}}$$
$$= -\alpha C_d \ln z - \alpha C_d \ln 4 + \alpha \frac{4^d}{d}.$$

In this way, for every $z < 1$, we have that:

$$\mathbb{P}\left(0 \in \mathcal{V}_\alpha[z]\right) \geq 4^{\alpha C_d} z^{\alpha C_d} \exp\left\{-\alpha \frac{4^d}{d}\right\}. \tag{4.1}$$

Now, when $h \in (0,1)$ by inclusion arguments, for every $z > 0$, $\alpha > 0$ and $x \in \mathbb{T}^d$, we have $\mathbb{P}\left(x \in \mathcal{V}_\alpha[z]\right) \leq \mathbb{P}\left(x \in \mathcal{V}_\alpha^{(h)}[z]\right)$. Then, we can use the equation (4.1) to find the lower bound for any $h > 0$, for this take $c = 4^{\alpha C_d} \exp\left\{-\alpha \frac{4^d}{d}\right\}$. For the upper bound when $h \in (0,1)$, if $z < \frac{1}{4(1+h)}$ then $z < \frac{1}{4} - hz$, and we can define the region:

$$\widehat{A_1} = \left\{(x,r) \in S : \, z \leq r < \frac{1}{4} \text{ and } 0 \in B(x, r - hz)\right\}.$$

Notice that whenever $\{0 \in \mathcal{V}_\alpha^{(h)}[z]\}$, we find that $\widehat{A_1}$ is empty. Computing its rate, by Bernoulli's inequality, we get:

$$\Lambda_\alpha(\widehat{A_1}) = \alpha \int_z^{\frac{1}{4}} \frac{C_d (r - hz)^d}{r^{d+1}} dr \geq \alpha C_d \int_z^{\frac{1}{4}} \frac{1}{r} - \frac{hzd}{r^2} dr$$
$$= \alpha C_d \left(-\ln 4 - \ln z + hd\left(4z - 1\right)\right).$$

Then, one can bound the probability for any $h \in (0,1)$ taking $C = 4^{\alpha C_d} \exp\{hd\}$, and this concludes the proof as desired. □

As mentioned above, the moment threshold α_M is fundamental for understanding the subcritical phase. To compute such threshold, using the bounds of Lemma 15, one can conclude that:

Proposition 11. *We have* $\alpha_M = \frac{d}{C_d} = \frac{d}{\pi^{d/2}} \Gamma\left(\frac{d}{2} + 1\right)$, *where* $C_d = \pi^{d/2} \Gamma^{-1}\left(\frac{d}{2} + 1\right)$.

Proof of Proposition 11. Applying Lemma 15, for $h = 0$, and $z < 1/4$, there exists constants $C = C(d, \alpha) > 0$ and $c = c(d, \alpha) > 0$ such that:

$$c z^{-d + \alpha C_d} \leq z^{-d} \mathbb{E}\left(\lambda(\mathcal{V}_\alpha[z])\right) \leq C z^{-d + \alpha C_d}.$$

Concluding the proof. □

Now, using the shrunk configuration and the bounds provided by Lemma 15, we can show that the mean threshold α_M belongs to the subcritical phase. Informally speaking, we will define a finite grid in the torus; in such a grid, if a point is missing in the original process, in the shrunk configuration, one will see a missing square. By this fact, instead of dealing with a non-enumerable number of points, one can use the finite number of squares in the grid to perform a finite union bound that will give the polynomial decay of the covering probability.

Fix any $\ell > 0$, then in the $1-$dimensional torus $\mathbb{T}^1$ define the disjoint intervals:

$$\begin{cases} Q_i = [\ell i, \ell(i+1)), & \text{for } i \in \mathbb{N} \text{ such that } \ell(i+1) \leq 1. \\ Q_i = \left[\ell \left\lfloor \frac{1}{\ell} \right\rfloor, 1\right) & \text{for } i \in \mathbb{N} \text{ such that } \ell(i+1) \geq 1. \end{cases}$$

Using the intervals, one can define the $1-$dimensional grid as the set:

$$\mathcal{Q}(\ell) = \left\{ Q_i : i \in \left\{ 1, \cdots, \left\lfloor \frac{1}{\ell} \right\rfloor \right\} \right\}. \tag{4.2}$$

For the torus $\mathbb{T}^d$, define a grid inside the torus to be $\mathcal{Q}^d(\ell)$ be the collection of elements in the form $\prod_{k=1}^d \mathcal{Q}_k(\ell)$ where $(\mathcal{Q}_k(\ell))_k$ are copies of $\mathcal{Q}(\ell)$, one for each dimension.

Using the grid, we can give bounds on the decay threshold α_+ defined in equation (1.14). More precisely, we have the following:

Proposition 12. *For every $d \geq 1$, we have $\alpha_c \leq \alpha_+ \leq \alpha_M$.*

Proof. Proof of Proposition 12 The fact that $\alpha_c \leq \alpha_+$ is trivial, since for every $\alpha > \alpha_+$ there exists a $\zeta_+ = \zeta_+(\alpha) > 0$, where:

$$\lim_{z \to 0} \mathbb{P}\left(\lambda(\mathcal{V}_\alpha[z]) > 0\right) \leq \lim_{z \to 0} z^{\zeta_+} = 0.$$

To see that $\alpha_+ \leq \alpha_M$, take any $h \in (0,1)$, and fixed $\ell \leq \frac{h}{\sqrt{d}}$, where $\sqrt{d}$ is the size of the main diagonal of the $d-$dimensional cube. By the choice of ℓ, for every $Q \in \mathcal{Q}^d(\ell z)$, and for any point $x \in Q$:

$$\{x \in \mathcal{V}_\alpha[z]\} \subset \{x \in Q \subset \mathcal{V}_\alpha^{(h)}[z]\}.$$

This means that missing a point implies missing a square in the shrunk configuration.

In particular, for small values of z using union bound by Lemma 15, there exists a $C = C(\alpha, h, d)$, such that:

$$\begin{aligned} \mathbb{P}\left(\mathcal{V}_\alpha[z] \neq \emptyset\right) &= \mathbb{P}\left(\bigcup_{Q \in \mathcal{Q}(\ell z)} \{Q \cap \mathcal{V}_\alpha[z] \neq \emptyset\} \right) \leq \mathbb{P}\left(\bigcup_{Q \in \mathcal{Q}(\ell z)} \{Q \subset \mathcal{V}_\alpha^{(h)}[z]\} \right) \\ &\leq \frac{1}{(\ell z)^d} \mathbb{P}\left(Q \subset \mathcal{V}_\alpha^{(h)}[z]\right) \leq \frac{1}{(\ell z)^d} \mathbb{P}\left(0 \in \mathcal{V}_\alpha^{(h)}[z]\right) \leq C \frac{z^{\alpha C_d - d}}{\ell^d}. \end{aligned}$$

Thus, for every $\alpha < \alpha_M$, we can find a value $\zeta = \alpha C_d - d > 0$ such that:

$$\mathbb{P}\left(\mathcal{V}_\alpha[z] \neq \emptyset\right) \leq \frac{C}{\ell^d} z^\zeta,$$

concluding the proof. □

A second-moment argument is necessary and sufficient to finish the proof. Interestingly, the random variable that concentrates is not related to the number of missing squares in the grid; instead, it is related to the measure of $\lambda(\mathcal{V}_\alpha[z])$. The technique developed by Billard in [22] is capable of indicating whether the event $\{\mathcal{V} \neq \emptyset\}$ has a positive probability. This strong result fills an important hole in the area, but it is not sufficient to provide quantitative information on the number of missing points or the rate at which the probability converges. Therefore, Propositions 12 and 13 are, in fact, a new contribution to the field. Moreover, it is worth to mention that for $d \geq 2$ the necessarily and sufficient condition to cover the space using fixed-size objects remains an open question, being described only when the object has the shape of a simplex; see [19, 16].

Theorem 4 (Billard's)**.** *Consider in $\mathbb{T}^d$, a set of balls with volumes $1 > v_1 > v_2 > \cdots > v_n > \cdots$. If*

$$\sum_n v_n^2 \exp\{v_1 + \ldots + v_n\} < \infty.$$

Then, when placing these balls uniformly over the torus, there exists a chance to not cover the space $\mathbb{P}\left(\mathcal{V} \neq \emptyset\right) > 0$.

Billard's theorem can be found in [19], and is valid for high-dimensional spaces. It is the simplest result that relates the mean threshold α_M to the critical point α_c. More precisely, we will show that $\alpha_c = \alpha_M$, and for this we will prove that for $\alpha < \alpha_M$ there exists a positive probability of not covering the space.

Proposition 13. *We have $\alpha_M \leq \alpha_c$, and in particular $\alpha_c = \alpha_+ = \alpha_M$.*

Proof. We will use Theorem 4. Fixed any $\alpha \in (0, \alpha_M)$, let $v_0 = 1$, and define inductively for every configuration ω_α the set of volumes $(v_n)_n$:

$$v_n = \inf\{v < v_{n-1} : \exists \xi \in S \text{ s.t. } \omega_\alpha(\xi) = 1 \text{ and } \lambda(\Pi(\xi)) = v\}.$$

For any $\eta \in (0,1)$ and $\alpha = (1-\eta)\frac{d}{C_d}$, define the increasing regions:

$$B_n^\eta = \mathbb{T}^d \times \left[\sqrt[d]{\frac{1-\eta^2}{C_d n}}, \infty \right),$$

for $n \in \{1, 2, \cdots\}$.

Notice that $\omega(B_n^\eta)$ is a Poisson random variable with rate $\frac{\alpha C_d n}{d\delta} = \frac{n}{1+\eta}$, in particular by Chernoff bounds over the rate of the Poisson random variable, one gets that:

$$\sum_n \mathbb{P}\left(\omega(B_n^\eta) \geq n\right) < \infty. \tag{4.3}$$

Since, we have that:

$$\{\omega(B_n^\eta) < n\} = \left\{ v_n < \frac{1-\eta^2}{n} \right\}.$$

Using equation (4.3), together with Borel Cantelli we conclude that:

$$\mathbb{P}\left(v_n < \frac{1-\eta^2}{n}, \forall n > 1 \right) > 0.$$

This means that there exists a positive probability that all volumes of objects are dominated by $\frac{1-\eta^2}{n}$. A series that by Billard's Proposition 4 does not cover the space. In particular, for every $\alpha < \alpha_M = \frac{d}{C_d}$, we get that:

$$\lim_{z\to 0} \mathbb{P}\left(\mathcal{V}_\alpha[z] \neq \emptyset\right) > 0.$$

Then the proof is a direct consequence of the topological Lemma 16: □

Lemma 16. *In any dimension $d \geq 1$, for any parameter $\alpha > 0$, we have*

$$\mathbb{P}\left(\mathcal{V}_\alpha \neq \emptyset\right) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} \left\{ \mathcal{V}_\alpha \left[\frac{1}{n}\right] \neq \emptyset \right\} \right) = \lim_{z\to 0} \mathbb{P}\left(\mathcal{V}_\alpha[z] \neq \emptyset\right) > 0.$$

Proof of Lemma 16. Fixed $n > 0$, and $\alpha \geq 0$, note that:

$$\{\mathcal{V}_\alpha \neq \emptyset\} \subseteq \left\{ \mathcal{V}_\alpha \left[\frac{1}{n}\right] \neq \emptyset \right\}.$$

Therefore:

$$\mathbb{P}\left(\mathcal{V}_\alpha \neq \emptyset\right) \leq \mathbb{P}\left(\bigcap_{n=1}^{\infty} \left\{ \mathcal{V}_\alpha \left[\frac{1}{n}\right] \neq \emptyset \right\} \right). \tag{4.4}$$

To prove the equality, we need a topological argument. Observe that the set of truncated points $\omega_\alpha\left[\frac{1}{n}\right]$ is almost surely finite. Moreover, since the projection function of any point is an open set, we have that $\mathcal{V}_\alpha\left[\frac{1}{n}\right]$ is the complementary of a finite union of open set, thus it is almost surely closed in $\mathbb{T}^d$, i.e. compact. Finally, consider $m > n$, and notice that $\mathcal{V}_\alpha\left[\frac{1}{m}\right] \subseteq \mathcal{V}_\alpha\left[\frac{1}{n}\right]$, in particular, $\bigcap_n \mathcal{V}_\alpha\left[\frac{1}{n}\right]$ is the intersection of nested, compact sets of $\mathbb{T}^d$. Therefore, if all are non empty, there must be a point in the limit, and the space will not be covered at time α. Proving then the equality in (4.4). □

In particular, since Theorem 1 is a direct consequence of Proposition 12 and 13, we conclude describing the subcritical phase.

4.2 The supercritical phase

This Section focuses on giving information about the process when $\alpha < \alpha_c$. The goal of this section is to understand the way in which the covering process survives: The process, when it survives, has a well-behaved structure of empty sets that behave in some way as a classical branching process.

Before presenting the proof of the main results, we will show that the vacant set can indeed survive as a Branching Process. For this, consider the definition of the grid from equation (4.2), and fix $\mathcal{Q}^d(\ell) = \prod_{k=1}^d \mathcal{Q}_k(\ell)$ where $\mathcal{Q}_k(\ell)$ are orthogonal copies of $\mathcal{Q}(\ell)$. Moreover, for any open set $A \subset \mathbb{T}^d$, a height $z > 0$, a parameter $\ell > 0$, and $\alpha \geq 0$ define the random variable

$$Y_\alpha(z, \ell, A) = \sum_{Q \in \mathcal{Q}^d(\ell z), Q \subset A} \mathbf{1}\{Q \subset \mathcal{V}_\alpha[z]\}.$$

Intuitively, Y_α counts the number of squares of size ℓz that is completely not cover in the scale z. Concerning this random variable we have that:

Lemma 17. *For any $d \geq 1$, $\alpha \geq 0$, $\ell > 0$, and $A \subset \mathbb{T}^d$ with positive measure. There exists a constant $C = C(d, \ell, A) > 0$ such that:*

$$\mathbb{E}\left(Y_\alpha(z, \ell, A)\right) \geq C z^{-d+\alpha C_d}.$$

As a consequence, for every $M > 1$ and $\alpha < \alpha_c$ there exists an constant $k_0 = k_0(d, \ell, A, \alpha, M)$ such that for every $k > k_0$ we get $\mathbb{E}\left(Y_\alpha(2^{-k}, \ell, A)\right) > M$.

Later in the proof, in a set of scale heights $(2^{-kn})_n$, looking at the grid $\mathcal{Q}^d(2^{-kn})$, with any $Q \in \mathcal{Q}^d(2^{-kn})$, one can use the random variable $Y_\alpha(2^{-k(n+1)}, 1, Q)$ to find completely vacant elements of the grid $\mathcal{Q}^d(2^{-k(n+1)})$ within Q. By allowing the mean of $Y_\alpha(2^{-k(n+1)}, 1, Q)$ to be greater than $M > 1$, we will be able to relate this quantity to the number of children in a branching process. As a brief remark, notice that the number of children in neighboring regions is not independent; therefore, we take the mean greater than $M = M(d)$ large enough, so we find many well-separated regions that are completely independent.

Proof of Lemma 17. Fixed $\ell > 0$ and an open set with positive measure $A \subset \mathbb{T}^d$. There exists a value of $z_0 = z_0(A, \ell)$ and a constant $c_1 = c_1(A)$ such that for every $z < z_0$, we obtain a proportional number of squares from $\mathcal{Q}^d(\ell z)$ within A, this is:

$$\sum_{Q \in \mathcal{Q}^d(\ell z)} \mathbf{1}\{Q \subset A\} \geq c_1(\ell z)^{-d}.$$

Given $Q \in \mathcal{Q}^d(\ell z)$, consider the region:

$$R_Q = \{\xi = (x, r) : r > z, \text{ and } \Pi(\xi) \cap Q \neq \emptyset\}.$$

Notice that, $\{Q \subset \mathcal{V}_\alpha[z]\}$ if an only if R_Q is empty. In this way, we are going to finish the proof by bounding the rate of R_Q.

To obtain a bound on the rate of the region R_Q, start by bounding the rate from above by assuming that any object with size greater than $1/4$ automatically covers the squared region. Since such objects have a rate equal to $\alpha 4^d$, in general, for $d \geq 2$, one gets that:

$$\begin{aligned}\Lambda_\alpha(R_Q) &\leq \alpha \left(4^d + \int_z^{1/4} \frac{(\ell z)^d}{r^{d+1}} dr + 2d \int_z^{1/4} \frac{(\ell z)^{d-1} r}{r^{d+1}} dr + \int_z^{1/4} \frac{C_d r^d}{r^{d+1}} dr \right) \\ &, \leq \alpha \left(4^d + \frac{\ell^d}{d} + 2d \frac{\ell^{d-1}}{d-1} - C_d \ln(4) - C_d \ln(z) \right).\end{aligned}$$

Where the first integral concerns objects directly above Q. The second integral is related to objects whose minimum distance between its center and Q is achieved by a point in one of the $2d$ $(d-1)$-hypersurfaces of the hypercube. The last integral accounts for the points for which the minimum distance between its center and Q is achieved in one of the 2^d vertices of the hypercube. Moreover, if $d = 1$, the bound is true without the second term.

In particular, by evaluating the integral. Following the order of the last term, we find a constant $C = C(\ell, A, d) > 0$ such that for every z, we get that:

$$\mathbb{E}\left(Y_\alpha(z, \ell, A)\right) \geq C z^{-d + \alpha C_d}.$$

As desired. □

The random variable $Y_\alpha(z, \ell, A)$ can be difficult to work with because it requires completely vacant cubes of the grid. In essence, the vacant set can survive in each generation without those simply by using partially uncovered elements of $\mathcal{Q}^d$ instead. Our goal in this section is to show how the continuity of the covering function is related to a branching type of survival, and if the covering function is continuous, the random variable $Y_\alpha(z, \ell, A)$ can carry a branching in the vacant set.

The main argument of this Section is divided in two points:

1. Using the continuity points of the covering function, we can show that with high probability the set that survives between scales roughly behave as a branching process.

2. In the second part, we prove the continuity of the covering probability when $d = 1$, outside the critical point.

Together, these subsections will prove Theorem 2 and Theorem 3.

4.2.1 Counting the missing points

The aim of this Subsection is to find a portion of the vacant set that survive as a branching process. Later, for every continuity point of the covering function, we will use a sprinkling technique to clear regions that will reveal such branching processes.

Analogously to the random variable $Y_\alpha(z, \ell, A)$, with the grid $\mathcal{Q}^d(\ell) = \prod_{k=1}^d \mathcal{Q}_k(\ell)$ fixed, for any $z > 0$, $\ell > 0$, and $\alpha \geq 0$ define the random variable

$$X_\alpha(z, \ell) = \sum_{Q \in \mathcal{Q}^d(\ell z)} \mathbf{1}\{Q \cap \mathcal{V}_\alpha[z] \neq \emptyset\}.$$

Which corresponds to the number of squares of size ℓz that are not fully covered at height z in the space $\mathbb{T}^d$.

Unlike the random variable Y_α, the random variable X_α is always positive if the space is not covered and we can show that it eventually has a large value.

Lemma 18. *For every $\ell > 0$, $\alpha > 0$ and $N \in \mathbb{N}$, we have the following.*

$$\sum_{k=0}^{\infty} \mathbb{P}\left(X_\alpha(2^{-k}, \ell) \in \{1, ..., N\}\right) < \infty. \tag{4.5}$$

As a consequence, when $\alpha < \alpha_c$, conditioning in not covering, with high probability we eventually have more than N empty regions, that is $\lim_{k\to\infty} \mathbb{P}_\alpha\left(X_\alpha(2^{-k}, \ell) > N \middle| \mathcal{V}_\alpha \neq \emptyset\right) = 1$.

Proof of Lemma 18. Notice that $\{\mathcal{V}_\alpha[z] = \emptyset\}$ happens, if and only if $\{X_\alpha(z,\ell) = 0\}$ occurs for some $\ell > 0$. Then, partitioning the event $\{\mathcal{V}_\alpha = \emptyset\}$ in the first height in which its is covered, using the Lemma 16, we get for any $A \subset \mathbb{T}^d$ that:

$$\mathbb{P}\left(\mathcal{V}(\omega_\alpha) = \emptyset\right) \geq \mathbb{P}\left(\bigcup_{k=1}^{\infty}\{\mathcal{V}_\alpha[2^{-k}] = \emptyset\} \cap \{\mathcal{V}_\alpha[2^{-(k-1)}] \neq \emptyset\}\right) \tag{4.6}$$

$$\geq \sum_{k=1}^{\infty} \mathbb{P}\left(X_\alpha(2^{-k},\ell) = 0, X_\alpha(2^{-(k-1)},\ell) > 0\right)$$

$$\geq \sum_{k=1}^{\infty} \mathbb{P}\left(X_\alpha(2^{-k},\ell) = 0, X_\alpha(2^{-(k-1)},\ell) \in \{1,...,N\}\right). \tag{4.7}$$

To avoid auto-intersection of the objects, let k_0 be large enough such that for every $k > k_0$, we get $\ell 2^{-(k-1)} < 1/4$. Then, for each square $Q \in \mathcal{Q}^d(\ell 2^{-(k-1)})$, the rate of the region $Q \times [2^{-(k-1)}, 2^{-k}]$ is constant equal to $\frac{\alpha \ell^d (2^d - 1)}{d}$ and does not depend on k. Furthermore, it is always possible to cover an element $Q \in \mathcal{Q}^d(\ell 2^{-(k-1)})$ using objects that belong to the set $Q \times [2^{-(k-1)}, 2^{-k}]$, so there is a positive probability $p = p(\alpha,\ell)$ to cover Q using objects that belong only to $Q \times [2^{-(k-1)}, 2^{-k}]$. As an illustration, note that for $\ell = 1$, any sufficient centered ball with radius equal to one can cover $Q_1 \in \mathcal{Q}^d(1)$. However, when $\ell = 2$, it is possible to cover $Q_2 \in \mathcal{Q}^d(2)$ using 2^d balls with a radius equal to one, each centered in an element of $\mathcal{Q}^d(1) \cap Q_2$. In any case, the probability $p = p(\alpha,\ell)$ is a constant that does not depend on k.

Since p does not depend on k when $k > k_0$, and by the equation (4.7), if the vacant set is contained in at most N squares of $\mathcal{Q}^d(\ell 2^{-k})$, we can cover it with probability bounded by p^N. So:

$$\mathbb{P}\left(\mathcal{V}_\alpha = \emptyset\right) \geq \sum_{k=k_0}^{\infty} \mathbb{P}\left(X_\alpha(2^{-k},\ell) = 0, X(2^{-(k-1)},\ell) \in \{1,...,N\}\right)$$

$$\geq \sum_{k=k_0}^{\infty} p^N \mathbb{P}\left(X_\alpha(2^{-(k-1)},\ell) \in \{1,...,N\}\right).$$

With this bound on the tail sum, by adding the finite probabilities between $k \in \{1,...,k_0\}$, we conclude the proof of equation (4.5).

To conclude the Lemma, and prove the conditional limit, observe that we can bound:

$$\mathbb{P}\left(X_\alpha(2^{-k},\ell) \in \{1,\cdots,N\} | \mathcal{V}_\alpha \neq \emptyset\right) \leq \frac{\mathbb{P}\left(X_\alpha(2^{-k},\ell) \in \{1,\cdots,N\}\right)}{\mathbb{P}\left(\mathcal{V}_\alpha \neq \emptyset\right)}.$$

Therefore, by fixing $\alpha < \alpha_c$, since $\mathbb{P}(\mathcal{V}_\alpha \neq \emptyset) > 0$, we can use the bound on the sum in equation (4.5), to get:

$$\lim_{k\to\infty} \mathbb{P}\left(X_\alpha(2^{-k},\ell) > N | \mathcal{V}_\alpha \neq \emptyset\right) \geq \lim_{k\to\infty} 1 - \frac{\mathbb{P}\left(X_\alpha(2^{-k},\ell) \in \{1,\cdots,N\}\right)}{\mathbb{P}(\mathcal{V}_\alpha \neq \emptyset)} = 1.$$

As desired. □

Now, building on Lemma 17 that computes the mean of the random variable $Y_\alpha(z,\ell,A)$, we will define the branching process used in our construction and get a more quantitative argument.

Lemma 19. *Fixed $\ell = 1$, there exists a number $k_0 = k_0(\alpha, d)$, a value $\gamma = \gamma(\alpha, d) \in (0,1)$ and a probability $\theta = \theta(\alpha, d, \gamma) > 0$, such that for every fixed $k > k_0$:*

$$\mathbb{P}\left(Y(2^{-kn}, 1, \mathbb{T}^d) > 2^{\gamma n}\right) > \theta, \quad \forall n > 0.$$

Proof of Lemma 19. The proof consists in relating the number of $Y_\alpha(z,\ell,A)$ to a specific branching process. Fixed $\ell = 1$, for each given $k > 1$, notice that any element of the grid $Q \in \mathcal{Q}^d(2^{-kn})$ is completely contained in one element of the grid $Q_j \in \mathcal{Q}^d(2^{-kj})$, for every $j \in \{1,\cdots,n\}$. In particular, define the quantity:

$$Z_n(k) = \sum_{Q\in\mathcal{Q}^d(2^{-kn})} \prod_{j=1}^{n} \mathbf{1}\{Q_j \subset \mathcal{V}_\alpha[2^{-kj}] : Q \subset Q_j\}.$$

This is the number of completely empty regions that between heights $(2^{-kj})_{j=1}^n$ are also empty. In particular, we have $Z_n(k) < Y_\alpha(2^{-kn}, 1, \mathbb{T}^d)$, since the random variable $Y_\alpha(2^{-kn}, 1, \mathbb{T}^d)$ counts all empty regions, and $Z_n(k)$ counts those that are always empty between heights.

For each chosen k the process $(Z_n(k))_n$ behaves similarly to a branching process with finite dependencies, where the children are elements of the grid $\mathcal{Q}^d(2^{-kn})$. By the fixed choice of $\ell = 1$, the range of dependencies of the problem is always equal to one. In particular, we can take $M = M(d)$ so that if $Z_n(k) > M$, there exist at least two well separated and thus independent children.

To obtain a lower bound in the distribution of the process $Z_n(k)$, we will use the process Y_α. To show that the space maintains the distribution between different heights, fix $k > 2$, set $Q_0 \in \mathcal{Q}^d(2^{-k(n-1)})$, and set $Q_1 \in \mathcal{Q}^d(2^{-kn})$. Then, we get for every $x \geq 0$ and $n \geq 1$ that:

$$\mathbb{P}\left(Y_\alpha(2^{-kn}, 1, Q_0) > x \middle| Q_0 \subset \mathcal{V}_\alpha[2^{-k(n-1)}]\right) = \mathbb{P}\left(Y_\alpha(2^{-k(n+1)}, 1, Q_1) > x \middle| Q_1 \subset \mathcal{V}_\alpha[2^{-kn}]\right).$$

Therefore, conditioning in a vacant region, the random variable Y_α for the next generation preserves the distribution. Consequently, it also preserves its mean.

By Lemma 17, there exists a value $k_0(\alpha, d, M(d))$ such that for every $k > k_0$ we get $\mathbb{E}\left(Y_\alpha(2^{-k}, 1, \mathbb{T}^d)\right) > M$, and by conditioning ourselves in the scale 2^{-kn} to not have objects greater than $2^{-k(n-1)}$, we get the following.

$$\mathbb{E}\left(Y_\alpha(2^{-kn}, 1, Q)\middle|Q \subset \mathcal{V}_\alpha[2^{-k(n-1)}]\right) > \mathbb{E}\left(Y_\alpha(2^{-k}, 1, \mathbb{T}^d)\right) > M.$$

Fixing $Q \in \mathcal{Q}^d(2^{-kn})$, note that when conditioning on the event $\{Q \in Z_n(k))\}$, the distribution of the number of children of Q counted by $Z_{n+1}(k)$ is equal to the distribution of $Y_\alpha(2^{-k(n+1)}, 1, Q)$ conditioning on the event $\{Q \subset \mathcal{V}_\alpha[2^{-kn}]\}$. Furthermore, choosing $M = M(d)$, if $\{Y_\alpha(2^{-k(n+1)}, 1, Q) > M\}$ occurs, then there exist at least two independent children within the process $Z_n(k)$. Therefore, in relation to independent children, we can argue that the process $Z_n(k)$ is expected to have on average at least 2^n completely disjoint independent regions that have not yet been explored at time 2^{-kn}.

Since the children preserve distributions, by a classical branching process argument, see [21] Theorem 2, page 9, the process $Z_n(k)$ can survive with a large number of individuals. Therefore, there exist a $\gamma \in (0, 1)$ and a probability $\theta = \theta(\alpha, d, \gamma) > 0$ such that for every $n > 0$:

$$\mathbb{P}\left(Y_\alpha(2^{-kn}, 1, \mathbb{T}^d) > 2^{\gamma n}\right) > \mathbb{P}\left(Z_n > 2^{\gamma n}\right) > \theta > 0.$$

As desired. □

To prove Theorem 2 and get information about the vacant set of the process in the continuity points of the covering function π, we need the following diminishing/enhancing construction. For every configuration $\omega = \sum_{i\in I} \delta_{(x_i, r_i)}$ in the space S, and for every $\eta > 0$, define

$$\omega * \eta = \sum_{i\in I} \delta_{(x_i, \eta r_i)}.$$

Notice that the number of objects in any region R after a diminishing/enhancing by η is described by the number of objects in $\omega * \eta(R) = \omega(\{(x, r) : (x, r/\eta) \in R\}$. Moreover, if R is a rectangular region, the set $\{(x, r) : (x, r/\eta) \in R\}$ corresponds to a region of objects with radius η times larger, thus at time α one gets that $\omega_\alpha * \eta(R)$ has the same distribution as $\omega_{\alpha\eta^d}(R)$. Since this is true for every rectangular region, we get that $\omega_\alpha * \eta$ have the same distribution as $\omega_{\alpha\eta^d}$. Therefore, the diminishing/enhancing construction can induce a coupling between different time configurations by just stretching the objects radii. If it is the case that $\eta < 1$, we are going to say that we are going to diminish the objects, otherwise if $\eta > 1$, we are going to enhance them.

Also note that when we diminish the configurations, vacant points end up becoming empty regions. More precisely, if $\ell < 1$, and if $Q \in \mathcal{Q}^d(z\ell)$ and $\{Q \cap \mathcal{V}_\alpha[z] \neq \emptyset\}$ happens, then we get $\left\{Q \subset \mathcal{V}\left[\omega_\alpha * \left(\frac{1-\ell}{\sqrt{d}}\right)\right]\right\}$, for this, observe that the radius of each empty ball decreases by $(1-\ell)/\sqrt{d}$.

With this coupling, we get the following.

Proposition 14. *For every $\alpha < \alpha_c$ a continuity point of the function π, and $\eta > 0$, there exist $\gamma_- = \gamma_-(\alpha) > 0$, $\ell = \ell(\eta)$, and $k_0 = k_0(\eta, \alpha)$ such that for every $k > k_0$, we get:*

$$\mathbb{P}\left(Y_\alpha(2^{-k}\ell, 1, \mathbb{T}^d) > 2^{k\gamma_-} \middle| \mathcal{V}_\alpha \neq \emptyset\right) \geq 1 - \eta.$$

In particular, here exists $\gamma_a = \gamma_a(\alpha) > 0$, such that:

$$\lim_{z \to 0} \mathbb{P}\left(Y_\alpha(z, 1, \mathbb{T}^d) > z^{-d\gamma_a} \middle| \mathcal{V}_\alpha \neq \emptyset\right) = 1.$$

Proof of Proposition 14. This proof will use the continuity of the cover function π in α. Start by fixing $\alpha < \alpha_c$ a continuity point for the function π, take any $\eta > 0$, and choose $\varepsilon = \varepsilon(\eta, \alpha, \alpha_c) > 0$ so that two things happen: we have $\alpha + \varepsilon < \alpha_c$ and $\pi(\alpha+\varepsilon) - \pi(\alpha) < \frac{\eta(1-\pi(\alpha))}{4}$. In particular, we get the following.

$$\mathbb{P}\left(\mathcal{V}_{\alpha+\varepsilon} \neq \emptyset | \mathcal{V}_\alpha \neq \emptyset\right) = 1 - \mathbb{P}\left(\mathcal{V}_{\alpha+\varepsilon} = \emptyset | \mathcal{V}_\alpha \neq \emptyset\right) \geq 1 - \frac{\eta}{4}. \tag{4.8}$$

The idea of the proof consists in looking at the configuration at time $\alpha + \varepsilon$ and guaranteeing the existence of many partially vacant regions $X_{\alpha+\varepsilon}(z, \ell)$. Then, by diminishing the configuration back to time α, these regions guarantee some empty elements of the grid in $Y_\alpha(z, \ell, \mathbb{T}^d)$. Then, since the number of such regions is high, we will eventually survive as a branching with high probability, finishing the proof.

More specifically, consider the configuration ω_α and take $\omega_{\alpha+\varepsilon} = \omega_\alpha * \left(\frac{\alpha+\varepsilon}{\alpha}\right)^{1/d}$, in particular, we have $\omega_\alpha = \omega_{\alpha+\varepsilon} * \left(\frac{\alpha+\varepsilon}{\alpha}\right)^{-1/d}$. Now, fix $\ell = \ell(\alpha, \varepsilon, d) \in (0,1)$ so that for each $x \in Q \in \mathcal{Q}(\ell z)$ such that $\{x \in \mathcal{V}_{\alpha+\varepsilon}[z]\}$ occurs, we can guarantee that $\{Q \subset \mathcal{V}_\alpha[z]\}$ happens in the diminished configuration ω_α.

With $\ell \in (0,1)$ fixed, consider the probability $p = p(\ell, \alpha, \varepsilon) > 0$that for any fixed $Q \in \mathcal{Q}(z\ell)$, there are no objects with heights between z and ℓz intercepting with Q. This corresponds to the probability of the region

$$R_Q = \{(x, r) \in S : \Pi(x, r) \cap Q \neq \emptyset, r \in (z\ell, z)\}$$

being empty. Notice that the rate of R_Q does not depend on z. And in particular, with probability p, one can recover a square $Q \in \mathcal{Q}(\ell z)$ in the height ℓz, i.e., $Y_\alpha(z\ell, 1, \mathbb{T}^d) \geq 1$.

Therefore, for every quantity $M > 0$ of vacant regions in the height $z\ell$, one can find a number $N = N(M, p, \ell, d, \eta, \alpha)$, such that:

$$\mathbb{P}\left(Y_\alpha(z\ell, 1, \mathbb{T}^d) \geq M | Y_\alpha(z, \ell, \mathbb{T}^d) > N, \mathcal{V}_\alpha \neq \emptyset\right) > 1 - \frac{\eta}{2}. \tag{4.9}$$

Then, if $\alpha + \varepsilon < \alpha_c$, using Lemma 18, we can find $k_0 = k_0(\eta) > 0$ such that:

$$\mathbb{P}\left(X_{\alpha+\varepsilon}(2^{-k}, \ell) > N | \mathcal{V}_{\alpha+\varepsilon} \neq \emptyset\right) > 1 - \frac{\eta}{4}, \tag{4.10}$$

Thus, using equations (4.8) and (4.10), by total probability, we get that:

$$\begin{aligned}&\mathbb{P}\left(X_{\alpha+\varepsilon}(2^{-k}, \ell) > N | \mathcal{V}_\alpha \neq \emptyset\right) \geq \mathbb{P}\left(X_{\alpha+\varepsilon}(2^{-k}, \ell) > N, \mathcal{V}_{\alpha+\varepsilon} \neq \emptyset | \mathcal{V}_\alpha \neq \emptyset\right)\\ &= \mathbb{P}\left(X_{\alpha+\varepsilon}(2^{-k}, \ell) > N | \mathcal{V}_{\alpha+\varepsilon} \neq \emptyset\right) \mathbb{P}\left(\mathcal{V}_{\alpha+\varepsilon} \neq \emptyset | \mathcal{V}_\alpha \neq \emptyset\right) > 1 - \frac{\eta}{2}.\end{aligned}$$

Therefore, using the diminishing, we get that $\{X_{\alpha+\varepsilon}(2^{-k}, \ell) \leq Y_\alpha(2^{-k}, \ell, \mathbb{T}^d)\}$ almost sure. Thus:

$$\mathbb{P}\left(Y_\alpha(2^{-k}, \ell, \mathbb{T}^d) > N | \mathcal{V}_\alpha \neq \emptyset\right) > 1 - \frac{\eta}{2}. \tag{4.11}$$

By the choice of N, we get by equations (4.11) and (4.9), that:

$$\begin{aligned}&\mathbb{P}\left(Y_\alpha(2^{-k}\ell, 1, \mathbb{T}^d) \geq M | \mathcal{V}_\alpha \neq \emptyset\right) \geq \mathbb{P}\left(Y_\alpha(2^{-k}\ell, 1, \mathbb{T}^d) \geq M, Y_\alpha(2^{-k}, \ell, \mathbb{T}^d) > N | \mathcal{V}_\alpha \neq \emptyset\right)\\ &= \mathbb{P}\left(Y_\alpha(2^{-k}\ell, 1, \mathbb{T}^d) \geq M \middle| Y_\alpha(2^{-k}, \ell, \mathbb{T}^d) > N, \mathcal{V}_\alpha \neq \emptyset\right) \mathbb{P}\left(Y_\alpha(2^{-k}, \ell, \mathbb{T}^d) > N | \mathcal{V}_\alpha \neq \emptyset\right)\\ &\geq 1 - \eta.\end{aligned}$$

Fixing $k > 2k_0$, and truncating the space at height $2^{-k/2}$, since $k/2 > k_0$, we will have a high probability of having more than M children. Then, between the scales $2^{k/2}$ and 2^k, we should try to survive as a branching. Therefore, by Lemma 19, there exists $\gamma_- \in (0,1)$, and a probability $\theta = \theta(\alpha, d, \gamma) > 0$ such that:

$$\begin{aligned}\mathbb{P}\left(Y_\alpha(2^{-k}\ell, 1, \mathbb{T}^d) > 2^{\frac{k}{2}\gamma_-} \middle| \mathcal{V}_\alpha \neq \emptyset\right) &< \mathbb{P}\left(Y_\alpha(2^{-k/2}\ell, 1, \mathbb{T}^d) \geq M \middle| \mathcal{V}_\alpha \neq \emptyset\right)(1 - (1-\theta)^M)\\ &\leq (1-\eta)(1-(1-\theta)^M)\end{aligned}$$

Since M is arbitrary, θ depends only on α, d, and γ, and k diverge as η goes to zero, we get that:

$$\lim_{k\to\infty} \mathbb{P}\left(Y_\alpha(2^{-k}\ell, 1, \mathbb{T}^d) > 2^{\frac{k}{2}\gamma_-} \middle| \mathcal{V}_\alpha \neq \emptyset\right) = 1.$$

To finish the proof, again using the probability $p' = p(\ell, \alpha, \varepsilon) > 0$, the box $Q \in \mathcal{Q}^d(2^{-k}\ell)$ can survive between scales. In particular we get for any height $z > 0$ there exists an value $\gamma_a = \gamma_a(\alpha) \in (0,1)$ such that:

$$\lim_{z \to 0} \mathbb{P}\left(Y_\alpha(z, 1, \mathbb{T}^d) > z^{-d\gamma_a} \middle| \mathcal{V}_\alpha \neq \emptyset\right) = 1.$$

Finishing the proof. □

Now, we are able to proof Theorem 2.

Proof of Theorem 2. By Proposition 14, for every continuity point α of the function π, we can find γ_a to limit the number of $Y_\alpha(z, 1, \mathbb{T}^d)$ and thus $\lambda(\mathcal{V}_\alpha[z])$. Now, using the first moments of Lemma 17, we can find by Markov's inequality a value of $\gamma_b > d - \alpha d/C_d$, and concludes:

$$\lim_{z \to 0} \mathbb{P}\left(z^{\gamma_a} < \lambda(\mathcal{V}_\alpha[z]) < z^{\gamma_b} | \mathcal{V}(\omega_\alpha) = \emptyset\right) = 1.$$

Proving that α is also a well-behaved point, as desired. □

4.2.2 Continuity of $pi(\alpha)$ d=1 in the supercritical phase

To prove that the covering function is a continuous function, we will proof that it is continuous on the left and right. As a direct consequence of Lemma 16, one can prove the left-continuity of $\pi(\alpha)$. This proof also works in dimension $d \geq 1$.

Lemma 20. *$\pi(\alpha)$ is left-continuous.*

Proof of Lemma 20. Notice that $\mathcal{V}_\alpha$ is the intersection of fitted compact sets $\mathcal{V}_\alpha[z]$, thus $\mathcal{V}_\alpha$ is compact for every $\alpha > 0$. Then, by taking $\alpha < \beta$, by inclusion $\mathcal{V}_\beta \subseteq \mathcal{V}_\alpha$, we get:

$$\lim_{\varepsilon \to 0} \pi(\alpha) - \pi(\alpha - \varepsilon) = \mathbb{P}\left(\mathcal{V}_{\alpha - \varepsilon} \neq \emptyset \, \forall \varepsilon > 0, \mathcal{V}_\alpha = \emptyset\right) = 0,$$

since the sets $(\mathcal{V}_\alpha)_\alpha$ are also fitted non empty compact sets. □

The proof of the right-continuity of the covering probability is more complex and involves a quantitative argument that is only available for dimension $d = 1$. In the article [6], more precisely in Lemma 13. and Proposition 8., a quantitative description of the vacant set for dimension $d = 1$ is given. By analyzing this description, we can conclude the right-continuity of the probability of coverage for $d = 1$ in the supercritical phase.

The model treated in [6] uses arcs that start at points U and have length R, in this text, we use objects that have center x and have radius r. To be precise, in both models, the Poisson point process in S is the same, and with the same rate, the only difference is the projection function of the models. In [6], the projection of a point (x, r) is the interval $(x, x+r)$. Here, the projection is the interval $(x-r, x+r)$. Notice that our model at rate α corresponds to their model at rate 2α. To prove this, just take the bijection function that takes $(x, r) \in S$ and sends it to $(x-r, 2r) \in S$, which sends a point in our model that covers some interval $(x-r, x+r)$ to one point that in the model treated in [6] that covers the same interval. The application doubles the rate of the space, but since it preserves the same covering, both models have the same covering distribution.

With the relation between the models stated, as a direct corollary of Lemma 13. and Proposition 8., we can conclude for our model in dimension $d = 1$ that:

Proposition 15. *For $d = 1$ and any fixed $\alpha < \alpha_c = 1/2$, we have that:*

$$\lim_{z\to 0} \mathbb{P}\left(Y_\alpha(z, 1, \mathbb{T}) > z^{-\frac{\alpha_c-\alpha}{2}} | \mathcal{V}_\alpha \neq \emptyset\right) = 1.$$

As a consequence, we get:

Theorem 5. *For $d = 1$, we find that $\pi(\alpha)$ is a continuous function.*

Proof. Take any $\varepsilon > 0$ and $\alpha < \alpha_c$. Then, by Proposition 15, there exists a z_0 such that for every $z < z_0$ we get $\mathbb{P}\left(Y_\alpha(z, 1, \mathbb{T}) < z^{-\frac{\alpha_c-\alpha}{2}} | \mathcal{V}_\alpha[z] \neq \emptyset\right) < \frac{\varepsilon}{2}$. Now, for any $\delta_0 > 0$ such that $\alpha + \delta_0 < \alpha_c$, we get the following:

$$\begin{aligned}\pi(\alpha + \delta_0) - \pi(\alpha) &= \mathbb{P}\left(\mathcal{V}_{\alpha+\delta_0} = \emptyset, \mathcal{V}_\alpha \neq \emptyset\right) \\ &\leq \mathbb{P}\left(\mathcal{V}_{\alpha+\delta} = \emptyset, Y_\alpha(z, 1, \mathbb{T}) > z^{-\frac{\alpha_c-\alpha}{2}} | \mathcal{V}_\alpha \neq \emptyset\right) + \frac{\varepsilon}{2}.\end{aligned}$$

To conclude that the probability is small we need to control two probabilities: the probability to cover a region of size z using objects smaller than z, and the probability to not have any new object of size greater than z between times α and $\alpha + \delta$.

Since $\alpha + \delta_0 < \alpha_c$, the probability of covering the space is not one. In particular, assuming without loss of generality that $z < \frac{1}{2}$, there exists a positive probability $\eta = \eta(\alpha + \delta_0)$ of not fully covering an interval of size z using objects of size smaller than z. Here, we take $z < \frac{1}{2}$, so that θ is a constant that does not depend on the value of z. Also note that by taking intervals disjoint by a distance of $2z$, the probability of not covering each interval is independent. In particular:

$$\mathbb{P}\left(\mathcal{V}_{\alpha+\delta_0} = \emptyset, Y_\alpha(z, 1, \mathbb{T}) > z^{-\frac{\alpha_c-\alpha}{2}} | \mathcal{V}_\alpha \neq \emptyset\right) \leq \exp\left\{\ln(1-\eta)\frac{z^{-\frac{\alpha_c-\alpha}{2}}}{4}\right\} + (1 - e^{-\delta_0 z^{-1}}).$$

Where, $e^{-\delta z^{-1}}$ is the probability to have an new object with size greater than z between times α and $\alpha + \delta_0$. In words, to cover the space between times α and $\alpha + \delta$ when $\{Y_\alpha(z, 1, \mathbb{T}) > z^{-\frac{\alpha_c - \alpha}{2}}\}$ happens, either there exists a new object greater than z and we assume that the space will be covered, or there isn't, and using just objects with size less than z, you cover every $\frac{z^{-\frac{\alpha_c-\alpha}{2}}}{4}$ interval of size z that is separeted by at least a distance of $2z$.

Now, fix $z_1 = z_1(\theta)$ such that for every $z < z_1$, the first term is less than $\frac{\varepsilon}{4}$. Then, with $z < z_1$ fixed, one can now find $\delta_1 = \delta_1(z) < \delta_0$ such that $1 - e^{-\delta_1 z^{-1}} < \frac{\varepsilon}{4}$. Notice that by monotonicity of the probability we get $\eta(\alpha + \delta_1) > \eta(\alpha + \delta_0)$, and we can choose a smaller value of δ without changing the bound of the first term. So, we get that:

$$\pi(\alpha + \delta_1) - \pi(\alpha) \leq \varepsilon.$$

And since $\varepsilon > 0$ is arbitrary π is right continuous for every $\alpha < \alpha_c$ as desired, thus π is continuous for every $\alpha < \alpha_c$. □

In conclusion, we get the proof of Theorem 3, as a direct corollary of Theorem 2.

CHAPTER 5

Post-textual

5.1 Open questions

During this paper, we encountered many questions that were left open. To state them, consider the cover process X_t, with radius R distributed as $f(r)$, then:

1. Describe the exact number of different phases of the covering process in the one dimensional case. To be more specific, complete the Gumbel phase, showing that if $R \in L^1$, there exists a constant c such that $\frac{T_n}{n \ln n}$ converges in probability to c, and find the limits in distribution for all functions of the form $f(r) = \frac{1}{r \ln^b(r)}$, with $b > 1$; Between the Gumbel phase and the compact phase it is expected a new phase, for this take $f(r) = \frac{1}{r \ln r \ln \ln r}$ and find its limit in distribution (notice that it is not expected to be like Gumbel since it does not have first moment, and $1/f(n) > n \ln n$, so it is not expected to be on the same scale as the compact phase; In the compact phase, find the exact conditions or proof that conditions (2.19) and (2.20) fully describe this phase.

2. Find the explicit function of π and prove that the function is continuous for any dimension.

3. The main model makes sense in high dimensions, and we expect to see in such process the same phases.

4. Define the subcritical rate threshold:

$$\alpha_- = \sup\left\{\alpha > 0 : \begin{array}{l} \exists\gamma(\alpha) > 0 \text{ where } \forall\varepsilon > 0,\ \exists\zeta_- = \zeta_-(\alpha,\varepsilon) > 0 \text{ such that} \\ \mathbb{P}_\alpha\left(z^{\gamma+\varepsilon} < \lambda(\mathcal{V}^d\left(\omega[z]\right)) < z^{\gamma-\varepsilon}\middle|\mathcal{V}^d(\omega) \neq \emptyset\right) \geq 1 - z^{\zeta_-} \end{array}\right\}.$$

We expect $\alpha_c = \alpha_-$ for every dimension.

5. Find the exact exponents of the thresholds α_+ and α_-.

5.2 Appendix

5.2.1 Useful propositions

Here, we are going to fill out some details that are left from the proof, or just enunciate them and indicate where the proof is.

Proof of Lemma 1. To prove that 1 is equivalent to 2, observe that R is discrete. Then, if $R \in \mathrm{L}^p(\mathbb{R})$ for some $p > 1$:

$$\mathbb{E}\left(R^p\right) = \sum_{y=1}^{\infty} y^p \mathbb{P}\left(R = y\right) = \int_0^\infty p\frac{y^p f(y)}{y} dy.$$

Therefore, if $y^p f(y) \to 0$, then for every $p' \in [1,p)$, it is true that $\mathbb{E}\left(R^{p'}\right) < \infty$. And if, $\mathbb{E}\left(R^p\right) < \infty$ for some $p > 1$, then by Markov $y^p f(y) < \mathbb{E}\left(R^p\right)$, and for every $p'' \in [1,p)$ it is true that $y^{p''} f(y) \to 0$.

To prove that 2 equivalent to 1 just check that for every $\lambda > 0$, exists a $k_0 = k_0(\lambda)$ such that for every $k > k_0$:

$$f(k)k^{1+\frac{\lambda}{2}} < f(k)k^{1+\frac{\lambda}{2}}\ln k < f(k)k^{1+\lambda} < f(k)k^{1+\lambda}\ln k.$$

Finish the proof of the Lemma. □

Proof of Lemma 6. Using straightforward calculations, one may get:

$$\mathbb{P}\left(T_K^\ell < t\right) = \mathbb{P}\left(\max_{k=1,\dots,K}\{\xi_k\} < t\right) = \prod_{k=1}^{K}\mathbb{P}\left(\xi_k < t\right) = (1 - e^{-\frac{tp}{K}})^K.$$

Therefore:

$$\mathbb{P}\left(\frac{p}{K}T_K^\ell - \ln K < t\right) = \left(1 - \exp\left\{\frac{-Kp(t+\log K)}{Kp}\right\}\right)^K = \left(1 - \frac{e^t}{K}\right)^K$$

That equation converges to the Gumbel distribution when K goes to infinity and does not depend on the parameter p. □

Proof of the Lemma 12. For a Poisson Random variable X with rate λ, using concentration, [7], it is true that for every $x > 0$:

$$\mathbb{P}\left(|X-\lambda| > x\right) < 2\exp\left\{-\frac{x^2}{2(\lambda+x)}\right\}, \tag{5.1}$$

Now, with $\alpha > 0$ notice that $\omega(R_n^1)$ is a Poisson random variable with rate αn, and $\omega(R_n^\delta)$ is a Poisson random variable with rate $\alpha n/\delta$. So, we have:

$$\mathbb{P}_{1+\varepsilon}\left(|\omega(R_n^1) - n(1+\varepsilon)| > \frac{\varepsilon}{2}n\right) \leq 2\exp\left\{-n\frac{\varepsilon^2}{2(2+3\varepsilon)}\right\}, \text{ and}$$
$$\mathbb{P}_{1-\varepsilon}\left(\left|\omega(R_n^\delta) - \frac{n}{1+\varepsilon}\right| > \frac{\varepsilon}{2(1+\varepsilon)}n\right) \leq 2\exp\left\{-n\frac{\varepsilon^2}{4(1+\varepsilon)(2(1-\varepsilon^2)+\varepsilon)}\right\}$$

Therefore

$$\mathbb{P}_{1+\varepsilon}\left(\omega(R_n^1) < n\right) \leq 2\exp\left\{-n\frac{\varepsilon^2}{4(2+3\varepsilon)}\right\}, \text{ and}$$
$$\mathbb{P}_{1-\varepsilon}\left(\omega(R_n^\delta) > n\right) \leq 2\exp\left\{-n\frac{\varepsilon^2}{4(1+\varepsilon)(2(1-\varepsilon^2)+\varepsilon)}\right\}.$$

Since, both of them are summable, we conclude the Lemma. □

Some basic propositions about regular variation functions are exposed here, and the proof of them can be found in [27].

Proposition 16. *Then the following hold:*

1. *If $L \in RV_0$, we have that $\lim_{x\to\infty} L(tx)/L(x) = 1$ uniformly on each compact $t-$set of $(0,\infty)$.*

2. *For every $U(x) \in \mathrm{RV}_p$, we have that $U(x)x^{-p}$ is slowly varying. Therefore, $U(x) = x^p L(x)$, for some $L \in \mathrm{RV}_0$.*

3. *If $L \in RV_0$, then for every $\alpha > 0$, $x^\alpha L(x) \to \infty$, and $x^{-\alpha}L(x) \to 0$.*

4. *If $L \in RV_0$, then for every $\alpha > 0$, $(L(x))^\alpha$ and $(L(x))^{-\alpha}$ are slowly varying.*

5. *Let $L, L' \in RV_0$, then $L + L'$ and LL' are slowly varying.*

Also, proof in [27], Karamata's theorem, and Karamata's representation Theorem

Theorem 4 (Karamata's Theorem). *Considering the space of regular varying functions, we have that:*

(a) If $p \geq -1$ then $U \in \mathrm{RV}_p$ implies $\int_0^x U(t)dt \in \mathrm{RV}_{p+1}$ and

$$\lim_{x\to\infty} \frac{xU(x)}{\int_0^x U(t)dt} = p + 1. \tag{5.2}$$

If $p < -1$ (or if $p = -1$ and $\int_x^\infty U(s)ds < \infty$) then $U \in \mathrm{RV}_p$ implies $\int_x^\infty U(t)dt$ is finite, $\int_x^\infty U(t)dt \in \mathrm{RV}_{p+1}$ and

$$\lim_{x\to\infty} \frac{xU(x)}{\int_x^\infty U(t)dt} = -p - 1. \tag{5.3}$$

(b) If U satisfies

$$\lim_{x\to\infty} \frac{xU(x)}{\int_0^x U(t)dt} = \lambda \in (0, \infty) \tag{5.4}$$

then $U \in \mathrm{RV}_{\lambda-1}$. If $\int_x^\infty U(t)dt < \infty$ and

$$\lim_{x\to\infty} \frac{xU(x)}{\int_x^\infty U(t)dt} = \lambda \in (0, \infty) \tag{5.5}$$

then $U \in \mathrm{RV}_{-\lambda-1}$.

Proof of Lemma 9. The proof uses the same branching construction presented in Subsection 2.2.2. Since the proof is analogous to what was done there, some details will not be filled in.

Fix the tree $\widehat{\mathcal{T}} \subset \mathcal{T}_4$ defined in Subsection 2.2.2. Now, indexed in the vertices $v(i,h) \in \widehat{\mathbb{V}}$, define the regions in $[0,1) \times (0,\infty)$ and intervals in $[0,1)$ as:

$$\widetilde{R}(i,h) = \left[\frac{|i|}{4^h}, \frac{|i|+2}{4^h}\right) \times \left[\frac{1}{4^{h+1}}, \frac{1}{4^h}\right)$$
$$\widetilde{I}(i,h) = \left[\frac{|i|+1}{4^h}, \frac{|i|+2}{4^h}\right).$$

Also, for $v(0,0)$, define $\widetilde{R}(0,0) = [0,1) \times [1/4, \infty)$, and $\widetilde{I}(0,0) = [0,1)$. See Figure 2.3, where a representative of this regions in another scale happens.

To define the Branching process, fix a configuration ω_α, and define $(Z_h)_h$, where $Z_0 = 1$, and associated with it, we have the vertex $v(0,0)$. For other values of h, define inductively $Z_{h+1} = \sum_{i=1}^{Z_h} 2 \cdot \mathbf{1}\{\omega(\widetilde{R}(|v(i)|, h) = 0\}$, where $\{v(1), ..., v(Z_h)\}$ are the vertex associated with the $h-th$ generation of the branching process. Moreover, define the vertex associated with the next generation as the union of the two children in $\widehat{\mathcal{T}_4}$ of each $v \in \{v(i)\}_{i=1}^{Z_h}$ such that $\{\omega(\widetilde{R}(|v|, h) = 0\}$ happens.

As before, using the intervals $\widetilde{I}$ it is simple to show that if $\{Z_h > 0\}$ then the event $\{\mathcal{V}(\omega[4^{-h}]) \neq \emptyset\}$ happens. About probabilities, notice that the process is more homogeneous now, then:

$$\mathbb{P}_\alpha\left(\omega(\widetilde{R}(i,h)) = 0\right) = e^{-6\alpha} \text{ for } h > 0,$$
$$\mathbb{P}_\alpha\left(\omega(\widetilde{R}(0,0)) = 0\right) = e^{-4\alpha}.$$

So despite the origin, by classical branching arguments for $\alpha < \frac{\ln(2)}{6}$ the event $\{Z_h > 0, \forall h\}$ has positive probability. So clearly, by Lemma 10:

$$\mathbb{P}_\alpha\left(\mathcal{V}(\omega) \neq \emptyset\right) = \mathbb{P}_\alpha\left(\bigcap_h \{\mathcal{V}(\omega[4^{-h}]) \neq \emptyset\}\right) > \mathbb{P}\left(Z_h > 0, \forall h\right) > 0.$$

As desired. □

Proof of Billard's Theorem. Set λ be the Lesbegue measure.Fix the size of the objects to be $1 > v_1 > v_2 > ...$, and let g_n be open sets of $\mathbb{T}_d$, such that $\lambda(g_n) = v_n$. Consider χ_n the characteristic functions of the set g_n defined as:

$$\chi_n(x) = \begin{cases} 1, & x \in g_n \\ 0, & x \notin g_n. \end{cases}$$

Let $(\omega_n)_n$ be a set of i.i.d. uniform points on $\mathbb{T}_d$. Now, define:

$$O_n = \{x \in \mathbb{T}_d : x + \omega_n \in g_n\}.$$

And with that set the vacant set to be

$$\mathcal{V}_n = \mathbb{T}_d \setminus \bigcup_{i=1}^n O_n.$$

The characteristic function of the vacant set is

$$\chi(\mathcal{V}_n) = \prod_{i=1}^n (1 - \chi_i(x - \omega_n))dx,$$

and the measure of it is

$$\lambda(\mathcal{V}_n) = \int \prod_{i=1}^n (1 - \chi_i(x - \omega_n))dx.$$

Notice that, both of these are random functions.

Also we denote:

$$v_i = \lambda(g_i) = \int \chi_i(x)dx$$
$$\xi_j(x) = \chi_j * \tilde{\chi_j}(x) = \int \chi_j(x+y)\chi_j(y)dy$$
$$\tau = \lambda * \tilde{\lambda},$$

that is:

$$\int f(x)d\tau(x) = \int\int f(x-y)dxdy.$$

Using that notation, we have that:

$$\mathbb{E}\left(\lambda(\mathcal{V}_n)\right) = \int \prod_{i=1}^{n} \mathbb{E}\left((1-\chi_i(x-\omega_i)\right) dx = \prod_{i=1}^{n}(1-v_i)$$
$$\mathbb{E}\left(\lambda(\mathcal{V}_n)^2\right) = \int\int \prod_{i=1}^{n} \mathbb{E}\left((1-\chi_i(x-\omega_i))(1-\chi_i(y-\omega_i))\right) dxdy$$
$$= \int\int \prod_{i=1}^{n}(1-2v_i+\xi_i(x-y))dxdy$$
$$= \int \prod_{i=1}^{n}(1-2v_i+\xi_i(x))d\tau(x)$$

Now, observe that $v_i < v_1 < 1$, and $\xi_i < v_i$. So, for any the case, we have that:

$$1-2v_i+\xi_i \le (1+\xi_i)(1+C_iv_i^2)(1-v_i)^2,$$

where $C_i = \max\left\{1, \frac{2}{1-v_i}\right\}$. Since $1 > v_1 > v_2 > ...$ we can take $C_i < \frac{2}{1-v_1}$.

Now, supposing that $\sum_i v_i^2 < \infty$, we can find $C = C(v_1) < \infty$ such that:

$$\mathbb{E}\left(\lambda(\mathcal{V}_n)^2\right) \le C\left[\mathbb{E}\left(\lambda(\mathcal{V}_n)\right)\right]^2 \int \prod_{i=1}^{n}(1+\xi_i(x))d\tau(x)$$

therefore:

$$\mathbb{P}\left(\lambda(\mathcal{V}_n) > 0\right) \ge \frac{\left(\mathbb{E}\left(\lambda(\mathcal{V}_n)\right)\right)^2}{\mathbb{E}\left(\lambda(\mathcal{V}_n)^2\right)}$$
$$\ge C^{-1}\left[\int \prod_{i=1}^{\infty}(1+\xi_i(x))d\tau(x)\right]^{-1}$$

If for instance:

$$\int \prod_{i=1}^{\infty}(1+\xi_i(x))d\tau(x) < \infty$$

Then, we have that $\lim_{n\to\infty} \mathbb{P}\left(\lambda(\mathcal{V}_n)=0\right) < 1$, and we do not cover. To find the Billard's condition lets use the following lemma

Lemma 21. *For every set of numbers $\{a_n\}_n$, and for every $K>0$, we have that*

$$\begin{aligned}\prod_{n=1}^{K}(1+a_n) &= a_K\prod_{n=1}^{K-1}(1+a_n) + \prod_{n=1}^{K-1}(1+a_n)\\ &= \sum_{n=1}^{K-1} a_n \prod_{i=1}^{n-1}(1+a_i).\end{aligned}$$

Therefore, we have that

$$\sum_{n=1}^{\infty}\int \xi_n(x)\prod_{i=1}^{n-1}(1+\xi_i(x))d\tau(x) < \infty$$

Implies:

$$\int \prod_{i=1}^{\infty}(1+\xi_i(x))d\tau(x) < \infty$$

Since $\xi_n \le v_n$ for every $x \in \text{supp}\{\xi_n\}$, we have that:

$$\begin{aligned}\prod_{i=1}^{n-1}(1+\xi_i) &\le \exp\{v_1+...+v_{n-1}\}\\ \int \xi_n(x)d\tau(x) &\le v_n\lambda(\text{supp}\{\xi_n\})\\ &= v_n^2.\end{aligned}$$

Therefore, we have that:

$$\sum_{n=1}^{\infty} v_n^2 \exp\{v_1+...+v_n\} < \infty \implies \mathbb{P}\left(\mathcal{V}\neq\emptyset\right) > 0.$$

□

5.2.2 Proving theorems in the discrete time case

This subsection shows that theorems A, B, C, and D, are also true for the discrete cover model. For this, we need to show that the cover time of the problem in the continuous and in the discrete case have the same limit in distribution.

The equivalence of the statements is due to Proposition 17, taking the pair (a_n, b_n) as $(f^{-1}(n), 0)$ or $(n, \ln n)$.

Proposition 17. *Let $a_n, b_n > 0$, two sequences of numbers, with $a_n \to \infty$. Then, we have that:*

$$\frac{T_n}{a_n} - b_n \xRightarrow{D} Y, \text{ if, and only if, } \frac{\tau_n}{a_n} - b_n \xRightarrow{D} Y.$$

Proof of Proposition 17. The proof follows by using concentration inequalities for the difference of τ_n and T_n. To start, by the Poisson construction, set

$$T_n = \sum_{i=1}^{\tau_n} \eta_i,$$

where η_i is a sequence of i.i.d. exponential random variables with rate 1. By Chebyshev's inequality, we have

$$\mathbb{P}\left(|k - \sum_{i=1}^{k} \eta_i| > k^{3/4}\right) = \mathbb{P}\left(|\sum_{i=1}^{k}(\eta_i - 1)| > k^{3/4}\right) \leq k^{-1/2}. \tag{5.6}$$

Since the cover times τ_n and T_n are diverging in n, we have that condition on the value of τ_n, T_n will not oscillate $(\tau_n)^{3/4}$ from this value. More than this, by symmetry we also have that given T_n, the value of τ_n will not oscillate more than $T_n^{3/4}$ of this value.

To control both of this inequalities, we will make standard arguments using any open set $A \subset \mathbb{R}$, define A_n such that:

$$A_n = \{x \in A : a_n(x + b_n)\left[1 + a_n^{-1/4}(x + b_n)^{-1/4}\right] \in A\}.$$

We have that $A_n \uparrow A$, and:

$$\mathbb{P}\left(T_n \in a_n(A + b_n)\right) \geq \mathbb{P}\left(T_n \in a_n(A + b_n), \tau_n \in a_n(A_n + b_n)\right).$$

Then:

$$\mathbb{P}\left(T_n \in a_n(A + b_n)\right) \geq \mathbb{P}\left(|T_n - \tau_n| < (\tau_n)^{3/4}, \tau_n \in a_n(A_n + b_n)\right).$$

Taking the limit, we can use the independence of the exponential variables, together with equation (5.6), to show that:

$$\liminf_{n\to\infty} \mathbb{P}\left(T_n \in a_n(A + b_n)\right) \geq \mathbb{P}\left(Y \in A\right).$$

The other affirmation, have an analogous proof, just substitute T_n by τ_n. □

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