Singular Lagrangian Torus Fibrations on the smoothing of algebraic cones

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Abstract

In this thesis, we present a construction of special Lagrangian fibrations on the smoothing of Gorenstein singularities. This construction was initially described by Gross [30], with further details studied under the perspective of the SYZ conjecture of mirror symmetry in subsequent works [17, 18, 42]. Our approach presents a fresh description of the Gross fibration using global coordinates tied to Altmann's characterization of the smoothing [6]. This enables us to provide alternative proofs for known facts concerning these fibrations, construct a convex base diagram generalizing those constructed by Symington [51], and prove a theorem that completely characterizes its shape. Moreover, we use techniques from [48] to recover the potential for certain monotone fibers derived in [42], and discuss their non-displaceability by precisely describing the cases where there exist local systems on the monotone Lagrangian for which their Floer homology are non-vanishing. We provide several interesting examples along the way and end by discussing some future developments for this project.

More specifically, given a lattice polytope $Q \subset \mathbb{R}^n$, Altmann's theorem in [6] states that the versal deformation space of the affine toric variety Y_σ related to $\sigma = C(Q) = \{\lambda(q,1) \in \mathbb{R}^{n+1} | \lambda \in \mathbb{R}{\geq} 0, q \in \mathbb{Q}\} \subset \mathbb{R}^{n+1}$ can be described by the Minkowski decomposition of the polytope Q. Under certain conditions on Q, we can derive a smooth deformation Y_ϵ of Y_σ utilizing Altmann's theorem. In this thesis, we consider Y_ϵ inside some $\mathbb{C}^N$ and construct a complex fibration on Y_ϵ, with general fiber $(\mathbb{C}^*)^n$ and a finite number of singular fibers, each one associated with a term in the Minkowski decomposition of Q. Moreover, the neighborhood of each singularity is explicitly described here in terms of its corresponding component in the Minkowski decomposition using global coordinates. We construct a singular Lagrangian torus fibration from this complex fibration, as done in [8, 9, 30, 42], and show that it is represented by a convex base diagram whose image is the dual cone of $C(Q)$. These fibrations contain a 1-parameter family of monotone Lagrangian tori, which are important to the study of symplectic geometry. Using the wall-crossing formula [48], we describe the potential associated with this family in terms of the Minkowski decomposition of Q, thereby recovering the result of [42], and study their Floer homology.

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Chapter 1

Introduction

Symplectic geometry, with its rich structure and profound connections to dynamics and other branches of mathematics and physics, has emerged as a central field in modern mathematics. In particular, it plays a vital role in the ongoing developments in homological mirror symmetry, a field that connects algebraic-complex geometry and symplectic geometry. This thesis aims to study singular Lagrangian fibrations on the smoothing of algebraic cones, a subject of special interest due to its relation with the Strominger-Yau-Zaslow (SYZ) conjecture in mirror symmetry and the theory of almost toric manifolds.

A symplectic manifold is a smooth manifold M equipped with a non-degenerate closed two-form, usually denoted by ω, i.e., $d\omega = 0$ and $\omega^{\#} : TM \to T^*M$ sending v to $\omega(v, \cdot)$ is an isomorphism on every fiber. Studying their properties and structure-preserving transformations, or morphisms, enhances comprehension of the manifold's inherent dynamics and topology.

Symplectomorphisms are the natural class of morphisms in the category of symplectic manifolds. They are diffeomorphisms between two symplectic manifolds, (M_1, ω_1) and (M_2, ω_2), that preserve the symplectic form: $\phi^*(\omega_2) = \omega_1$. The significance of symplectomorphisms lies in their ability to preserve the symplectic structure, thereby enabling a meaningful equivalence of different symplectic manifolds.

To further understand the symplectic universe, we examine submanifolds within symplectic manifolds. These submanifolds are essential in studying global invariants and rigidity properties. We will particularly focus on Lagrangian submanifolds, which are n-dimensional smooth manifolds in a symplectic manifold (M^{2n}, ω) satisfying $\omega|_L = 0$. This interesting class of submanifolds is central to our exploration due to their significance in symplectic geometry and their intricate role in homological mirror symmetry.

In 1980, Arnold conjectured a fascinating property related to Lagrangian submanifolds, involving Hamiltonian symplectomorphisms. Given a smooth function $H : M \to \mathbb{R}$, we associate to H a vector field V_H defined as the unique vector field such that $\omega(V_H, \cdot) = dH$. We call H a Hamiltonian and V_H a Hamiltonian vector field. The flow ϕ_H^t of a Hamiltonian vector field is known as a Hamiltonian isotopy. One can prove that ϕ_H^1 is a symplectomorphism called a Hamiltonian symplectomorphism.

The Hamiltonian symplectomorphisms form a group denoted by $\mathrm{Ham}(M, \omega)$.

The Arnold-Givental conjecture reads as follows: given a symplectic manifold (M^{2n}, ω) and a closed Lagrangian submanifold $L \subset M$, let $\psi \in \mathrm{Ham}(M^{2n}, \omega)$ be a Hamiltonian diffeomorphism such that $\psi(L)$ and L intersect transversally, denoted by $\psi(L) \pitchfork L$. In these conditions, the intersection $L \cap \psi(L)$ is a finite set of points. The conjecture then states that $|\psi(L) \cap L|$ is bounded below by $\sum \dim H^i(L, \mathbb{Z}/2)$, the sum of the dimensions of the cohomology groups of L with coefficients in $\mathbb{Z}/2$.

While the conjecture is not universally valid, it has been shown to hold under certain conditions. To approach this conjecture, we delve into Floer's (co)homology theory for Lagrangian submanifolds. This pioneering methodology, first introduced by Andreas Floer in the late 1980s, furnishes evidence in favor of the Arnold conjecture in particular situations [24]. The Floer complex is generated by the intersection points in $L_0 \cap L_1$, with its differential defined by the count of J-holomorphic discs between them. A more detailed discussion is presented in Section 2.1.

Floer (co)homology expands the machinery of Morse theory into the infinite-dimensional setting of the loop space of a manifold. Since its inception, this theory has been extended and refined by various researchers, notably including Oh [46]. Moreover, Fukaya, Oh, Ohta, and Ono have developed a sophisticated method that assists in defining Floer cohomology in more challenging circumstances [27]. This technique has emerged as a formidable tool for investigating a range of problems in symplectic geometry.

In the remarkable contributions of Fukaya, Oh, Ohta, and Ono [27], they notably elucidated the structure of the Floer complex of a Lagrangian submanifold as a curved or obstructed A_∞-algebra. This structure, characterized by non-vanishing higher Massey products, provides a more intricate understanding of the interaction between algebraic and geometric aspects of symplectic manifolds.

At the core of this discussion is the moduli space of J-holomorphic discs, specifically those with boundaries in the selected Lagrangian submanifold. In this thesis, we will be concerned with the counts of holomorphic discs of Maslov index 2 that intersect a given point on the Lagrangian (refer to Section 3.3). This particular aspect has a direct link to the degree 0 part of the obstruction term $\mathfrak{m}_0$, as detailed in the Fukaya-Oh-Ohta-Ono framework.

Symplectic geometry has seen tremendous advancements with the introduction of the Fukaya category by Fukaya [25] and Kontsevich's proposal of homological mirror symmetry [40]. Notably, the Fukaya category has deepened our understanding of the structure of symplectic manifolds, while homological mirror symmetry has provided key insights into the dualities between symplectic and complex geometries, thereby enriching our comprehension of Floer cohomology.

Furthermore, the study of open string theory and D-branes, as elucidated in pivotal physics papers [37, 38], has offered valuable cross-disciplinary perspectives. These contributions underscore the relevance of theoretical physics in mathematical exploration, particularly in our discussion about Floer cohomology. As a result, the theory of Floer cohomology continues to develop into an increasingly intricate and multifaceted field.

Mirror symmetry, a conjectured equivalence between certain pairs of Calabi-Yau manifolds, emerged from physicists' quest for a unifying theory of fundamental forces [13,14,29]. In the context of string theory, this unification is proposed to be achieved by considering a spacetime constructed as a product of a four-dimensional Minkowski space and a compact six-dimensional Calabi-Yau manifold. The latter serves as the arena for string dynamics, potentially leading to observable phenomena in spacetime. Mirror symmetry claims that for specific pairs of Calabi-Yau manifolds $(X, \check{X})$, a conjectured bijective correspondence exists between their topological and geometric properties, reflecting certain symmetries. Notably, the Hodge diamonds of X and $\check{X}$ mirror each other. This symmetry has profound implications in geometry, topology, and mathematical physics.

Kontsevich [40] proposed a mathematical formulation of mirror symmetry through an equivalence between the Fukaya category of X and the derived category of coherent sheaves on $\check{X}$. This conjecture, known as homological mirror symmetry, suggests that the symplectic geometry of X is mirrored in the complex geometry of $\check{X}$, and has been supported in many cases beyond the Calabi-Yau setting. Despite this, the conjecture remains largely unproven in its most general form due to the complexities involved in constructing the necessary categories and establishing the isomorphisms between them. Furthermore, identifying the mirror of a given manifold is a complex task that requires deep geometric and algebraic insight.

The Strominger-Yau-Zaslow (SYZ) conjecture provides a potential avenue to address these challenges in validating homological mirror symmetry. The SYZ conjecture proposes a more geometrical and tangible interpretation of mirror symmetry based on string theory principles. It suggests that two Calabi-Yau manifolds, which are mirrors of each other, should both admit a special Lagrangian torus fibration over the same base, indicating a dual relationship. This conjecture led to numerous subsequent refinements, such as the necessity of considering singular Lagrangian fibrations. These refinements, some of which goes beyond the Calabi-Yau setting, have been developed through extensive work over the years [1,2,4,8,9,18,20,26,31–36,41,42,52].

In the present thesis, we describe the smoothing of certain Gorenstein singularities based on [6]. We then construct a special Lagrangian torus fibration in the complement of a specific type of divisor as in [8,9]. Some of the results and examples in this thesis align with those found in [18,42], which also investigate Altmann's smoothing of cones and Lagrangian torus fibrations. However, the proofs of the main theorems are different. In particular, we employ an embedding of the Y_ϵ into a certain $\mathbb{C}^N$, allowing us to describe both a complex fibration and the singular Lagrangian fibration using global coordinates tied to the components of the Minkowski decomposition. One of the main contributions of the thesis is to provide a local model for the base of these Lagrangian torus fibrations, akin to Symington's almost toric base diagrams. This thesis provides several illustrative examples.

1.1 Main Results

Given a lattice polytope $Q \subset \mathbb{R}^n$, we can consider $\sigma = C(Q) = \{\lambda(q,1) \in \mathbb{R}^{n+1} | \lambda \in \mathbb{R}_{\geq 0}, q \in Q\}$, the cone of Q in $\mathbb{R}^{n+1}$. Since σ is a polyhedral rational cone, we can associate an algebraic toric variety Y_σ, singular in general and known as a Gorenstein singularity. In [6], Altmann showed that the versal deformation space of Y_σ can be described in terms of a maximal Minkowski decomposition of the polytope Q. Using Altmann's result, we obtain a deformation Y_ϵ of Y_σ. In this setup, Gross [30] gave a recipe to construct special Lagrangian fibrations, and in [18, 42] the authors studied the special torus Lagrangian fibration with singularities resulting from [30]. This construction can be understood using a complex fibration $\pi : Y_\epsilon \to \mathbb{C}$, where the preimage of a point is isomorphic to $(\mathbb{C}^*)^n$ except at k singular points, where k is the number of components on the Minkowski decomposition. In Theorem 1.1.1, under certain conditions, we describe the local behavior of the singularities of π, using global coordinates related to the Minkowski decomposition, which guarantees that Y_ϵ is smooth.

Theorem 1.1.1. *Given some conditions in Q, a neighborhood of the singular fibres of $\pi : Y_\epsilon \to \mathbb{C}$ is biholomorphic to the preimage of a neighborhood of 0 under the map $x_0 \cdots x_m$ defined in $\mathbb{C}^{m+1} \times (\mathbb{C}^*)^{n-m}$, for some m. In particular, the conditions imply that Y_ϵ is smooth.*

In [8, 9], Auroux constructed a special singular Lagrangian fibration in the complement of a divisor of $\mathbb{C}^n$. The mentioned fibration is constructed from the complex fibration $\widetilde{\pi} : \mathbb{C}^k \to \mathbb{C}$ given by $\widetilde{\pi}(x_1, \ldots, x_k) = x_1 \cdots x_k$. We showed that the local model of the singular fibres of the complex fibration π is of the form $\mathbb{C}^k \times (\mathbb{C}^*)^{n+1-k} \to \mathbb{C}^k \xrightarrow{\widetilde{\pi}} \mathbb{C}$. So the analogous approach in [8, 9] produces a singular Lagrangian fibration as in [18, 42]. To visualize symplectic aspects of this singular Lagrangian fibration, we describe it by a convex base diagram with cuts, which can be thought of as analogous to the moment map of a toric action, and is a generalization of the base diagrams constructed by Symington [51] to describe almost toric fibrations. In particular, we show that the image of the diagram is the polyhedral dual to σ.

Theorem 1.1.2. *With the same assumptions as in Theorem 1.1.1, there is a singular Lagrangian fibration in Y_ϵ that can be represented by a convex diagram $Y_\epsilon \to \mathbb{R}^{n+1}$ with cuts, such that the image of the convex diagram is $\sigma^\vee$ (the dual cone of σ). In addition, there is a one-parameter family of monotone Lagrangian torus fibres in Y_ϵ.*

These results allow us to carry out the following immediate applications:

- Using the wall-crossing formula [48], we can describe the potential of the one-parameter family of monotone fibres in terms of the Minkowski decomposition of Q. This potential was also described in [42] using a different technique.

- We present a criteria to determine the non-vanishing of $HF(L, L)$ for the monotone Lagrangian fibers and hence provide several examples of families of non-displaceable monotone tori.

- When Q is itself a polytope associated with the fan of a Fano algebraic variety, we can immediately describe a singular Lagrangian skeleton for which the smoothing Y_ϵ retracts to. Moreover, there is a natural compactification of Y_ϵ, and by the work of [22] the potential of the monotone Lagrangian on the compactification $\overline{Y_\epsilon}$ of Y_ϵ has a term associated with the relative Gromov Witten invariant between $\overline{Y_\epsilon}$ and the compactifying divisor. This term can be readily identified in the potential for the Lagrangian in Y_ϵ, which is described in terms of the Minkowski decomposition of Q.

Some future developments arise naturally from this work. In [20], the authors discussed homological mirror symmetry for the conifold beginning with a singular Lagrangian fibration as in this thesis. In a work in progress with Diogo and Vianna, analogous homological mirror symmetry results for Y_ϵ. Also, we hope to be able to use the fibration to describe a Weinstein structure on Y_ϵ and work towards describing symplectic homology and Wrapped Floer homology for these spaces. We also want to describe a Gelfand-Cetlin fibration in the sense of [50] originated as a limit of the singular Lagrangian fibrations described in this thesis. These limits can be thought of as moving the singular fibres toward the boundary of the convex base diagram with cuts. We intend to provide some applications using these Gelfand-Cetlin fibrations.

In Chapter 2, we give some background in toric geometry in the algebro-geometric framework and singular Lagrangian fibrations. In Chapter 3, we present our description of the singular Lagrangian fibration in Y_ϵ. In Chapter 4 we present future research developments

Chapter 2

Background

In Section 2.1 we follow [10] to present an introductory exposition to Lagrangian Floer homology. In Section 2.2 we discuss Toric Varieties. Most parts of the definitions are classical and taken from [21]. In Section 2.3 we comment on Lagrangian fibrations, as in [51], and later we provide definitions for what we will call restricted almost toric fibrations.

2.1 Lagrangian Floer homology

The aim of this section is to present basic definitions and results about Lagrangian Floer homology. For now on, we fix (M, ω) a symplectic manifold. We begin this section by introducing the concept of the Floer complex. Given two Lagrangian submanifolds L_0 and L_1 intersecting transversally, we define the Floer complex $CF^*(L_0, L_1)$ as the free Λ-module generated by the intersection points of L_0 and L_1:

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda \cdot p.$$

In this context, Λ represents the Novikov field, which is defined as follows:

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R},\ \lim_{i \to \infty} \lambda_i = +\infty \right\},$$

where $\mathbb{K}$ is the ground field, usually $\mathbb{C}$, or $\mathbb{Z}/2$.

One can consider a richer version of the Fukaya category, whose objects are Lagrangian submanifolds equipped with local systems, i.e., flat vector bundles $\mathcal{E} \to L$ with unitary holonomy (over the Novikov field over $\mathbb{K} = \mathbb{C}$). In this situation, the Floer complex is defined by:

$$CF((L_0, \mathcal{E}_0), (L_1, \mathcal{E}_1)) = \bigoplus_{p \in L_0 \cap L_1} \hom(\mathcal{E}_0|_p, \mathcal{E}_1|_p).$$

The Floer complex is endowed with a differential used to define the cohomology, denoted by ∂. Its construction relies heavily on the concept of J-holomorphic curves of certain Maslov index.

In order to give a clear exposition, we turn our attention to the Maslov index, denoted as μ. The Maslov index is a map:

$$\mu : \pi_2(M, L) \to \mathbb{Z}$$

which, for a map $u : (\mathbb{D}^2, S^1) \to (M, L)$ representing a class $[u] \in \pi_2(M, L)$, is defined as follows: We trivialize the pullback of the tangent bundle u^*TM over $\mathbb{D}^2$, thereby obtaining a trivial rank $2n$ bundle. Restricting the tangent bundle TL to S^1 under this trivialization produces a loop γ in the Lagrangian Grassmannian, denoted as $LGr(n)$. The fundamental group of $LGr(n)$ is isomorphic to $\mathbb{Z}$, and thus, we can associate an integer with $[\gamma] \in \pi_1(LGr(n))$. This integer is referred to as the Maslov index of $[u]$, $\mu([u])$. Subsequently, we define the minimal Maslov number N_L as the smallest positive integer in the image of the map μ, or ∞ if the Maslov index μ is identically zero.

Delving deeper into the construction of Floer (co)homology, we particularly consider monotone Lagrangians that have a minimal Maslov number of 2 or greater. We say that a Lagrangian submanifold $L \subseteq M$ is monotone if there exists a fixed positive constant $\lambda \in \mathbb{R}_{>}0$, such that it satisfies the following condition:

$$\int_A \omega = \lambda \cdot \mu_L(A)$$

for every $A \in \pi_2(M, L)$. Here, $\int_A \omega$ refers to the symplectic area of the class A, and $\mu_L(A)$ is the Maslov class, a topological invariant. This monotonicity condition equates these two distinct measures, thus establishing an essential interplay between the symplectic and topological properties of $L \subset M$. From now on, we restrict our attention to monotone Lagrangians with $N_L \geq 2$.

Before we describe the differential ∂, let's quickly review the concept of an almost complex structure. On a smooth manifold M, an almost complex structure is an automorphism $J : TM \to TM$ such that $J^2 = J \circ J = -\mathrm{Id}$. For a symplectic manifold (M, ω), an almost complex structure is said to be ω-tame if it satisfies $\omega(v, Jv) > 0$ for every nonzero $v \in TM$. Furthermore, we say that it is compatible with ω if it is ω-tamed and satisfies $\omega(Jv, Jw) = \omega(v, w)$. We denote the space of ω-compatible almost-complex structures by $\mathcal{J}(M, \omega)$.

The Floer differential, represented as $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$, is defined by counting J-holomorphic strips in (M, J), where J is an ω-compatible almost-complex structure. More specifically, given intersection points $p, q \in L_0 \cap L_1$, the coefficient of q in ∂p is calculated by considering the moduli space of maps $u : \mathbb{R} \times [0, 1] \to M$, such that u is a solution to the Cauchy-Riemann equation $\bar{\partial}_J u = 0$, meaning that these maps represent J-holomorphic strips with boundary on L_0 and L_1. The Cauchy-Riemann equation can be written in coordinates $(s, t) \in \mathbb{R} \times [0, 1]$:

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0, \tag{2.1}$$

and the maps u are subject to the boundary conditions

$$\begin{cases} u(s, 0) \in L_0 \text{ and } u(s, 1) \in L_1 \ \forall s \in \mathbb{R}, \\ \lim_{s \to +\infty} u(s, t) = p, \ \lim_{s \to -\infty} u(s, t) = q, \end{cases} \tag{2.2}$$

and the finite energy condition

$$E(u) = \int u^*\omega = \int\int \left|\frac{\partial u}{\partial s}\right|^2 ds\ dt < \infty. \tag{2.3}$$

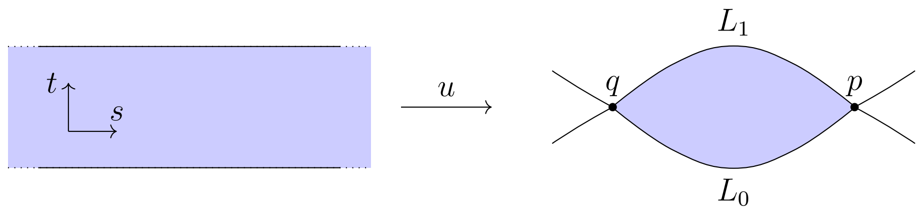


Figure 2.1: Pseudo-holomorphic strip u contributing to the Floer differential on $CF(L_0, L_1)$.

The set of solutions to equations (2.1), (2.2), and (2.3) representing a given homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$ forms a space, which we denote as $\widehat{\mathcal{M}}(p, q; [u], J)$. Furthermore, if we consider the quotient of this space under the action of $\mathbb{R}$ by reparametrization, we obtain a new space denoted as $\mathcal{M}(p, q; [u], J)$.

We now are going to extend the definition of the Maslov index for the case of strips $u : \mathbb{R} \times [0,1] \to (M, L_0 \cup L_1)$ as in (2.2). Since $\mathbb{R} \times [0,1]$ is contractible, we can fix a trivialization in u^*TM and using this trivialization view $l_0 = u^*|_{\mathbb{R}\times\{0\}}TL_0$ and $l_1 = u^*|_{\mathbb{R}\times\{1\}}TL_1$ as paths oriented with s going from $+\infty$ to $-\infty$ in $LGr(n)$. The Maslov index of u is then the number of times (counting with signs and multiplicities) at which $l_0(s)$ is not transversal to $l_1(s)$.

The boundary value problem described by equations (2.1), (2.2), and (2.3) constitutes a Fredholm problem. The Fredholm index of this problem can be calculated via the Maslov index. In the scenario where all solutions to (2.1), (2.2), and (2.3) are regular, the space of solutions $\widehat{\mathcal{M}}(p, q; [u], J)$ forms a smooth manifold whose dimension corresponds to the Fredholm index $\mathrm{ind}([u])$. Moreover, three critical aspects require careful attention to ensure the proper definition of Floer cohomology: transversality, compactness, and orientability.

Assuming that all the issues related to transversality, compactness, and orientability are properly addressed, the moduli space $\mathcal{M}(p, q; [u], J)$ forms a compact, oriented 0-manifold in the case where $\mathrm{ind}([u]) = 1$. With these conditions fulfilled, we can tentatively define the Floer differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ as the Λ-linear map given by

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u]: ind([u])=1}} (\#\mathcal{M}(p, q; [u], J)) T^{\omega([u])} q. \tag{2.4}$$

In the case where the Lagrangians are equipped with local systems, we fix objects $(L_0, \mathcal{E}_0)$, $(L_1, \mathcal{E}_1)$, intersection points p, q, and a homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$. Using parallel transport along the boundary of $[u]$, we obtain isomorphisms

$\gamma_0 \in \hom(\mathcal{E}_0|_q, \mathcal{E}_0|_p)$ and $\gamma_1 \in \hom(\mathcal{E}_1|_p, \mathcal{E}_1|_q)$. Given $\rho \in \hom(\mathcal{E}_0|_p, \mathcal{E}_1|_p)$, we define $\eta_{[u],\rho} = \gamma_1 \cdot \rho \cdot \gamma_0 \in \hom(\mathcal{E}_0|_q, \mathcal{E}_1|_q)$. We define the differential by:

$$\partial(\rho) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u]: ind([u])=1}} (\#\mathcal{M}(p,q;[u],J)) T^{\omega([u])} \eta_{[u],\rho}. \tag{2.5}$$

Here, $\#\mathcal{M}(p,q;[u],J) \in \mathbb{Z}$ (or $\mathbb{Z}_2$) signifies the signed (or unsigned) total count of points within the moduli space of pseudo-holomorphic strips linking points p and q in the class denoted by $[u]$. Also, the notation $\omega([u]) = \int u^*\omega$ is used to represent the symplectic area of the mentioned strips.

In most cases, to attain transversality, we modify the Cauchy-Riemann equation by incorporating a perturbation term. Consequently, the Floer differential is, in reality, counting the perturbed pseudo-holomorphic strips that link the perturbed points of intersection of L_0 and L_1. Floer demonstrated the subsequent outcome when $\mathbb{K} = \mathbb{Z}_2$:

Theorem 2.1.1 ([24]). *Assume that $[\omega].\pi_2(M, L_0) = 0$ and $[\omega].\pi_2(M, L_1) = 0$. Furthermore, when char$(\mathbb{K}) \neq 2$, assume that both L_0 and L_1 are oriented and are equipped with spin structures. Under these conditions, the Floer differential ∂ is well-defined and satisfies $\partial^2 = 0$. The resulting Floer cohomology $HF(L_0, L_1) = H^*(CF(L_0, L_1), \partial)$, up to isomorphism, does not depend on the chosen almost-complex structure and remains invariant under Hamiltonian isotopies of L_0 or L_1.*

Remark 2.1.2. *In the case of exact symplectic manifolds, where $\omega = d\lambda$, we consider a Lagrangian submanifold L to be exact if there exists a smooth function $f \in C^\infty(L, \mathbb{R})$ for which $\lambda|_L = df$ holds. Then, by applying Stokes' theorem, we obtain $[\omega]\ \pi_2(M, L) = 0$.*

The proof of $\partial^2 = 0$ relies on the compactification of the moduli spaces of J-holomorphic strips, which is governed by Gromov's compactness theorem. According to this theorem, any sequence of J-holomorphic curves with uniformly bounded energy admits a subsequence that converges, up to reparametrization, to a nodal tree of J-holomorphic curves. The components of the limit curve are obtained as limits of different reparametrizations of the given sequence of curves, focusing on the different regions of the domain in which a non-zero amount of energy concentrates ("bubbling"). In the case of a sequence of $J-$holomorphic strips $u_n : \mathbb{R} \times [0,1] \to M$ with boundaries on Lagrangian submanifolds L_0 and L_1, there are three types of phenomena to consider:

1. *Strip breaking:* energy concentrates at either end $s \to \pm\infty$, i.e., there exists a sequence $a_n \to \pm\infty$ such that the translated strips $u_n(s - a_n, t)$ converge to a non-constant limit strip. See Figure 2.2.

2. *Disc bubbling:* energy concentrates at a point on the boundary of the strip, where suitable rescalings of u_n converge to a J-holomorphic disc in M with the boundary entirely contained in either L_0 or L_1. See Figure 2.3.

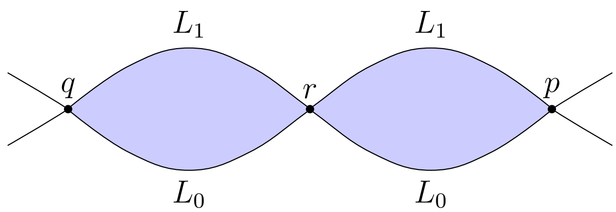


Figure 2.2: Strip breaking.

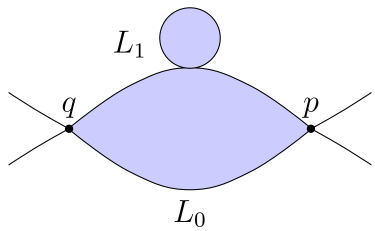


Figure 2.3: Disc bubbling.

3. *Sphere bubbling:* energy concentrates at an interior point of the strip, where suitable rescalings of u_n converge to a J-holomorphic sphere in M. See Figure 2.4.

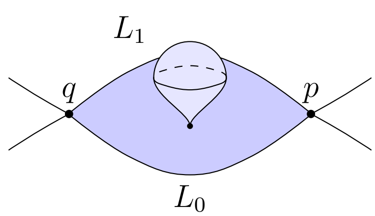


Figure 2.4: Sphere bubbling.

Strip breaking is the key geometric factor in the proof that the Floer differential squares to zero, provided that disc bubbling can be excluded. Within the hypothesis of Theorem 2.1.1, the absence of disc and sphere bubbles is ensured by the condition that $[\omega].\pi_2(M, L_i) = 0$. A broader context in which $\partial^2 = 0$ is when bubbling can be excluded for index reasons, for instance, when all bubbles are guaranteed to have a Maslov index greater than 2. Sphere bubbling does not interfere on ∂^2 for generic J, due to dimensional arguments.

The Floer cohomology $HF^*(L, L)$ is defined as $HF^*(L, \phi_H(L))$, where $\phi_H(L)$ is Hamiltonian isotopic to L, and is chosen so that $\phi(L) \pitchfork L$. This is well defined because Floer cohomology is invariant under Hamiltonian perturbations. Under appropriate conditions, we can see that the Lagrangian Floer cohomology is isomorphic to the Morse homology, which in turn, is isomorphic to the singular homology.

Specifically, when L is a monotone Lagrangian submanifold in a monotone symplectic manifold M, and $N_L \geq 2$, the Floer cohomology $HF^*(L, L)$ is well defined, as detailed in Proposition 5.1.17 in [27]. In such cases, disc bubbles either do not appear, or they appear in pairs that cancel each other out. Under these conditions, Oh [46, 47] has shown that there exists a spectral sequence which starts with the Morse cohomology and converges to the Floer cohomology.

We note that the differential can be seen as an operation between two Lagrangians, defined by counting J-holomorphic strips. Similarly, we can construct operations between multiple Lagrangians. These operations provide us with additional insights into our original symplectic variety. They satisfy a specific relation and are crucial in the definition of the Fukaya category.

Suppose L_0, L_1, L_2 are three Lagrangian submanifolds of (M, ω). We make the assumption that these submanifolds intersect transversely and do not bound any pseudo-holomorphic discs. We can establish a product operation on their complexes in the following manner:

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_2)$$

Consider the intersection points $p_1 \in L_0 \cap L_1$, $p_2 \in L_1 \cap L_2$, and $q \in L_0 \cap L_2$. The coefficient of q in the product $p_1 \cdot p_2$ is determined by a weighted count of pseudo-holomorphic discs within M, with boundary in $L_0 \cup L_1 \cup L_2$ and corners situated at p_1, p_2, q.

Let D be the closed unit disc minus three boundary points, given an almost-complex structure J on M and a homotopy class $[u]$, we denote by $\mathcal{M}(p_1, p_2, q; [u], J)$ the space of finite energy J-holomorphic maps $u : D \to M$ which extend continuously to the closed disc, mapping the boundary arcs from z_0 to z_1, z_1 to z_2, z_2 to z_0 to L_0, L_1, L_2 respectively, and the boundary punctures z_0, z_1, z_2 to q, p_1, p_2 respectively in the given homotopy class $[u]$.

Analogous to the strip situation, the expected dimension of $\mathcal{M}(p_1, p_2, q; [u], J)$ is dictated by the index of the linearized Cauchy-Riemann operator $D_{\bar{\partial}_J}$. The index can be expressed via the Maslov index, as done previously: now we concatenate the paths given by the tangent spaces to L_0, L_1, L_2 moving counterclockwise along the boundary of u, along with the appropriate canonical short paths at p_1, p_2, q, to obtain a closed loop in $LGr(n)$. The Maslov index of this loop equals $\mathrm{ind}(u)$. If $c_1(TM)$ is 2-torsion and the Maslov classes of L_0, L_1, L_2 vanish, then after choosing graded lifts of the Lagrangians we have $\mathbb{Z}$-gradings on the Floer complexes, and one can verify that

$$\mathrm{ind}(u) = \deg(q) - \deg(p_1) - \deg(p_2). \tag{2.6}$$

Assuming transversality is satisfied, the moduli spaces $\mathcal{M}(p_1, p_2, q; [u], J)$ are smooth. Moreover, if $char(\mathbb{K}) \neq 2$, we assume additional orientations and spin structures have been selected for L_0, L_1, L_2 in order to establish the orientations of the moduli spaces. Subsequently, we can establish the following:

Definition 2.1.3. The *Floer product* is the $\Lambda-$linear map $CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$ defined by

$$p_2 \cdot p_1 = \sum_{\substack{q \in L_0 \cap L_2 \\ [u]: ind([u])=0}} (\#\mathcal{M}(p_1, p_2, q; [u], J)) T^{\omega([u])} q. \tag{2.7}$$

Similar to the prior discussion, this approach typically requires adjustments, including the incorporation of domain-dependent almost-complex structures and Hamiltonian perturbations, to ensure transversality. Also, it can be enhanced to take into account the presence of local systems on the Lagrangians. For the moment, let's work under the assumption that transversality is achieved without needing additional perturbations and focus on exploring the characteristics of the Floer product.

Proposition 2.1.4. *Assuming that $[\omega] \cdot \pi_2(M, L_i) = 0$ for all i, the Floer product adheres to the Leibniz rule (with appropriate signs) in respect to the Floer differentials:*

$$\partial(p_2 \cdot p_1) = \pm(\partial p_2) \cdot p_1 \pm p_2 \cdot (\partial p_1), \tag{2.8}$$

As a result, it gives rise to a well-defined product $HF(L_1, L_2) \otimes HF(L_0, L_1) \to HF(L_0, L_2)$. Additionally, this induced product on Floer cohomology groups is independent of the selected almost-complex structure and any Hamiltonian perturbations, and it is associative.

For any collection of $k+1$ Lagrangian submanifolds, namely $L_0, ..., L_k$, we can implement an analogous procedure to the one discussed above, thus enabling us to establish an operation.

$$\mu^k : CF(L_{k-1}, L_k) \otimes ... \otimes CF(L_1, L_2) \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_k).$$

In situations where the Floer complexes are graded, this operation has a degree of $2-k$, with μ^1 representing the Floer differential and μ^2 representing the product.

We can establish $\mu^k(p_k, ..., p_1)$ using generators $p_i \in L_{i-1} \cap L_i; (i = 1, ..., k)$ and $q \in L_0 \cap L_k$. The coefficient of q in this operation is calculated by counting the (perturbed) pseudo-holomorphic discs in M, adjusted for the area, which possesses boundaries on $L_0 \cup ... \cup L_k$ and corners at $p_1, ..., p_k, q$.

In more detail, we examine maps $u : D \to M$ where the domain D is the closed unit disc, excluding $k+1$ boundary points $z_0, z_1, ..., z_k$ located sequentially along the unit circle. These marked points aren't fixed. We consider the moduli space $\mathcal{M}_{0,k+1}$ of conformal structures on the domain D, which is the quotient of the space of ordered $(k+1)$-tuples of points on S^1 by the action of $\mathrm{Aut}(D^2)$. This space is a contractible manifold of dimension $(k-2)$.

In the context of a chosen almost complex structure J on M and a homotopy class $[u]$, we symbolize by $\mathcal{M}(p_1, ..., p_k, q; [u], J)$ the moduli space of J-holomorphic maps $u : D \to M$ such that:

1. These maps, where the positions of $z_0, ..., z_k$ are not predetermined, continuously extend to the closed disc.

2. The boundary arcs, from z_i to z_{i+1} (or z_0 for $i = k$), map to L_i and the boundary punctures $z_1, ..., z_k, z_0$ map to $p_1, ..., p_k, q$ correspondingly, within the specified homotopy class $[u]$.

This is up to the action of $Aut(D^2)$ through reparametrization.

Given a stationary conformal structure on D, the index of the linearized Cauchy-Riemann operator follows the Maslov index, as discussed earlier. Therefore, when taking into account deformations of the conformal structure on D and assuming transversality, the anticipated dimension of the moduli space can be expressed as:

$$\dim \mathcal{M}(p_1, ..., p_k, q; [u], J) = k - 2 + \mathrm{ind}([u]) = k - 2 + \deg(q) - \sum_{i=1}^{k} \deg(p_i) \quad (2.9)$$

Assuming transversality, and defining orientations and spin structures on $L_0, ..., L_k$ if $\mathrm{char}(\mathbb{K}) \neq 2$, we can now define:

Definition 2.1.5. The operation $\mu^k : CF(L_{k-1}, L_k) \otimes ... \otimes CF(L_0, L_1) \to CF(L_0, L_k)$ is a Λ-linear map defined by

$$\mu^k(p_k, ..., p_1) = \sum_{\substack{q \in L_0 \cap L_k \\ [u]: ind([u]) = 2-k}} (\#\mathcal{M}(p_1, ..., p_k, q; [u], J)) T^{\omega([u])} q \quad (2.10)$$

Again, it can be enhanced to take into account the presence of local systems on the Lagrangians. The algebraic attributes of μ^k can be inferred from examining the boundary cases that appear in the compactification of 1-dimensional moduli spaces of (perturbed) pseudo-holomorphic discs, with a focus on stable maps. Besides strip breaking, we can now have other potential situations where the domain D collapses. We are led to the moduli space of conformal structures $\overline{\mathcal{M}}_{0,k_1}$, known as the Stasheff associahedron. Its top-dimensional facets correspond to nodal degenerations of D into a pair of discs $D_1 \cup D_2$, each disc containing at least two of the marked points $z_0, ..., z_k$. The faces with higher codimensions correspond to nodal degenerations with more components. The following proposition provides a concise framework for the interaction between these operations.

Proposition 2.1.6. *If $[\omega] \cdot \pi_2(M, L_i) = 0$ for all i, then the operations μ^k satisfy the A_∞-relations*

$$\sum_{l=1}^{k} \sum_{j=0}^{k-l} (-1)^* \mu^{k+1-l}(p_k, ..., p_{j+l+1}, \mu^l(p_{j+l}, ..., p_{j+1}), p_j, ..., p_1) = 0 \quad (2.11)$$

where $* = j + \deg(p_1) + ... + \deg(p_j)$.

Numerous variations of the Fukaya category of a symplectic manifold exist, each with its unique degree of generality and specific implementation attributes. Their shared characteristics are: the objects of the Fukaya Category are suitable Lagrangian submanifolds, furnished with additional parameters such as grading, spin structure, and local systems. The morphism spaces are defined by Floer complexes, which are further equipped with the Floer differential. The composition of morphisms is defined by the Floer product, which displays associativity up to homotopy. Hence, the Fukaya category constitutes an A_∞-category. This indicates that the differential and composition represent the first two in a series of operations:

$$\mu^k : \hom(L_{k-1}, L_k) \otimes ... \otimes \hom(L_0, L_1) \to \hom(L_0, L_k),$$

which are of degree $2-k$ when a $\mathbb{Z}$-grading is available. These operations satisfy the A_∞-relations. In this context, the operation μ^k forms a multi-linear map describing interactions among the hom spaces.

In light of the framework within which we have elaborated Floer's theory above, the most intuitive definition of the Fukaya category can be presented as follows:

Definition 2.1.7. Let (M,ω) be a symplectic manifold with $2c_1(TM) = 0$. The objects of the (compact) Fukaya category $\mathcal{F}(M,\omega)$ are compact, closed, oriented, spin Lagrangian submanifolds $L \subset M$ such that $[\omega]\cdot\pi_2(M,L) = 0$ and with vanishing Maslov class $\mu_L = 0 \in H^1(L,\mathbb{Z})$, together with additional data, the choice of a spin structure and a graded lift of L. The Fukaya category is defined as follows:

1. For every pair of objects (L, L'), we select perturbation data $H_{L,L'} \in C^\infty([0,1]\times M,\mathbb{R})$ and $J_{L,L'} \in C^\infty([0,1],\mathcal{J}(M,\omega))$.

2. For all tuples of objects $(L_0,...,L_k)$ and all moduli spaces of discs, we choose consistent perturbation data (H,J) compatible with the choices made for the pairs objects (L_i,L_j). The purpose of this perturbation data is to achieve transversality for all moduli spaces of perturbed J-holomorphic discs.

3. Given this, we define $\hom(L,L') = CF(L,L';H_{L,L'},J_{L,L'})$; and the differential μ^1, composition μ^2, and higher operations μ^k are given by counts of perturbed pseudo-holomorphic discs.

In a broader context where we are not assuming $[\omega]\cdot\pi_2(M,L) = 0$, we are met with disc bubbling. This prompts us to take into account Lagrangians that are equipped with local systems within a curved A_∞-category. In this setting, each object, denoted as $(L,\mathcal{E})$, is associated with an element $\mu^0_{L,\mathcal{E}} \in \hom((L,\mathcal{E}),(L,\mathcal{E}))$. This element represents a weighted count of J-holomorphic discs bounded by L [27]. The established A_∞ relations (2.11) are deformed by this term. For instance, for $k=1$ we derive the following relationship:

$$\mu^1(\mu^1(\rho)) + (-1)^{\deg\rho}\mu^2(\mu^0_{L_1,\mathcal{E}_1},\rho) + \mu^2(\rho,\mu^0_{L_0,\mathcal{E}_0}) = 0. \tag{2.12}$$

The last two terms here correspond to disc bubbling occurring along either edge of an index 2 strip. Within this framework, it's commonplace to consider what we term as 'weakly unobstructed Lagrangians'. These are characterized by their property that $\mu^0_{L,\mathcal{E}}$ is a scalar multiple of the (cohomological) unit of $\hom((L,\mathcal{E}),(L,\mathcal{E}))$, this multiple ($W_{L,\mathcal{E}}$) is known as the potential or superpotential within the context of mirror symmetry. For instance, when $N_L \geq 2$ and index 2 discs are regular, a Lagrangian is weakly unobstructed. In such a case, utilizing (2.12), it's demonstrated that the differential μ^1 squares to zero if and only if $\mu^0_{L_0,\mathcal{E}_0} = \mu^0_{L_1,\mathcal{E}_1}$, i.e., if $W_{L_0,\mathcal{E}_0} = W_{L_1,\mathcal{E}_1}$.

When the inclusion-induced map $i_* : H_1(L) \to H_1(M)$ is null, the corresponding map $\delta_* : H_2(M,L) \to H_1(L)$, arising in the long exact sequence for relative singular homology, becomes surjective. For a Lagrangian torus, we can fix a basis $e_1,\dots,e_n$

of $H_1(L)$, which enables us to express $W_{L,\mathcal{E}}$ as

$$W_{L,\mathcal{E}} = \sum_{\substack{\beta \in H_2(M,L) \\ \mu(\beta)=2}} n_\beta \cdot z^{\partial\beta}.$$

Here, $\partial\beta = \sum a_i e_i$, and $z^{\partial\beta} = z_1^{a_1} \cdots z_n^{a_n}$, with $z_i = \mathrm{hol}_{\mathcal{E}}(e_i)$ representing the holonomy of the local system $\mathcal{E}$. Additionally, n_β denotes the count of holomorphic discs within the class β.

Additionally, as demonstrated in the works of Fukaya et al. and Biran-Cornea [11,27], the Floer homology, $HF(L, L)$, can be computed using a model that initiates with singular cohomology. In this model, the differential is calculated by counting the number of holomorphic discs with boundaries on L.

For a Lagrangian torus, L, the differential is completely determined by the count of $\mu = 2$ holomorphic discs, and under the above assumptions, by

$$\sum z_i \frac{\partial W}{\partial z_i}.$$

In particular, critical points of W_L corresponds to local systems for which $HF(L, L) \neq 0$.

2.2 Toric Algebraic Geometry

Definition 2.2.1. The affine variety $(\mathbb{C}^*)^n$ is a group under component-wise multiplication. An **algebraic torus** T is an affine variety isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure from the isomorphism.

There are two essential groups associated with an arbitrary algebraic torus T: the group of characters and the group of one-parameter subgroups.

Definition 2.2.2. A **character** of an algebraic torus T is a morphism $\chi : T \to \mathbb{C}^*$, as an algebraic variety, that is a group homomorphism.

For example, if $T = (\mathbb{C}^*)^n$, $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ gives a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi^m(t_1, \ldots, t_n) = t_1^{a_1} \ldots t_n^{a_n}. \tag{2.13}$$

Definition 2.2.3. A **one-parameter subgroup** of an algebraic torus T is a morphism $\lambda : \mathbb{C}^* \to T$, as an algebraic variety, that is a group homomorphism.

For example, if $T = (\mathbb{C}^*)^n$, $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup $\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by

$$\lambda^u(t) = (t^{b_1}, \ldots, t^{b_n}). \tag{2.14}$$

For an arbitrary algebraic torus T, its characters and the one-parameter subgroups form free abelian groups of rank equal to the dimension of T. Denote the group of characters and the group of one-parameter subgroups as M and N, respectively. We say that $m \in M$ gives the character χ^m and that $u \in N$ gives the one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T$.

There is a natural bilinear pairing $\langle , \rangle : M \times N \to \mathbb{C}$ defined as follows: Given a character χ^m and a one-parameter subgroup λ^u, the composition $\chi^m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$ is a character of $\mathbb{C}^*$, which is given by $t \mapsto t^l$ for some $l \in \mathbb{Z}$. Then $\langle m, u \rangle = l$.

By [39, §16], all characters and one-parameter subgroups of $(\mathbb{C}^*)^n$ arise as in (2.13) and (2.14), respectively. Therefore, we can identify M and N with $\mathbb{Z}^n$. The bilinear pairing obtained is the usual dot product

$$\langle m, u \rangle = \sum_{i=1}^{n} a_i b_i.$$

These two groups are important because we can construct affine toric varieties from subsets of them, as we will see below. In what follows, we will denote by T an algebraic torus isomorphic to $(\mathbb{C}^*)^n$, M its character lattice, and N its group of one-parameter subgroups.

Definition 2.2.4 ([21, Definition 1.1.3])**.** An **affine toric variety** is an irreducible affine variety V containing an algebraic torus T as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on V.

A set $\mathscr{A} = \{m_1, \ldots, m_s\} \subset M$ gives characters $\chi^{m_i} : T \to \mathbb{C}^*$. Consider the map

$$\Phi_{\mathscr{A}} : T \to (\mathbb{C}^*)^s$$

defined by

$$\Phi_{\mathscr{A}}(t) = (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) \in (\mathbb{C}^*)^s$$

Definition 2.2.5 ([21, Definition 1.1.7])**.** Given a finite set $\mathscr{A} \subseteq M$, the affine toric variety $Y_{\mathscr{A}}$ is defined to be the Zariski closure in $\mathbb{C}^s$ of the image of the map $\Phi_{\mathscr{A}}$.

The following proposition ensures that $Y_{\mathscr{A}}$ is an affine toric variety.

Proposition 2.2.6 ([21, Proposition 1.1.8])**.** *Given $\mathscr{A} \subset M$ as above, let $\mathbb{Z}\mathscr{A} \subset M$ be the sublattice generated by $\mathscr{A}$. Then $Y_{\mathscr{A}}$ is an affine toric variety whose algebraic torus (the image of $\Phi_{\mathscr{A}}$) has character lattice $\mathbb{Z}\mathscr{A}$. In particular, the dimension of $Y_{\mathscr{A}}$ is the rank of $\mathbb{Z}\mathscr{A}$.*

Example 2.2.7. Set

$$\mathscr{A} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

The Zariski closure in $\mathbb{C}^4_{(x,y,z,w)}$ of the image of the map $\Phi_{\mathscr{A}}(t_1, t_2, t_3) = (t_1^{-1} t_3, t_2^{-1} t_3, t_2, t_1)$ is $Y_{\mathscr{A}} = V(xw - yz) \subseteq \mathbb{C}^4$.

From the finite set $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$, we can also get a projective variety by considering the homomorphism

$$\begin{aligned} \pi : (\mathbb{C}^*)^s &\longrightarrow \mathbb{P}^{s-1} \setminus V(x_0 \ldots x_{s-1}) \\ (t_1, \ldots, t_s) &\longmapsto [t_1 : \cdots : t_s] \end{aligned}$$

Definition 2.2.8 ([21, Definition 2.1.1])**.** Given a finite set $\mathscr{A} \subseteq M$, the **projective toric variety** $X_{\mathscr{A}}$ is the Zariski closure in $\mathbb{P}^{s-1}$ of the image of the map $\pi \circ \Phi_{\mathscr{A}}$.

Proposition 2.2.9 ([21, Proposition 2.1.4])**.** *Given $Y_{\mathscr{A}}$ and $X_{\mathscr{A}}$ as above, the following are equivalent:*

1. *$Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ is the affine cone of $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$.*
2. *The ideal of $Y_{\mathscr{A}}$ is homogeneous.*
3. *There is $u \in N$ and $k > 0$ in $\mathbb{N}$ such that $\langle m_i, u \rangle = k$ for $i = 1 \ldots, s$.*

Example 2.2.10. Following Example 2.2.7, we can consider:

$$\pi \circ \Phi_{\mathscr{A}}(t_1, t_2, t_3) = [t_1^{-1} t_3 : t_2^{-1} t_3 : t_2 : t_1] \subseteq \mathbb{P}^3.$$

The Zariski closure in $\mathbb{P}^3_{(x:y:z:w)}$ of the image of $\pi \circ \Phi_{\mathscr{A}}$ is $X_{\mathscr{A}} = V(xw - yz) \subseteq \mathbb{P}^3$. In this case, $Y_{\mathscr{A}}$ is the affine cone of $X_{\mathscr{A}}$

To describe another way to obtain affine toric varieties, we will need the following definitions:

Definition 2.2.11. A **semigroup** is a set S with an associative binary operation and an identity element. To be an **affine semigroup**, we also require that:

- The binary operation on S is commutative. We will write the operation as $+$ and the identity element as 0. Thus a finite set $\mathscr{A} \subset S$ gives

$$\mathbb{N}\mathscr{A} = \left\{ \sum_{m \in \mathscr{A}} a_m m \,\middle|\, a_m \in \mathbb{N} \right\} \subseteq S$$

- The semigroup is finitely generated, meaning that there is a finite set $\mathscr{A} \subset S$ such that $\mathbb{N}\mathscr{A} = S$.
- The semigroup can be embedded in a lattice M.

Definition 2.2.12. Given an affine semigroup $S \subset M$, the **semigroup algebra** $\mathbb{C}[S]$ is the vector space over $\mathbb{C}$ with S as basis and multiplication induced by the semigroup structure of S.

To make this precise, recall that $m \in M$ gives the character χ^m. Then

$$\mathbb{C}[S] = \left\{ \sum_{m \in S} c_m \chi^m \,\middle|\, c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m \right\}$$

with multiplication induced by

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}$$

If $S = \mathbb{N}\mathscr{A}$ for $\mathscr{A} = \{m_1, \dots, m_s\}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$.

The following proposition will tell us how to obtain an affine toric variety from an affine semigroup.

Proposition 2.2.13 ([21, Proposition 1.1.14])**.** *Let $S \subset M$ be an affine semigroup. Then:*

1. *$\mathbb{C}[S]$ is an integral domain and finitely generated as a $\mathbb{C}$-algebra.*
2. *Spec$(\mathbb{C}(S))$ is an affine toric variety whose algebraic torus has character lattice $\mathbb{Z}S$, and if $S = \mathbb{N}\mathscr{A}$ for a finite set $\mathscr{A} \subset M$, then Spec$(\mathbb{C}(S)) = Y_{\mathscr{A}}$.*

A fundamental definition in this thesis is the idea of a convex polyhedral cone and how to obtain an affine toric variety from it. Set $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.2.14 ([21, Definition 1.2.1])**.** A **convex polyhedral cone** in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S') = \left\{ \sum_{u \in S'} \lambda_u u \,\middle|\, \lambda_u \geq 0 \right\} \subset N_{\mathbb{R}}$$

where $S' \subset N_{\mathbb{R}}$ is finite. We say that σ is **generated** by S'. Also set $\mathrm{Cone}(\emptyset) = \{0\}$. We say that σ is **rational** if $S' \subset N$.

Definition 2.2.15 ([21, Proposition 1.2.12])**.** A convex polyhedral cone σ is **strongly convex** if $\sigma \cap (-\sigma) = \{0\}$.

Definition 2.2.16 ([21, Definition 1.2.3])**.** Given a polyhedral cone $\sigma \subset N_{\mathbb{R}}$, its **dual cone** is defined by

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}$$

Given a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, the lattice points

$$S_{\sigma} := \sigma^{\vee} \cap M \subset M$$

form a semigroup. A key fact is that this semigroup is finitely generated.

Proposition 2.2.17 (Gordan's Lemma. [21, Proposition 1.2.17])**.** *S_{σ} is finitely generated and is an affine semigroup.*

Theorem 2.2.18 ([21, Theorem 1.2.18]). *Let $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ be a rational polyhedral cone with affine semigroup S_σ. Then*

$$Y_\sigma := Y_{S_\sigma} = Spec(\mathbb{C}[S_\sigma])$$

is an affine toric variety.

From Theorem 2.2.18 and Proposition 2.2.13, we conclude that if $\mathscr{A} \subset M$ is a finite set such that $\mathbb{N}\mathscr{A} = S_\sigma$, then $Y_{\mathscr{A}} = Y_\sigma$. This is the motivation for the construction of the set $\mathscr{A}_{\mathscr{H}}$ in the proof of Theorem 3.2.4.

We say that an element $m \neq 0$ of S_σ is **irreducible** if $m = m' + m''$ for $m', m'' \in S_\sigma$ implies $m' = 0$ or $m'' = 0$.

Proposition 2.2.19 ([21, Proposition 1.2.23]). *Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone of maximal dimension. Then*

$$\mathscr{H} = \{m \in S_\sigma | m \text{ is irreducible } \}$$

has the following properties:

1. *$\mathscr{H}$ is finite and generates S_σ as a semigroup.*
2. *$\mathscr{H}$ contains the ray generators of the edges of $\sigma^\vee$.*
3. *$\mathscr{H}$ is the minimal generating set of S_σ with respect to inclusion.*

Definition 2.2.20. The set $\mathscr{H} \subset S_\sigma$ is called the **Hilbert basis** of S_σ and its elements are the **minimal generators** of S_σ.

In this thesis, we are interested in the affine varieties given by the cone of a polytope, as defined below.

Definition 2.2.21 ([21, Definition 1.2.2]). A **polytope** in $N_{\mathbb{R}}$ is a set of the form

$$Q = \text{Conv}(S') = \left\{ \sum_{u \in S'} \lambda_u u \,\middle|\, \lambda_u \geq 0, \sum_{u \in S'} \lambda_u = 1 \right\} \subset N_{\mathbb{R}}$$

where $S' \subset N_{\mathbb{R}}$ is finite. We say that Q is the **convex hull** of S'.

A polytope $Q \subset N_{\mathbb{R}}$ gives a polyhedral cone $C(Q) \subset N_{\mathbb{R}} \times \mathbb{R}$, called the **cone of** Q and defined by

$$C(Q) = \{\lambda(u, 1) \in N_{\mathbb{R}} \times \mathbb{R} | u \in Q, \lambda \geq 0\}.$$

Definition 2.2.22. If $\sigma = C(Q)$, where Q is a polytope, we also refer to Y_σ as the **cone of** Q.

Example 2.2.23. Let $Q := \text{Conv}\{(0,0),(1,0),(0,1),(1,1)\} \subseteq \mathbb{R}^2$ and $\sigma = C(Q)$. In this setting, we have

$$\sigma = \text{Cone}\{(0,0,1),(1,0,1),(0,1,1),(1,1,1)\} \text{ and}$$
$$\sigma^\vee = \text{Cone}\{(-1,0,1),(0,-1,1),(0,1,0),(1,0,0)\}.$$

Finally, we get $Y_\sigma = V(xw - yz)$ by Example 2.2.7.

Definition 2.2.24. The Minkowski sum of subsets $A_1, A_2 \subset M_\mathbb{R}$ is

$$A_1 + A_2 = \{m_1 + m_2 | m_1 \in A_1, m_2 \in A_2\}.$$

Given polytopes $M_1 = \text{Conv}(C_1)$ and $M_2 = \text{Conv}(C_2)$, their Minkowski sum $Q = M_1 + M_2 = \text{Conv}(C_1 + C_2)$ is again a polytope. If $Q = M_1 + \cdots + M_k$, we say that $M_1 + \cdots + M_k$ is a **Minkowski decomposition** of Q.

In [6], Altmann studied the cone $\sigma = C(Q)$, where $Q \subseteq \mathbb{R}^n$ is a lattice polygon, i.e. the vertices are contained in $\mathbb{Z}^n$. He gave a description of a set of generators of $\sigma^\vee$ as follows:

To each $c \in \mathbb{Z}^n$ we associate an integer by $\eta_0(c) := \max\{\langle c, -Q\rangle\}$. By the definition of η_0, we have

$$\partial\sigma^\vee \cap \mathbb{Z}^{n+1} = \{(c, \eta_0(c)) | c \in \mathbb{Z}^n\}.$$

Moreover, if $c_1, \ldots, c_w \in \mathbb{Z}^n \backslash \mathbf{0}$ are those elements producing irreducible pairs $(c, \eta_0(c))$ (i.e. not allowing any non-trivial lattice decomposition $(c, \eta_0(c)) = (c', \eta_0(c')) + (c'', \eta_0(c''))$), then the elements

$$(c_1, \eta_0(c_1)), \ldots, (c_w, \eta_0(c_w)), (\mathbf{0}, 1)$$

form a generator set for $\sigma^\vee \cap \mathbb{Z}^{n+1}$ as a semigroup.

In order to study the versal deformation of Y_σ, in [6], Altmann used the cone

$$\widetilde{\sigma} = \text{Cone}\left(\bigcup_{i=1}^{k}((M_i \cap \mathbb{Z}^n) \times \{e_i\})\right) \subseteq \mathbb{R}^n \times \mathbb{R}^k.$$

where $Q = M_1 + \cdots + M_k$ is a Minkowski decomposition of Q.

In this setup, we define a function ϕ that will play the same role as η_0.

Definition 2.2.25. Let $Q = M_1 + M_2 + ... + M_k$ be a Minkowski decomposition of Q, and $n = \dim Q$. Define

$$\begin{aligned} \phi : \mathbb{Z}^n &\longrightarrow \mathbb{Z}^k \\ v &\longmapsto \sum_{i=1}^{k} \max\{\langle v, -M_i\rangle\} e_i \end{aligned}$$

where $e_1, \ldots, e_k$ is the standard basis of $\mathbb{Z}^k$.

2.3 Symplectic Geometry. On base diagrams for singular Lagrangian fibrations

Definition 2.3.1 ([51, Definition 2.1]). A locally trivial fibration of a symplectic manifold is a **regular Lagrangian fibration** if the fibres are smooth Lagrangians. An smooth map $\widetilde{\pi} : M^{2n} \to \widetilde{B}^n$ is a **Lagrangian fibration** if it restricts to a regular Lagrangian fibration over $\widetilde{B} \setminus \widetilde{\Sigma}$, where $\widetilde{B} \setminus \widetilde{\Sigma}$ is an open dense set of $\widetilde{B}$.

Remark 2.3.2. *We assume that our fibres are compact and connected, then by Arnold-Liouville, the regular fibres are isomorphic to T^n.*

The action coordinates, given by flux, allow us to locally identify an open simply connected neighborhood of $b \in \widetilde{B} \setminus \widetilde{\Sigma}$ with an open set in $H^1(F_b; \mathbb{R})$. Hence, $T_bB \cong H^1(F_b; \mathbb{R})$ endows a lattice $H^1(F_b; \mathbb{Z})$, inducing an affine structure in $\widetilde{B} \setminus \widetilde{\Sigma}$.

As in [51], we will consider cuts in the base to obtain an affine embedding to $\mathbb{R}^n$ in the complement of the cuts. The topology of these cuts will be the topology of a Whitney stratified space.

Definition 2.3.3. A **cover** of a set X is a collection of subsets of X whose union is all of X.

Definition 2.3.4 ([44, §5]). An **stratification** $\mathscr{X}$ of $X \subseteq M$ is a cover of X by pairwise disjoint smooth submanifolds X_α, $\alpha \in A$.

Let Δ_M be the diagonal subset of $M \times M$. Denote by $F(N)$ the blowing up of $M \times M$ along Δ_M. We will use the following identity

$$F(M) = PTM \sqcup ((M \times M) \setminus \Delta_M),$$

where PTM denotes the projective tangent bundle of M.

Definition 2.3.5 ([44, §5]). $\mathscr{X}$ is a **Whitney stratification** if it satisfies:

1. (Locally finite) Each point $x \in M$ has a neighborhood U_x such that $U_x \cap X_\alpha \neq \emptyset$ for at most finitely many $\alpha \in A$.

2. (Condition of the frontier) For each $\alpha \in A$, its frontier $(\overline{X_\alpha} \setminus X_\alpha) \cap X = \bigcup_{\beta \in B} X_\beta$ for some $B \subseteq A$.

3. (Whitney's condition B) Let $\{x_i\}$ be a sequence of points in X_α and $\{y_i\}$ be a sequence of points in X_β such that $x_i \neq y_i$. If $\{x_i\} \to y$, $\{y_i\} \to y$, $\{(x_i, y_i)\}$ converges to a line $l \subseteq PTM_y$ in $F(M)$, and $\{TX_{x_i}\}$ converges to an $\dim X$-plane $\tau \subseteq TM_y$. Then $l \subset \tau$.

Definition 2.3.6. A **Whitney stratified space** X of M is a subset of M with a Whitney stratification. The dimension of X is its dimension as a subset of M.

Definition 2.3.7. A Lagrangian fibration $\widetilde{\pi}$ admits a **convex base diagram** if there exists a homeomorphism $\psi : \widetilde{B} \to B \subseteq \mathbb{R}^n$ such that $\pi = \psi \circ \widetilde{\pi} : M \to \mathbb{R}^n$ satisfy that there exists a collection $\mathscr{C} \supseteq \widetilde{\Sigma}$ of codimension one Whitney stratified spaces, called cuts, such that the symplectic affine structure on $\psi(\widetilde{B} \setminus \mathscr{C})$ agrees with the standard affine structure on $\mathbb{R}^n$.

Remark 2.3.8. *Almost toric fibrations over disks defined in [51] admit a convex base diagram by taking cuts on eigendirections.*

Definition 2.3.9. A **restricted almost-toric fibration** is a Lagrangian fibration that admits a convex base diagram whose singular fibres are toric, homeomorphic to T^n/T^k (the group quotient that appears as the standard local model in toric fibrations), or homeomorphic to $T^n/\mathfrak{T}^k$ (the quotient space obtained by collapsing the k-cycle $\mathfrak{T}^k$). We refer to the cycles of $\mathfrak{T}^k$ as **collapsing cycles**.

Remark 2.3.10. *There exist ideas of what an almost-toric fibration should be in higher dimensions, in particular, including singular fibres described by Matessi and Castaño-Bernard [15]. Not all are like our local model, that is why we named these restricted.*

The Lagrangian fibrations we will consider in this thesis all have a convex base diagram satisfying the following assumptions on its cuts.

Assumption 2.3.11. Let $\pi : M^{2n} \to B \subseteq \mathbb{R}^n$ be a convex base diagram for a singular Lagrangian fibration. Let Σ be the image of singular fibres, S_i's the connected components of $\Sigma = \bigcup_i S_i$, and $\mathscr{C} = \bigcup_i C_i \subseteq B$ is the collection of cuts. Assume that each cut C_i is the cone of S_i with respect to some $v_i \in \mathbb{R}^n$, i.e.,

$$C_i = \{x + tv_i | x \in S_i, t \in \mathbb{R}_{\geq 0}\} \cap \mathrm{Im}(\pi).$$

Also that the image of C_i in $\mathbb{R}^{n-1} = v_i^{\perp}$ under the projection with respect to v_i is of the form

$$\left\{ \sum_{\sigma \in \mathscr{A}_i} \lambda_\sigma \sigma \,\middle|\, \lambda_\sigma \geq 0 \right\},$$

for some finite set $\mathscr{A}_i \subseteq \mathbb{R}^{n-1}$, satisfying the balancing condition $\sum_{\sigma \in \mathscr{A}_i} \sigma = 0$. Let $C_i^{\mathbb{R}} = \{x + tv_i | t \in \mathbb{R}\}$ and $\mathrm{Im}(\pi) \setminus C_i^{\mathbb{R}} = \bigcup_{j=0}^{N} D_{i,j}$. We assume that $\overline{D_{i,k}} \cap \overline{D_{i,l}} \neq \emptyset, \forall k, l$.

Example 2.3.12 ([9, §3]). Consider the complex fibration

$$\begin{aligned} f : \mathbb{C}^3 &\longrightarrow \mathbb{C} \\ (x, y, z) &\longmapsto xyz. \end{aligned}$$

There is an action of T^2 on $\mathbb{C}^3$ that preserves the fibres of f, let μ be its moment map and let $\gamma(r) \subseteq \mathbb{C}$ be the circle with center in 1 and radius r. Given real numbers δ_1 and δ_2, we define:

$$T_{\gamma(r),\delta_1,\delta_2} = f^{-1}(\gamma(r)) \cap \mu^{-1}(\delta_1, \delta_2).$$

$T_{\gamma(r),\delta_1,\delta_2}$ is an embedded Lagrangian torus in $\mathbb{C}^3$, except possibly when $0 \in \gamma(r)$ or in the limit when $r = 0$ and we have an isotropic T^2.
We have collapsing cycles in the following cases:

1. When $x = y = 0$ and $z \neq 0$, the collapsing cycle is $\theta_1 - \theta_2$.
2. When $x = z = 0$ and $y \neq 0$, the collapsing cycle is θ_1.
3. When $y = z = 0$ and $x \neq 0$, the collapsing cycle is θ_2.
4. When $x = y = z = 0$, all the T^2 collapses.

The singular Lagrangian fibration yields a convex base diagram with codimension-one cuts; away from these cuts, we will have a toric structure. If $r < 1$, $T_{\gamma(r),\delta_1,\delta_2}$ determine a toric structure, and we can take action coordinates $(\lambda_1, \lambda_2, \lambda_3)$ given by the symplectic flux concerning a reference fibre $T_{\gamma(r_0),0,0}$, as in the action-angle coordinates given in [23]. Considering the limit when $r_0 = 0$, we can interpret λ_3 as the symplectic area of a disk with boundary in a cycle of $T_{\gamma(r),\delta_1,\delta_2}$ that collapses as $r \to 0$.
For $r \geq 1$ we then consider cuts (i.e., we disregard fibres) for $\lambda_1 \geq 0, \lambda_2 = 0$, or $\lambda_1 = 0, \lambda_2 \geq 0$, or $\lambda_1 = \lambda_2 \leq 0$. This way, we killed monodromies around singular fibres and, hence, possible ambiguity to extend λ_3 via symplectic flux. Note that we can extend λ_3 to a continuous (but not smooth) function on the whole $\mathbb{C}^3$.
We call the image B of $\pi : \mathbb{C}^3 \to \mathbb{R}^3$ given by $(\lambda_1, \lambda_2, \lambda_3)$ and denote by Σ the locus on B corresponding to singular Lagrangian fibres. Let S_1 be the image of the singular fibres corresponding to the case when $0 \in \gamma(r)$. The cone C_1 is defined with $v_1 = (0, 0, 1)$. We obtain the left diagram of Figure 2.5 as the convex base diagram with cuts.

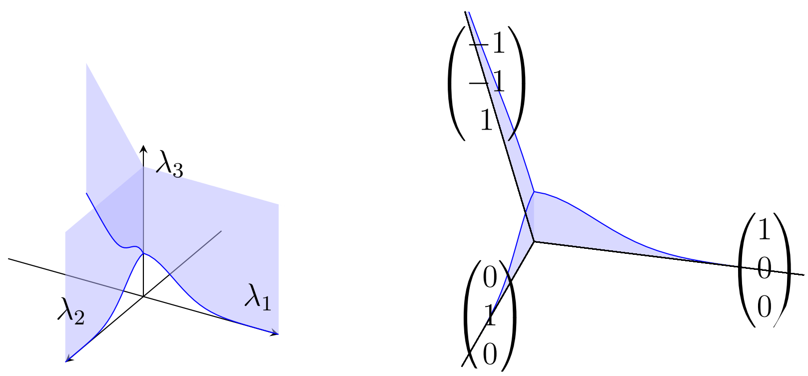


Figure 2.5: Convex base diagrams corresponding to the restricted Lagrangian fibration in Example 2.3.12

Definition 2.3.13. An element $[\gamma] \in \pi_1(B \setminus \Sigma, b)$ gives us an isotopy class of self-diffeomorphism of the torus fibre F_b. This self-diffeomorphism induces an automorphism on $H_1(F_b, \mathbb{Z})$. We call the map from $\pi_1(B \setminus \Sigma)$ to $\mathrm{Aut}(H_1(F_b))$ **topological monodromy**. A choice of basis for $H_1(F_b)$ allows us to see the topological monodromy as a map from $\pi_1(B \setminus \Sigma)$ to $GL(n, \mathbb{Z})$.

To study the topological monodromy of our restricted Lagrangian fibration, consider $\widetilde{C}_i = \{x - tv_i | x \in S_i, t \in \mathbb{R}_{\geq 0}\} \cap \mathrm{Im}(\pi), C_i^{\mathbb{R}} = \{x + tv_i | t \in \mathbb{R}\}$ and $\mathrm{Im}(\pi) \setminus C_i^{\mathbb{R}} = \bigcup_{j=0}^{N} D_{i,j}$. Fix $D_{i,0}$, and by the assumption, all $D_{i,j}$ are adjacent to $D_{i,0}$. For all $j = 1, \ldots, N$, consider the loop $\mathfrak{b}_{i,j} \subset \overline{D_{i,0}} \cup \overline{D_{i,j}}$, passing through C_i once, through $\widetilde{C}_i$ once, and with orientation given by passing through C_i from $D_{i,0}$ to $D_{i,j}$. Finally, let $M_{\mathfrak{b}_{i,j}}$ be the topological monodromy around the loop $\mathfrak{b}_{i,j}$.

Because $H^1(F_b, \mathbb{Z}) = \hom(H_1(F_b, \mathbb{Z}), \mathbb{Z})$, we see that the affine monodromy (the monodromy for the affine structure in $B \setminus \Sigma$) concerning each loop $\mathfrak{b}_{i,j} \in \pi_1(B \setminus \Sigma)$ is given by the transpose inverse of the monodromies $M_{\mathfrak{b}_{i,j}}$. Denote the affine monodromy by $M^{\mathrm{af}}_{\mathfrak{b}_{i,j}}$.

We now generalize the notion of transferring the cut move defined in [55]. We want to change the direction of all the cuts associated with a connected family of singularities, obtaining a different convex base diagram for the same singular Lagrangian fibration.

Definition 2.3.14. A **transferring the cut** operation with respect to the cut C_i consists in applying a piecewise linear map $\psi_i : \mathbb{R}^n \to \mathbb{R}^n$, preserving C_i, obtaining $\pi_i = \psi_i \circ \pi : M \to \mathbb{R}^n$ with associated cuts C_j' such that $C_j' = \psi_i(C_j)$ for $j \neq i$ and $C_i' = \widetilde{C}_i$. The map ψ_i is defined by applying $M^{af}_{\mathfrak{b}_{i,j}}$ to $\overline{D_{i,j}}$ for all j. Note that this operation fixes $C_i^{\mathbb{R}}$ and that as a result of applying the monodromies, the affine structure of π_i agrees with the affine structure of $\mathbb{R}^n$ on $C_i \setminus \bigcup_{j \neq i} \psi_i(C_j) \subseteq \mathrm{Im}(\pi_i)$, but is not defined in C_i'.

Example 2.3.15. The two convex base diagrams in Figure 2.5 are related by transferring the cut operation.

Chapter 3

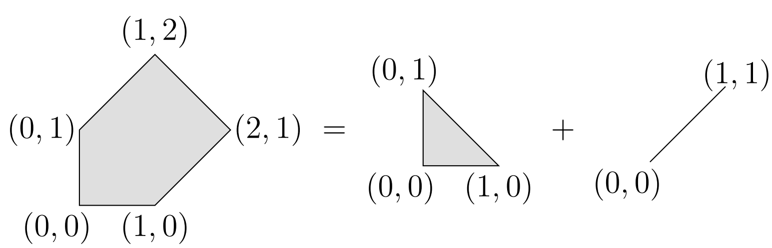
Singular Lagrangian fibrations in Y_ϵ

In Section 3.1, we describe a toy example to illustrate how the proofs of Theorem 1.1.1 and Theorem 1.1.2 work. In Section 3.2, we give the proofs of the main theorems. In Section 3.3, we describe the potential function of families of monotone Lagrangians in Y_ϵ. In Section 3.4, we present more examples of the application of the main results. In Section 3.5, we show how to obtain a compactification of Y_ϵ compatible with the $\mathbb{C}$-fibration under certain conditions (see Theorem 3.5.2).

3.1 Toy example

In this section, we will work in detail on an example to illustrate how to obtain a restricted Lagrangian fibration on the smoothing of a cone associated with a toric manifold Q. This will help us settle notation and introduce concepts used in the statement and proof of the main theorems.

Consider $Q_5 := \text{Conv}\{(0,0),(1,0),(0,1),(2,1),(1,2)\} \subset \mathbb{R}^2$ and the following Minkowski decomposition:



We write $Q_5 = M_1 + M_2$, where $M_1 = \text{Conv}\{(0,0),(1,0),(0,1)\}$ and $M_2 = \text{Conv}\{(0,0),(1,1)\}$. Let $\sigma = C(Q_5) \subseteq N_{\mathbb{R}} \times \mathbb{R} \cong \mathbb{R}^3$ (see Definition 2.2.22), and Y_σ the associated affine toric variety, which is singular. Note that

$$\sigma = \text{Cone}(\{(0,0,1),(1,0,1),(0,1,1),(2,1,1),(1,2,1)\})$$

Following [6], we will describe a way to obtain a deformation of the cone Y_σ from the Minkowski decomposition $Q = M_1 + M_2$. We begin by considering $\mathscr{A} =$

$((M_1 \cap \mathbb{Z}^2) \times \{e_1\}) \cup ((M_2 \cap \mathbb{Z}^2) \times \{e_2\}) \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$ and $\widetilde{\sigma} = \operatorname{Cone}(\mathscr{A})$ where e_1, e_2 denote the standard basis of $\mathbb{Z}^2$. We see that

$$\widetilde{\sigma} = \operatorname{Cone}(\{(0,0;1,0),(1,0;1,0),(0,1;1,0),(0,0;0,1),(1,1;0,1)\})$$

It contains the cone σ via the diagonal embedding $\mathbb{R}^2 \times \mathbb{R} \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^2$ given by $(a,1) \mapsto (a,1,1)$.

We use the computer program Normaliz [12] to obtain the Hilbert Basis of $S_{\widetilde{\sigma}} = \sigma^{\vee} \cap \mathbb{Z}^4$:

$$\begin{aligned}\mathscr{H}_{\widetilde{\sigma}} =& \{(-1,-1,1,2),(-1,0,1,1),(-1,1,1,0),(0,-1,1,1),\\ & (1,-1,1,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}\end{aligned}$$

To understand better Altmann's deformation of Y_σ, we will describe it using characters associated with $Y_{\widetilde{\sigma}}$, see also [30, §3]. In addition, we will systematically associate some elements of the Hilbert Basis $\mathscr{H}_{\widetilde{\sigma}}$ with each term of the Minkowski decomposition in the following manner: To M_i, we associate a $\dim M_i \times 2$ matrix V_i, such that the rows of V_i are the coordinates of the non-zero vertices of M_i:

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Each V_i can be completed to a 2×2 matrix X_i such that the rows of X_i form a basis of $\mathbb{Z}^n$. Let $Z_i = X_i^{-1}$, define A_i a $2 \times \dim M_i$ matrix and, when $\dim M_i < n$, C_i a $2 \times (n - \dim M_i)$ matrix given by $Z_i = (A_i \quad C_i)$. In our case, we choose

$$X_1 = Z_1 = A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_2 = Z_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Recall our definition of the map $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ associated to a Minkowski decomposition (Definition 2.2.25) and let $a_{i,j} \in \mathbb{Z}^2$ be the j-th column of A_i, then we associate $\widetilde{a}_{i,j} = (a_{i,j}, \phi(a_{i,j})) \in \mathbb{Z}^4$ to $a_{i,j}$. In the same way, let $c_{i,l}$ be the l-th column of C_i and associate $(\widetilde{c}^+)_{i,l} = (c_{i,l}, \phi(c_{i,l})) \in \mathbb{Z}^4$, $(\widetilde{c}^-)_{i,l} = (-c_{i,l}, \phi(-c_{i,l})) \in \mathbb{Z}^4$ to $c_{i,l}$. We also associate to M_i the vector:

$$\widetilde{b}_i = \left(-\sum_{j=1}^{\dim M_i} a_{i,j}, \phi\left(-\sum_{j=1}^{\dim M_i} a_{i,j} \right) \right) \in \mathbb{Z}^4.$$

Set

$$\mathscr{A}_1 = \left\{ \widetilde{a}_{1,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \widetilde{a}_{1,2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \widetilde{b}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 2 \end{pmatrix} \right\},$$

associated with M_1,

$$\mathscr{A}_2 = \left\{ \widetilde{a}_{2,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \widetilde{b}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, (\widetilde{c}^+)_{2,1} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, (\widetilde{c}^-)_{2,1} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

associated with M_2, and note that:

$$\mathscr{H}_{\widetilde{\sigma}} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We now associate the characters $z = \chi^{(0,-1,1,1)}$, $x_{i,j} = \chi^{\widetilde{a}_{i,j}}$, $y_i = \chi^{\widetilde{b}_i}$, $w^+_{i,j} = \chi^{(\widetilde{c}^+)_{i,j}}$, $w^-_{i,j} = \chi^{(\widetilde{c}^-)_{i,j}}$, $t_1 = \chi^{(0,0,1,0)}$, and $t_2 = \chi^{(0,0,0,1)}$, to each element of the Hilbert basis $\mathscr{H}_{\widetilde{\sigma}}$. The characters t_i, associated with the vectors in $M_i \times \mathbb{Z}^2$ of the form $(0,0,e_i)$, will be viewed as deformation parameters of Y_σ.

Let $Y_{\widetilde{\sigma}} = \operatorname{Spec}(\mathbb{C}[S_{\widetilde{\sigma}}]) = Y_{\mathscr{H}_{\widetilde{\sigma}}}$, then the ideal $I(Y_{\widetilde{\sigma}})$ is generated by:

$$\begin{array}{lll}
x_{1,1}t_1 - x_{1,2}w^+_{2,1} & x_{1,1}y_1 - zt_2 & x_{1,1}y_2 - t_1t_2 \\
x_{1,1}w^-_{2,1} - x_{1,2}t_1 & x_{1,1}z - w^+_{2,1}t_2 & x_{1,2}y_1 - y_2t_2 \\
x_{1,2}y_2 - w^-_{2,1}t_2 & x_{1,2}z - t_1t_2 & w^+_{2,1}w^-_{2,1} - t_1^2 \\
y_2z - y_1t_1 & w^-_{2,1}z - y_2t_1 & y_2w^+_{2,1} - zt_1 \\
y_1w^-_{2,1} - y_2^2 & y_1w^+_{2,1} - z^2 & y_2^2w^+_{2,1} - w^-_{2,1}z^2
\end{array} \tag{3.1}$$

The inclusion $\sigma \subseteq \widetilde{\sigma}$ induces a embedding $Y_\sigma \subseteq Y_{\widetilde{\sigma}}$ such that $Y_\sigma = Y_{\widetilde{\sigma}} \cap \{t_1 - t_2 = 0)\}$, as it is described in [6]. Consider now $\Psi : Y_{\widetilde{\sigma}} \to \mathbb{C}$ given by $t_1 - t_2$, and $Y_{\widetilde{\sigma},\epsilon} := \Psi^{-1}(\epsilon)$ with $\epsilon \neq 0$, a deformation of Y_σ (see [30, §3]). In this case, $Y_{\widetilde{\sigma},\epsilon}$ is smooth. We aim to study the complex fibration $f : Y_{\widetilde{\sigma},\epsilon} \to \mathbb{C}$ given by the projection to t_1, which we will eventually use as an auxiliary fibration for constructing a singular Lagrangian fibration as in [8, 9, 53].

Note that the singular fibres of f lie over $t_1 = 0$ and $t_1 = \epsilon$. Indeed, if $t_1 \neq 0, \epsilon$, $f^{-1}(t_1) \cong (\mathbb{C}^*)^2$ because

$$x_{1,1}y_2 = t_1t_2 \neq 0$$

$$w^+_{2,1}w^-_{2,1} = t_1^2 \neq 0$$

and from the other relations in (3.1), we can show that all the characters in $\mathscr{H}_{\widetilde{\sigma}}$ will depend only in the non-zero characters $x_{1,1}$ and $w^+_{2,1}$.

We then look at the singular fibre $f^{-1}(0)$ and identify it with a subset of $\mathbb{C}^3$ given by:

$$x_{1,1}x_{1,2}y_1 = t_1t_2^2 = 0$$

All the characters in $\mathscr{H}_{\widetilde{\sigma}}$ will depend only in $x_{1,1}, x_{1,2}, y_1$ by the relations:

$$y_2 = x_{1,2}y_1(-\epsilon)^{-1}$$

$$w^+_{2,1} = x_{1,1}^2y_1(\epsilon)^{-2}$$

$$w^-_{2,1} = x_{1,2}^2y_1(\epsilon)^{-2}$$

$$z = x_{1,1} y_1 (-\epsilon)^{-1}$$

obtained from (3.1).

We identify the singular fibre $f^{-1}(\epsilon)$ with a subset of $\mathbb{C}^2 \times \mathbb{C}^*$ given by:

$$x_{1,1} y_2 = t_1 t_2 = 0$$

$$w_{2,1}^+ w_{2,1}^- = t_1^2 = \epsilon^2$$

All the characters in $\mathscr{H}_{\tilde{\sigma}}$ will depend only in $x_{1,1}, y_2, w_{2,1}^+$ by the relations:

$$\begin{aligned} x_{1,2} &= x_{1,1} w_{2,1}^- \epsilon^{-1} \\ y_1 &= y_2^2 w_{2,1}^+ \epsilon^{-2} \\ z &= y_2 w_{2,1}^+ \epsilon^{-1} \end{aligned}$$

obtained from (3.1).

Now that we understand the complex fibration f, we will construct a Lagrangian fibration associated with it using the same methods used in [8,9,53]. First, we endow $Y_{\tilde{\sigma},\epsilon}$ with the symplectic form coming from the embedding as a subset of $\mathbb{C}^{|\mathscr{H}_{\tilde{\sigma}}|}$. Then, we study the Hamiltonian action of T^2 on $Y_{\tilde{\sigma},\epsilon}$ that preserves the fibres of f given by

$$(x_{1,1}, x_{1,2}, y_1, y_2, w_{2,1}^+, w_{2,1}^-, z) \to$$

$$(e^{i\theta_1} x_{1,1}, e^{i\theta_2} x_{1,2}, e^{-i\theta_1} e^{-i\theta_2} y_1, e^{-i\theta_1} y_2, e^{i\theta_1} e^{-i\theta_2} w_{2,1}^+, e^{-i\theta_1} e^{i\theta_2} w_{2,1}^-, e^{-i\theta_2} z).$$

The moment maps corresponding to the action of $S^1 \times \{1\}$ and $\{1\} \times S^1$ are:

$$\lambda_1(x_{1,1}, x_{1,2}, y_1, y_2, w_{2,1}^+, w_{2,1}^-, z) = (|x_{1,1}|^2 + |w_{2,1}^+|^2 - |y_1|^2 - |y_2|^2 - |w_{2,1}^-|^2)/2$$

$$\lambda_2(x_{1,1}, x_{1,2}, y_1, y_2, w_{2,1}^+, w_{2,1}^-, z) = (|x_{1,2}|^2 + |w_{2,1}^-|^2 - |z|^2 - |y_1|^2 - |w_{2,1}^+|^2)/2$$

respectively.

We will consider Lagrangian torus, which is contained in $f^{-1}(\gamma(r))$ for the circle $\gamma(r) \subset \mathbb{C}$ with center in 1 and radius r, and consist of a single $S^1 \times S^1-$orbit inside each fibre of a point of $\gamma(r)$. Our Lagrangian torus will be given by symplectic parallel transport of the T^2 orbits along $\gamma(r)$.

Definition 3.1.1. Given the circle $\gamma(r) \subset \mathbb{C}$ and real numbers δ_1, δ_2, we define

$$T_{\gamma(r),\delta_1,\delta_2} = \{(x_{1,1}, x_{1,2}, y_1, y_2, w_{2,1}^+, w_{2,1}^-, z) \in f^{-1}(\gamma(r)) | \lambda_1 = \delta_1, \lambda_2 = \delta_2\}.$$

We showed above that the general fibre of f is isomorphic to $(\mathbb{C}^*)^2$, then $T_{\gamma(r),\delta_1,\delta_2}$ is an embedded Lagrangian torus in $Y_{\tilde{\sigma}}$, except possibly when $0 \in \gamma(r)$, $\epsilon \in \gamma(r)$, or in the limit when $r = 0$ and we have an isotropic T^2. Let $\lambda = 2(\lambda_1, \lambda_2)$,

$$\lambda = \begin{pmatrix}1\\0\end{pmatrix}|x_{1,1}|^2 + \begin{pmatrix}0\\1\end{pmatrix}|x_{1,2}|^2 + \begin{pmatrix}-1\\-1\end{pmatrix}|y_1|^2 + \begin{pmatrix}-1\\0\end{pmatrix}|y_2|^2$$
$$+ \begin{pmatrix}1\\-1\end{pmatrix}|w_{2,1}^+|^2 + \begin{pmatrix}-1\\1\end{pmatrix}|w_{2,1}^-|^2 + \begin{pmatrix}0\\-1\end{pmatrix}|z|^2.$$

We assume $\epsilon, 0$ are not in the same circle $\gamma(r)$. Let us now understand the singular Lagrangians for when $0 \in \gamma(r)$. Recall that we identify $f^{-1}(0)$ as a subset of $\mathbb{C}^3$ given by

$$x_{1,1}x_{1,2}y_1 = 0.$$

Note that if two of the characters $x_{1,1}, x_{1,2}, y_1$ are non-zero, we have a T^2-orbit, and hence the Lagrangian passing through it is a smooth T^3. When $x_{1,1} = x_{1,2} = y_1 = 0$, the whole T^2 collapses. Let us analyze the remaining cases, identifying the cycle in T^2 that collapses as we approach one of these singular points.

When $x_{1,1} = y_1 = 0$ and $x_{1,2} \neq 0$, looking at equations (3.1) we see that:

$$\lambda = \begin{pmatrix}0\\1\end{pmatrix}|x_{1,2}|^2.$$

So the collapsing class is the cycle generated by the θ_1 coordinate of the T^2 action. We will call it $[\theta_1]$ viewed in $H_1(T^3, \mathbb{Z}) \supset H_1(T^2, \mathbb{Z})$. Here we think of T^3 as a reference Lagrangian fibre, $H_1(T^3, \mathbb{Z}) = \langle [\theta_1], [\theta_2], [\theta_3] \rangle$ where $[\theta_3]$ is a cycle corresponding to a choice of a lift of the curve γ on the base.
When $x_{1,2} = y_1 = 0$ and $x_{1,1} \neq 0$, we see that

$$\lambda = \begin{pmatrix}1\\0\end{pmatrix}|x_{1,1}|^2,$$

so the collapsing class is $[\theta_2]$.
When $x_{1,2} = x_{1,1} = 0$ and $y_1 \neq 0$, we see that

$$\lambda = \begin{pmatrix}-1\\-1\end{pmatrix}|y_1|^2,$$

so the collapsing class is $[\theta_1 - \theta_2]$.

If $\epsilon \in \gamma(r)$, recall that we identify $f^{-1}(\epsilon)$ as a subset of $\mathbb{C}^2 \times (\mathbb{C})^*$ given by

$$x_{1,1}y_2 = 0$$

$$w_{2,1}^+ w_{2,1}^- \neq 0$$

Note that if one of the characters $x_{1,1}, y_2$ is non-zero, we have a T^2-orbit, and hence the Lagrangian passing through it is a smooth T^3. We will analyze the remaining case identifying the cycle in T^2 that collapses as we approach a point for which $x_{1,1} = y_2 = 0$. In this case:

$$\lambda = \begin{pmatrix} 1 \\ -1 \end{pmatrix} |w_{2,1}^+|^2 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} |w_{2,1}^-|^2,$$

so the collapsing class is $[\theta_1 + \theta_2]$.

In summarizing, the singular Lagrangians, which are topologically pinched torus times a circle, are the fibres $T_{\gamma(r),\delta_1,\delta_2}$ corresponding to:

1. $0 \in \gamma(r)$, $\lambda_2 = 0$, and $\lambda_1 > 0$
2. $0 \in \gamma(r)$, $\lambda_1 = 0$, and $\lambda_2 > 0$
3. $0 \in \gamma(r)$, $\lambda_1 - \lambda_2 = 0$, and $\lambda_1 < 0$
4. $\epsilon \in \gamma(r)$ and $\lambda_1 + \lambda_2 = 0$

Now we will study the convex base diagram produced by this singular Lagrangian fibration. For this purpose, we will follow the same approach as in [51], in the sense that we will consider codimension one cuts in the base of the Lagrangian fibration, in the complement of which we will have a toric structure. Let us now assume that $|\epsilon - 1| > 1$, so the circles $\gamma(r)$ pass first through 0 and then through ϵ, as r increases. So, if $r < 1$, $T_{\gamma(r),\lambda_1,\lambda_2}$ determine a toric structure, and we can take action coordinates $(\lambda_1, \lambda_2, \lambda_3)$ given by the symplectic flux concerning a reference fibre $T_{\gamma(r_0),0,0}$, as in the action-angle coordinates given in [23]. Considering the limit when $r_0 = 0$, we can interpret λ_3 as the symplectic area of a disk with boundary in a cycle of $T_{\gamma(r),\lambda_1,\lambda_2}$ that collapses as $r \to 0$.

We will introduce cuts to represent our fibration by a convex base diagram as in Figure 2.5. For $r \geq 1$ we then consider cuts (i.e., we disregard fibres) for $\lambda_1 \geq 0, \lambda_2 = 0$, or $\lambda_1 = 0, \lambda_2 \geq 0$, or $\lambda_1 = \lambda_2 \leq 0$. For $r \geq |1 - \epsilon|$, we take cuts for $\lambda_1 = -\lambda_2$. This way, we killed monodromies around singular fibres and, hence, possible ambiguity to extend λ_3 via symplectic flux. Note that we can extend λ_3 to a continuous (but not smooth) function on the whole $Y_{\widetilde{\sigma},\epsilon}$.

We call the image B of $\pi : Y_{\widetilde{\sigma},\epsilon} \to \mathbb{R}^3$ given by $(\lambda_1, \lambda_2, \lambda_3)$ and denote by Σ the locus on B corresponding to singular Lagrangian fibres, as in Section 2.3. Let S_1 be the image of the singular fibres corresponding to the case when $0 \in \gamma(r)$, and S_2 be the image of the singular fibres corresponding to the case when $\epsilon \in \gamma(r)$. The cones C_i are defined with $v_i = (0, 0, 1)$, for $i = 1, 2$. In Figure 3.1, we present the convex base diagram with cuts. The blue curves correspond to S_1 while the green curves represent S_2. Similarly, the blue planes depict the cut C_1 and the green planes illustrate the cut C_2.

Remark 3.1.2. *One can check that λ_3 parameter of singular fibres tend to zero as $|\lambda_1|^2 + |\lambda_2|^2 \to \infty$, with r fixed.*

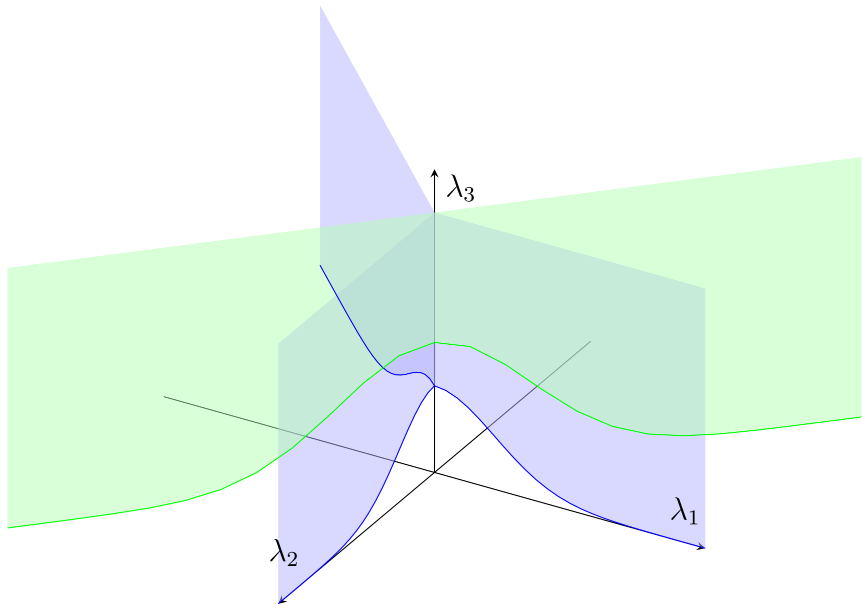


Figure 3.1: Convex base diagram corresponding to the restricted Lagrangian fibration related with Q_5

We are now going to study the topological monodromy, as defined in Section 2.3, as we go through the cuts of this fibration. Dropping, from now on, the brackets from the notation, let $\{\theta_1, \theta_2, \theta_3\}$ be a basis for $H_1(T_{\gamma(r),\delta_1,\delta_2})$ where θ_i is the circle class of orbit corresponding to λ_i for $i = 1, 2$, and θ_3 the collapsing class associated with λ_3. Let

$$D_{1,0} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 > 0, \lambda_2 > 0\},$$

$$D_{1,1} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_2 < 0, \lambda_1 > \lambda_2\},$$

$$D_{1,2} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 < 0, \lambda_2 > \lambda_1\},$$

$$D_{2,0} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 + \lambda_2 > 0\},$$

$$D_{2,1} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 + \lambda_2 < 0\},$$

and define $\mathfrak{b}_{i,j}$ an oriented loop from $D_{i,0}$ to $D_{i,j}$ and back, for $j \neq 0$, as in Section 2.3.

Taking into account that the topological monodromy leaves the T^2 orbit (generated by θ_1, θ_2) invariant and shears the θ_3 cycle concerning the collapsing cycle, we have that the topological monodromies $M_{\mathfrak{b}_{i,j}}$ concerning $\mathfrak{b}_{i,j}$ are:

$$M_{\mathfrak{b}_{1,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, M_{\mathfrak{b}_{1,2}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_{\mathfrak{b}_{2,1}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The action coordinates, given by flux, allow us to locally identify an open simply connected neighborhood of $b \in B \setminus \Sigma$ with an open set in $H^1(F_b;\mathbb{R})$. Hence, $T_bB \cong H^1(F_b;\mathbb{R})$ endows a lattice $H^1(F_b;\mathbb{Z})$, inducing an affine structure in $B \setminus \Sigma$. Since our Lagrangian fibration $\pi = (\lambda_1, \lambda_2, \lambda_3)$ is induced by flux, the affine structure on B in the complement of the cuts coincide with the standard affine structure in $\mathbb{R}^3$.

Because $H^1(F_b,\mathbb{Z}) = \hom(H_1(F_b,\mathbb{Z}),\mathbb{Z})$ and our choice of basis $\{\theta_1,\theta_2,\theta_3\}$ for $H_1(F_b,\mathbb{Z})$, associated with the $(\lambda_1,\lambda_2,\lambda_3)$ coordinates of $\mathbb{R}^3$, we see that the affine monodromy (the monodromy for the affine structure in $B \setminus \Sigma$) about each loop $\mathfrak{b}_{i,j} \in \pi_1(B \setminus \Sigma)$ is given by the transpose inverse of the monodromies $M_{\mathfrak{b}_{i,j}}$. The corresponding affine monodromies are:

1. $M^{af}_{\mathfrak{b}_{1,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ with eigenvectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

2. $M^{af}_{\mathfrak{b}_{1,2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ with eigenvectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

3. $M^{af}_{\mathfrak{b}_{2,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$ with eigenvectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

We want to change the direction of all the cuts associated with a connected family of singularities. The result of applying the transferring the cut operations (Definition 2.3.14) concerning the two cuts of our fibration $\pi : Y_{\widetilde{\sigma},\epsilon} \to \mathbb{R}^3$, is obtained by following the steps:

1. Apply the matrix $M^{af}_{\mathfrak{b}_{1,1}}$ to $\overline{D_{1,1}}$.

2. Apply the matrix $M^{af}_{\mathfrak{b}_{1,2}}$ to $\overline{D_{1,2}}$.

3. Apply the matrix $M^{af}_{\mathfrak{b}_{2,1}}$ to the image of $\overline{D_{2,1}}$ after the first two operations.

The resulting convex base diagram is presented in Figure 3.2.

Note that the cone presented in Figure 3.2 coincides with:

$$\sigma^\vee = \text{Cone}\left\{ \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

After applying the matrix

$$\mathscr{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in SL(3,\mathbb{Z})$$

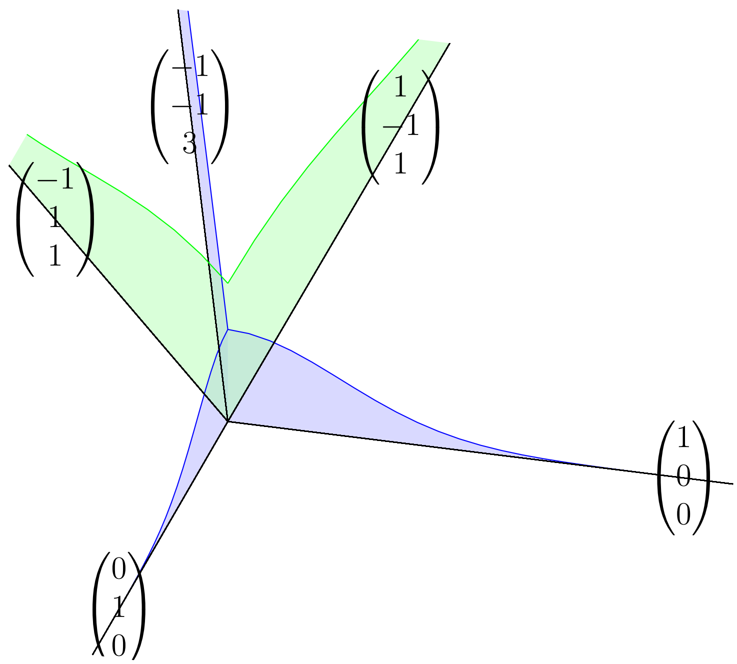


Figure 3.2: Convex base diagram related with Figure 3.1 after applying the transferring the cut operations.

to $\sigma^\vee$, we obtain:

$$\mathscr{M}\sigma^\vee = \mathrm{Cone}\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\},$$

where we see it as a cone having $Q_5^\vee$ (Figure 3.3) at height 1.

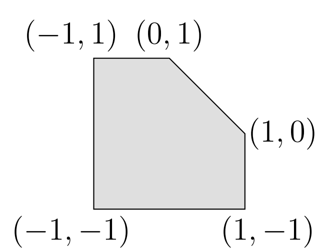


Figure 3.3: $Q_5^\vee$

Section 3.2 aims to construct a Lagrangian fibration on the smoothing of a cone singularity associated with the Minkowski decomposition in a general setting and prove a duality result relating the convex base diagram as the dual of the original cone.

3.2 Main Results

We are interested in lattice polytopes Q that have a particular type of Minkowski decomposition, as we define below.

Definition 3.2.1. We call a Minkowski decomposition $Q = M_1 + \cdots + M_k \subset \mathbb{R}^n$ **admissible** if:

- Each M_i is a lattice polytope that contains de origin as a vertex.
- The non-zero vertices $v_{i,1}, \ldots, v_{i,m_i}$ of M_i are linearly independent and there exist $v_{i,m_i+1}, \ldots, v_{i,n} \in \mathbb{Z}^n$ such that $v_{i,1}, \ldots, v_{i,n}$ are a basis for the lattice $\mathbb{Z}^n$ (all the M_i of the Minkowski decomposition are congruent to standard simplices of dimension m_i in $\mathbb{R}^{m_i} \times \mathbb{R}^{n-m_i}$ under the action of $SL_n(\mathbb{Z})$).

From now on, we use m_i to refer to the number of non-zero vectors of the lattice polytope M_i, and V_i to the $m_i \times n$ matrix whose rows are the non-zero vertices of M_i.

Remark 3.2.2. *Admissible decomposition of Q will imply that Altmann's versal deformation provides a smoothing of Y_σ, as shown in the following Theorem 3.2.4.*

Given a matrix V_i as in Definition 3.2.1, there is a $(n - m_i) \times n$ matrix E_i such that the rows of the matrix

$$X_i = \begin{pmatrix} V_i \\ E_i \end{pmatrix}$$

is a basis of $\mathbb{Z}^n$. The matrix X_i is invertible, and its inverse Z_i has integer entries. Let A_i be a $n \times m_i$ matrix and C_i be a $n \times (n - m_i)$ matrix defined by the formula:

$$Z_i = \begin{pmatrix} A_i & C_i \end{pmatrix}.$$

We obtain the following relations:

$$\begin{pmatrix} V_i \\ E_i \end{pmatrix} \begin{pmatrix} A_i & C_i \end{pmatrix} = \begin{pmatrix} V_i \cdot A_i & V_i \cdot C_i \\ E_i \cdot A_i & E_i \cdot C_i \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}. \tag{3.2}$$

Definition 3.2.3. After fixing a choice for E_i, we call the triple of matrices (A_i, C_i, E_i) satisfying (3.2), the matrices associated with V_i.

3.2.1 Auxiliar complex fibration

The Theorem below provides a complex fibration on a smoothing of Y_σ. This result was already known (see [42]), however, in our proof we find explicit global coordinates associated with the terms in the Minkowski decomposition. These coordinates are used to analyze the singular Lagrangian torus fibration in Section 3.2.2.

Theorem 3.2.4. *Let $Q = M_1 + M_2 + \ldots + M_k$ be an admissible Minkowski decomposition of Q. Then, there exists a complex fibration on a deformation $Y_{\widetilde{\sigma},\epsilon}$ of Y_σ, the toric variety associated with $\sigma = C(Q)$, over $\mathbb{C}$ such that it associates to each M_i a singular fibre that is isomorphic to the subvariety $x_0 \ldots x_{m_i} = 0$ inside $\mathbb{C}^{m_i+1} \times (\mathbb{C}^*)^{n-m_i}$, and the general fibre is $(\mathbb{C}^*)^n$ where $n = \dim Q$. In particular, we see that $Y_{\widetilde{\sigma},\epsilon}$ is smooth.*

Proof. Let $e_1, \dots, e_k$ denote the standard basis of $\mathbb{Z}^k$, and define

$$\widetilde{\sigma} = \text{Cone}\left(\bigcup_{i=1}^{k}((M_i \cap \mathbb{Z}^n) \times \{e_i\})\right) \subseteq \mathbb{R}^n \times \mathbb{R}^k.$$

Let $\mathscr{H}$ be the Hilbert Basis of $\widetilde{\sigma}^\vee$. To have a set of characters that describes our fibration, we are going to define a new set of generators $\mathscr{A}_{\mathscr{H}}$ as follows:

1. The vectors $(\mathbf{0}, e_i)$, $\mathbf{0} \in \mathbb{Z}^n$ for $i = 1, \dots, k$ are elements of $\mathscr{A}_{\mathscr{H}}$.

2. For each $(v, w) \in \mathscr{H}$ with $v \in \mathbb{Z}^n \setminus \mathbf{0}$ and $w \in \mathbb{Z}^k$, $(v, \phi(v))$ is an element of $\mathscr{A}_{\mathscr{H}}$.

3. Let V_i be the matrix associated to M_i, and consider its associated matrices A_i and C_i (Definition 3.2.3). Let $a_{i,j}$ be the columns of A_i, $c_{i,l}$ be the columns of C_i, and $b_i = -\sum_{j=1}^{m_i} a_{i,j}$. The vectors $\widetilde{a}_{i,j} = (a_{i,j}, \phi(a_{i,j}))$, $(\widetilde{c}^+)_{i,l} = (c_{i,l}, \phi(c_{i,l}))$, $(\widetilde{c}^-)_{i,l} = (-c_{i,l}, \phi(-c_{i,l}))$, and $\widetilde{b}_i = (b_i, \phi(b_i))$ are elements of $\mathscr{A}_{\mathscr{H}}$.

$$V_i = \begin{pmatrix} — & v_{i,1} & — \\ & \vdots & \\ — & v_{i,m_i} & — \end{pmatrix}, \quad A_i = \begin{pmatrix} | & & | \\ a_{i,1} & \dots & a_{i,m_i} \\ | & & | \end{pmatrix},$$

$$C_i = \begin{pmatrix} | & & | \\ c_{i,1} & \dots & c_{i,n-m_i} \\ | & & | \end{pmatrix}.$$

Lemma 3.2.5. *The set $\mathscr{A}_{\mathscr{H}}$ generates $\sigma^\vee$, i.e, $\mathbb{N}\mathscr{A}_{\mathscr{H}} = \widetilde{\sigma}^\vee \cap M$.*

Proof. Our strategy is to prove that $\mathscr{A}_{\mathscr{H}} \subseteq \mathbb{N}\mathscr{H}$, and $\mathscr{H} \subseteq \mathbb{N}\mathscr{A}_{\mathscr{H}}$. Then, we conclude that $\mathbb{N}\mathscr{A}_{\mathscr{H}} = \mathbb{N}\mathscr{H} = \widetilde{\sigma}^\vee \cap M$.

First, we are going to prove the inclusion $\mathscr{A}_{\mathscr{H}} \subseteq \mathbb{N}\mathscr{H} = \widetilde{\sigma}^\vee \cap M$. By Definition 2.2.16, it is equivalent to prove that $\langle a, m\rangle \geq 0, \forall a \in \mathscr{A}_{\mathscr{H}}, \forall m \in \widetilde{\sigma}$. Recall that in our construction the elements of $\mathscr{A}_{\mathscr{H}}$ are of the form $(\mathbf{0}, e_i)$, where $\mathbf{0} \in \mathbb{Z}^n$ and $e_1, \dots, e_k$ is the standard basis of $\mathbb{Z}^k$, or of the form $(v, \phi(v))$ where $v \in \mathbb{Z}^n \setminus \mathbf{0}$. It is clear that $\langle(\mathbf{0}, e_i), m\rangle \geq 0, \forall m \in \widetilde{\sigma}$. For $(v, \phi(v))$ with $v \neq 0$ and $(q_i, e_i) \in (M_i \cap \mathbb{Z}^n) \times \{e_i\}$, we have:

$$\begin{aligned}
\langle(v, \phi(v)), (q_i, e_i)\rangle &= \langle v, q_i\rangle + \langle\phi(v), e_i\rangle \\
&= \langle v, q_i\rangle + \max\{\langle v, -M_i\rangle\} \\
&\geq \langle v, q_i\rangle + \langle v, -q_i\rangle \\
&= 0
\end{aligned}$$

We conclude that $\mathscr{A}_{\mathscr{H}} \subseteq \mathbb{N}\mathscr{H} = \widetilde{\sigma}^\vee \cap M$.

Now, we are going to prove that $\mathscr{H} \subseteq \mathbb{N}\mathscr{A}_{\mathscr{H}}$. If $(v, w) \in \mathscr{H}$ with $v \in \mathbb{Z}^n$ and $w \in \mathbb{Z}^k$, then

$$\langle(v, w), (q_i, e_i)\rangle = \langle v, q_i\rangle + \langle w, e_i\rangle \geq 0$$

for all $(q_i, e_i) \in (M_i \cap \mathbb{Z}^n) \times \{e_i\}$. This implies that $\langle w, e_i\rangle \geq \langle v, -q_i\rangle, \forall q_i \in M_i$, therefore $\langle w, e_i\rangle \geq \max\{\langle v, -M_i\rangle\} = \langle \phi(v), e_i\rangle$. Therefore,

$$(v, w) = (v, \phi(v)) + \sum_{i=1}^{k} (\langle w, e_i\rangle - \langle \phi(v), e_i\rangle)(\mathbf{0}, e_i).$$

We conclude that $(v, w) \in \mathbb{N}\mathscr{A}_{\mathscr{H}}$, since $(\mathbf{0}, e_i), (v, \phi(v)) \in \mathscr{A}_{\mathscr{H}}$ and $\langle w, e_i\rangle - \langle \phi(v), e_i\rangle \geq 0$. $\square$

Let us now describe the deformation $Y_{\widetilde{\sigma},\epsilon}$ of Y_σ associated with the characters of $\mathscr{A}_{\mathscr{H}}$, and a complex fibration on $Y_{\widetilde{\sigma},\epsilon}$. Define

$$Y_{\widetilde{\sigma}} := Y_{\mathscr{A}_{\mathscr{H}}} = \mathrm{Spec}(\mathbb{C}([\widetilde{\sigma}^\vee \cap M]).$$

Let $t_i = \chi^{(\mathbf{0}, e_i)}$ and $\Psi : Y_{\widetilde{\sigma}} \to \mathbb{C}^{k-1}$ given by $(t_1 - t_2, \ldots, t_1 - t_k)$ similar to [30, §3] . From [6], $Y_\sigma = \Psi^{-1}(\mathbf{0})$. Let $\epsilon_1 = 0$, $\epsilon := (\epsilon_2, \ldots, \epsilon_k)$, where the ϵ_i are pairwise distinct and distinct from 1. Consider $Y_{\widetilde{\sigma},\epsilon} := \Psi^{-1}(\epsilon)$ and the complex fibration $f : Y_{\widetilde{\sigma},\epsilon} \to \mathbb{C}$ given by the projection to t_1.

We note that the singular fibres of f lie over $t_1 = \epsilon_i$ for $i = 1, \ldots, k$. Indeed, we have the following lemma.

Lemma 3.2.6. *If $t_1 \neq \epsilon_i$, we have that $f^{-1}(t_1) \cong (\mathbb{C}^*)^n$.*

Proof. The main idea is to find n non-zero characters of $Y_{\widetilde{\sigma},\epsilon}$ in $f^{-1}(t_1)$ such that they generate all the other characters. For instance, we are going to prove that for all $p = 1, \ldots, k$ the characters $x_{p,l} := \chi^{\widetilde{a}_{p,l}}$, $y_p := \chi^{\widetilde{b}_p}$,$w^+_{p,j} := \chi^{(\widetilde{c}^+)_{p,j}}$, and $w^-_{p,j} := \chi^{(\widetilde{c}^-)_{p,j}}$ generate all the other characters. Also, these characters satisfy the relations

$$y_p \prod_{l=1}^{m_p} x_{p,l} = t_p \prod_{\substack{j=1 \\ j\neq p}}^{k} t_j^{\beta_j(M_p)}, \tag{3.3}$$

for some $\beta_j(M_p) \in \mathbb{Z}$.

$$w^+_{p,j} w^-_{p,j} = \prod_{\substack{l=1 \\ l\neq p}}^{k} t_l^{\eta_{j,l}(M_p)}, \tag{3.4}$$

for all $j = 1, \ldots, n - m_p$, and for some $\eta_{j,l}(M_p) \in \mathbb{Z}$.

Since in $\Psi^{-1}(\epsilon)$, $t_j \neq 0, \forall j = 1, \ldots, k$, we get that $y_p \prod_{l=1}^{m_p} x_{p,l} \neq 0$, and $w^+_{p,j} w^-_{p,j} \neq 0$. So, we can choose $x_{p,1}, \ldots, x_{p,m_p}, w^+_{p,1}, \ldots, w^+_{p,n-m_p}$ as our n non-zero characters that generate all the others.

First, we are going to prove that equations (3.3) and (3.4) hold. They will also be useful later, in particular the fact that the exponent of t_p is 1 in (3.3), and $\eta_{j,p}(M_p) =$

0 in (3.4). Recall that we are using the multiplication induced by the semigroup structure of $\widetilde{\sigma}^\vee \cap \mathbb{Z}^{n+k}$. Therefore

$$b_p + \sum_{j=1}^{m_p} a_{p,j} = 0,$$

implies

$$y_p \prod_{l=1}^{m_p} x_{p,l} = \prod_{j=1}^{k} t_j^{\beta_j(M_p)},$$

for some $\beta_j(M_p) \in \mathbb{Z}$. In order to prove that $\beta_p(M_p) = 1$, we are going to check that

$$\langle \phi(b_p), e_p \rangle + \sum_{j=1}^{m_p} \langle \phi(a_{p,l}), e_p \rangle = 1.$$

By definition $\langle \phi(a_{p,l}), e_p \rangle = \max\{\langle a_{p,l}, -M_p \rangle\}$. Recall the relations in (3.2). Since $V_p \cdot A_p = Id$ we have that $\langle a_{p,l}, -v_{p,j} \rangle = 0$ for $l \neq j$, and $\langle a_{p,l}, -v_{p,l} \rangle = -1$. We conclude that $\langle \phi(a_{p,l}), e_p \rangle = 0$. On the other hand, $\langle \phi(b_p), e_p \rangle = \max\{\langle b_p, -M_p \rangle\}$ and it follows from $V_p \cdot A_p = Id$ that $\langle b_p, -v_{p,j} \rangle = 1$ for all j. Hence $\langle \phi(b_p), e_p \rangle = 1$. Therefore, we get $\beta_p(M_p) = 1$ and we obtain Equation (3.3).

As before, the first n coordinates of $\widetilde{c}^{+}_{p,l} - \widetilde{c}^{-}_{p,l}$ is $c_{p,l} - c_{p,l} = 0$, which implies that

$$w^{+}_{p,j} w^{-}_{p,j} = \prod_{l=1}^{k} t_l^{\eta_{j,l}(M_p)}$$

for some $\eta_{j,l}(M_p) \in \mathbb{Z}$. The fact that $\eta_{j,p}(M_p) = 0$ follows from $\langle \phi(c_{p,l}), e_p \rangle = \langle \phi(-c_{p,l}), e_p \rangle = 0$ since $c_{p,l} \in \ker V_p$, $\forall l = 1, \ldots, n - m_p$ ($V_p \cdot C_p = 0$, see (3.2)). This conclude the proof of the equations (3.3) and (3.4).

Now we are going to prove that all the characters of $Y_{\widetilde{\sigma},\epsilon}$ in $f^{-1}(t_1)$ depend only on

$$x_{p,1}, \ldots, x_{p,m_p}, w^{+}_{p,1}, \ldots, w^{+}_{p,n-m_p}.$$

By our construction, the elements of $\mathscr{A}_{\mathscr{H}}$, which are different from $(\mathbf{0}, e_i)$, are of the form $(\widehat{z}, \phi(\widehat{z}))$ with $\widehat{z} \in \mathbb{Z}^n \setminus \mathbf{0}$. There exists a unique way to write $\widehat{z}$ as a sum of the vectors $a_{p,1}, \ldots, a_{p,m_p}, c_{p,1}, \ldots, c_{p,n-m_p}$ over $\mathbb{Z}$, because $(A_p \quad C_p)$ is invertible, see (3.2). Hence, for all $\widehat{z} \in \mathbb{Z}^n \setminus \mathbf{0}$,

$$\widehat{z} = \sum_{l=1}^{m_p} \xi_l a_{p,l} + \sum_{j=1}^{n-m_p} \xi_{m_p+j} c_{p,j} \tag{3.5}$$

with $\xi_l \in \mathbb{Z}$ for $l = 1, \ldots, n$. Letting z be the character corresponding to $(\widehat{z}, \phi(\widehat{z}))$ we obtain:

$$z = \prod_{i=1}^{m_p} (x_{p,i})^{\xi_i} \prod_{l=1}^{n-m_p} (w^{+}_{p,l})^{\xi_{m_p+l}} \prod_{j=1}^{k} t_j^{\eta_{p,j}(z)} \tag{3.6}$$

for some $\eta_{p,j}(z) \in \mathbb{Z}$. Note that all the ξ_i and all the $\eta_{p,j}(z)$ are allowed to be negative since by Equation (3.3) and (3.4), we have $x_{p,l}$ and $w^+_{p,l}$ non-zero. Equation (3.6) proves the dependency that we want. $\square$

We conclude the proof of the proposition by analyzing the singular fibres.

Lemma 3.2.7. *$f^{-1}(\epsilon_p)$ is isomorphic to the subvariety $x_0 x_{p,1} \dots x_{p,m_p} = 0$ inside $\mathbb{C}^{m_p+1} \times (\mathbb{C}^*)^{n-m_p}$.*

Proof. In the same way as in the proof of Lemma 3.2.6, we are going to prove that all the characters of $Y_{\widetilde{\sigma},\epsilon}$, in $f^{-1}(\epsilon_p)$ depend only in

$$x_0 = y_p, x_{p,1}, \dots, x_{p,m_p}, w^+_{p,1}, \dots, w^+_{p,n-m_p}.$$

Note that Equations (3.3) and (3.4) hold in general, and that in $f^{-1}(\epsilon_p)$ we have $t_p = 0$. In this setting, we obtain the following relations:

$$y_p \prod_{l=1}^{m_p} x_{p,l} = 0,$$

$$w^+_{p,j} w^-_{p,j} = \prod_{\substack{l=1\\ l\neq p}}^{k} t_l^{\eta_{j,l}(M_p)} \neq 0,$$

for all $j = 1, \dots, n-m_p$ and some $\eta_{j,l}(M_p) \in \mathbb{Z}$.

Recall that the elements of $\mathscr{A}_{\mathscr{H}}$, different from $(\mathbf{0}, e_i)$, are of the form $(\widehat{z}, \phi(\widehat{z}))$ with $\widehat{z} \in \mathbb{Z}^n \setminus \mathbf{0}$. In this context, equation (3.5) holds, but we will need to modify it since now $x_{p,l}$ can be zero, so we need non-negative coefficients. For that, we will make use of the character y_p.

Recalling that $b_p = -\sum a_{p,l}$, define $\xi^+ = \max\{0, -\xi_1, \dots, -\xi_{m_p}\} \geq 0$ and $\xi^+_l = \xi_l + \xi^+ \geq 0$ for $l = 1, \dots, m_p$. We have:

$$\widehat{z} = \xi^+ b_p + \sum_{l=1}^{m_p} \xi^+_l a_{p,l} + \sum_{j=1}^{n-m_p} \xi_{m_p+j} c_{p,j}. \tag{3.7}$$

Now, since $t_p = 0$, we will not be able to correct the p-th coordinate of $\phi(\widehat{z})$ as we write the character z associated with $(\widehat{z}, \phi(\widehat{z}))$ in terms of $y_p, x_{p,l}, w^+_{p,j}, t_k$ $k \neq p$. It will suffice to show that:

$$\langle \phi(\widehat{z}), e_p \rangle = \xi^+ \tag{3.8}$$

because from $0 \in M_p$, $V_p \cdot A_p = \mathrm{Id}$ and $c_{p,j} \in \ker V_p$ we get that $\langle \phi(a_{p,l}), e_p \rangle = \langle \phi(c_{p,j}), e_p \rangle = 0$ and $\langle \phi(b_p), e_p \rangle = 1$. So equation (3.8) together with equation (3.7) implies:

$$z = y_p^{\xi^+} \prod_{i=1}^{m_p} (x_{p,i})^{\xi^+_i} \prod_{l=1}^{n-m_p} (w^+_{p,l})^{\xi_{m_p+l}} \prod_{\substack{j=1\\ j\neq p}}^{k} t_j^{\eta_{p,j}(z)} \tag{3.9}$$

for some $\eta_{p,j}(z) \in \mathbb{Z}$, which concludes the proof of Lemma 3.2.7.

Note that by definition, $\xi^+, \xi_1^+, \ldots, \xi_{m_p}^+$ are non-negative, and at least one is zero. If $\xi^+ = 0$, then:

$$\begin{aligned}
\langle \phi(\widehat{z}), e_p \rangle &= \left\langle \phi\left(\sum_{l=1}^{m_p} \xi_l^+ a_{p,l} + \sum_{j=1}^{n-m_p} \xi_{m_p+j} c_{p,j} \right), e_p \right\rangle \\
&= \max\left\{ \left\langle \sum_{l=1}^{m_p} \xi_l^+ a_{p,l} + \sum_{j=1}^{n-m_p} \xi_{m_p+j} c_{p,j}, -M_p \right\rangle \right\} \\
&= \max\left\{ \left\langle \sum_{l=1}^{m_p} \xi_l^+ a_{p,l}, -M_p \right\rangle \right\} \\
&\leq \sum_{l=1}^{m_p} \xi_l^+ \max\left\{ \langle a_{p,l}, -M_p \rangle \right\} \\
&= 0
\end{aligned}$$

since $\mathbf{0} \in M_p$ and $V_p \cdot A_p = \mathrm{Id}$. Therefore, $\langle \phi(\widehat{z}), e_p \rangle = 0$.

If $\xi^+ \neq 0$, then there exists a $j \in \{1, \ldots, m_p\}$ such that $\xi_j^+ = 0$, then

$$\begin{aligned}
\langle \phi(\widehat{z}), e_p \rangle &= \left\langle \phi\left(\xi^+ b_p + \sum_{l=1}^{m_p} \xi_l^+ a_{p,l} + \sum_{j=1}^{n-m_p} \xi_{m_p+j} c_{p,j} \right), e_p \right\rangle \\
&= \max\left\{ \left\langle \xi^+ b_p + \sum_{l=1}^{m_p} \xi_l^+ a_{p,l} + \sum_{j=1}^{n-m_p} \xi_{m_p+j} c_{p,j}, -M_p \right\rangle \right\} \\
&= \max\left\{ \left\langle \xi^+ b_p + \sum_{l=1}^{m_p} \xi_l^+ a_{p,l}, -M_p \right\rangle \right\} \\
&\leq \xi^+ \max\left\{ \langle b_p, -M_p \rangle \right\} + \sum_{l=1}^{m_p} \xi_l^+ \max\left\{ \langle a_{p,l}, -M_p \rangle \right\} \\
&= \xi^+.
\end{aligned}$$

Since $v_{p,j} \in M_p$ and $\xi_j^+ = 0$, we get:

$$\langle \phi(\widehat{z}), e_p \rangle \geq \langle \xi^+ b_p + \sum \xi_l^+ a_{p,l}, v_{p,j} \rangle = \xi^+,$$

recalling again that $V_p \cdot A_p = \mathrm{Id}$. Therefore, $\langle \phi(\widehat{z}), e_p \rangle = \xi^+$. $\square$

We see now that Lemma 3.2.6 and Lemma 3.2.7 imply Theorem 3.2.4.

$\square$

3.2.2 The singular Lagrangian fibration

As in Section 3.1, we will construct a Lagrangian fibration associated with the complex fibration f. First, we endow $Y_{\widetilde{\sigma},\epsilon}$ with the symplectic form coming from the embedding as a subset of $\mathbb{C}^{|\mathscr{A}_{\mathscr{H}}|}$. Next, we are going to study the action of T^n on each fibre of f.

Recall that in $Y_{\widetilde{\sigma}} = Y_{\mathscr{A}_{\mathscr{H}}}$ we have an algebraic torus given by the image of $\Phi_{\mathscr{A}_{\mathscr{H}}}$ (see Definition 2.2.5). We are interested in the induced action of the circles given by

$$\theta_i := \Phi_{\mathscr{A}_{\mathscr{H}}}(\{1\} \times \cdots \times \{1\} \times S^1 \times \{1\} \cdots \times \{1\}), \tag{3.10}$$

where $S^1 \subseteq \mathbb{C}^*$ is in the position i for $i = 1, \ldots, n$. Let μ_{θ_i} be the moment map of the action of θ_i and define:

$$\lambda_i := 2\mu_{\theta_i} = \sum_{q \in \mathscr{A}_{\mathscr{H}}} q_i |\chi^q|^2$$

where q_i is the i-th coordinate of q. We define $\lambda = (\lambda_1, \ldots, \lambda_n)$, by our construction, we have that

$$\lambda = \sum_{q \in \mathscr{A}_{\mathscr{H}}} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} |\chi^q|^2 = \sum_{(\widehat{z}, \phi(\widehat{z})) \in \mathscr{A}_{\mathscr{H}}} \widehat{z} \cdot |\chi^{(\widehat{z}, \phi(\widehat{z}))}|^2. \tag{3.11}$$

We will consider Lagrangian tori, which are contained in $f^{-1}(\gamma(r))$ for the circle $\gamma(r) \subset \mathbb{C}$ with center in 1 and radius r, and consist of a single $(S^1)^n$-orbit inside each fibre of $\gamma(r)$. Our Lagrangian torus will be given by symplectic parallel transport of the T^n orbits along $\gamma(r)$.

Definition 3.2.8. Given the circle $\gamma(r) \subset \mathbb{C}$ and $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$, we define

$$T_{\gamma(r),\delta} := f^{-1}(\gamma(r)) \cap \lambda^{-1}(\delta).$$

We showed above that the general fibre of f is isomorphic to $(\mathbb{C}^*)^n$, therefore $T_{\gamma(r),\delta}$ is an embedded Lagrangian torus in $Y_{\widetilde{\sigma}}$, except possibly when $\epsilon_i \in \gamma(r)$, or in the limit when $r = 0$, and we have an isotropic T^n.

We assume that there are not two ϵ_i in the same circle $\gamma(r)$. Let us understand the singular Lagrangian for when $\epsilon_p \in \gamma(r)$. Recall that we identify $f^{-1}(\epsilon_p)$ as a subset of $\mathbb{C}^{m_p+1} \times (\mathbb{C}^*)^{n-m_p}$ given by the characters

$$y_p, x_{p,1}, \ldots, x_{p,m_p}, w_1^+, \ldots, w_{n-m_p}^+ \tag{3.12}$$

and the equation

$$y_p \prod_{l=1}^{m_p} x_{p,l} = 0.$$

Note that if n of the characters in (3.12) are non-zero, we have a T^n-orbit in the fibre, and hence the Lagrangian passing through it is a smooth T^{n+1}. When $k \geq 2$ coordinates are zero, we get a singular Lagrangian $T^{n+1}/\mathfrak{T}_{k-1}$, where $\mathfrak{T}_{k-1}$ is a $k-1$ cycle inside $T^n \subseteq T^{n+1}$. This gives rise to a codimension k strata of the singular locus Σ of the Lagrangian fibration.

Singular fibres and collapsing cycles

We will analyze the monodromies around the codimension $k = 2$ strata of Σ, i.e., the cases where exactly $n-1$ of the characters in (3.12) are non-zero, identifying the cycle in T^n that collapses as we approach that point. We then get $\frac{m_p(m_p+1)}{2}$ collapsing 1-cycles associated with M_p.

Lemma 3.2.9. *In $f^{-1}(\epsilon_p)$,*

1. *$x_{p,i} = y_p = 0$ if and only if $\langle v_{p,i}, \lambda\rangle = 0$ and $\langle v_{p,l}, \lambda\rangle \geq 0, \forall l \neq i$.*
2. *$x_{p,i} = x_{p,j} = 0$ with $i \neq j$ if and only if $\langle v_{p,i} - v_{p,j}, \lambda\rangle = 0$, $\langle v_{p,i}, \lambda\rangle \leq 0$, and $\langle v_{p,i} - v_{p,l}, \lambda\rangle \leq 0, \forall l \neq i, j$.*

As a consequence, the collapsing cycles related with M_p are $v_{p,i} \cdot (\theta_1, \ldots, \theta_n)$ for $i = 1, \ldots, m_p$ and $(v_{p,i} - v_{p,j}) \cdot (\theta_1, \ldots, \theta_n)$ for $i \neq j$.

Proof. Recall that the elements of $\mathscr{A}_{\mathscr{H}}$, different from $(\mathbf{0}, e_i)$, are of the form $(\widehat{z}, \phi(\widehat{z}))$ and that we proved equation (3.7). In this setting, we have:

$$\chi^{(\widehat{z},\phi(\widehat{z}))} = y_p^{\xi^+(\widehat{z})} \prod_{l=1}^{m_p} (x_{p,l})^{\xi_l^+(\widehat{z})} \prod_{l=1}^{n-m_p} (w_{p,l}^+)^{\xi_{m_p+l}(\widehat{z})} \prod_{\substack{l=1 \\ j\neq p}}^{k} t_l^{\eta_{p,l}(z)},$$

and also from (3.11)

$$\lambda = \sum_{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}} \widehat{z} \cdot |\chi^{(\widehat{z},\phi(\widehat{z}))}|^2.$$

1. ($\Rightarrow$) If $x_{p,i} = y_p = 0$, we get:

$$\lambda = \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}} \\ \xi^+(\widehat{z})=\xi_i^+(\widehat{z})=0}} \widehat{z} \cdot |\chi^{(\widehat{z},\phi(\widehat{z}))}|^2.$$

   Using equation (3.7), $V_p A_p = Id$, and $c_{p,l} \in \ker V_p$, we have $\langle v_{p,i}, \widehat{z}\rangle = \xi_i(\widehat{z}) = \xi_i^+(\widehat{z}) - \xi^+(\widehat{z}) = 0$ and $\langle v_{p,l}, \widehat{z}\rangle = \xi_l(\widehat{z}) = \xi_l^+(\widehat{z}) - \xi^+(\widehat{z}) = \xi_l^+(\widehat{z}) \geq 0$, when $\xi^+(\widehat{z}) = \xi_i^+(\widehat{z}) = 0$. Therefore, $\langle v_{p,i}, \lambda\rangle = 0$ and $\langle v_{p,l}, \lambda\rangle \geq 0, \forall l \neq i$.

($\Leftarrow$) If $\langle v_{p,i}, \lambda\rangle = 0$ and $\langle v_{p,l}, \lambda\rangle \geq 0, \forall l \neq i$. As before, we have $\langle v_{p,i}, \widehat{z}\rangle = \xi_i(\widehat{z}) = \xi_i^+(\widehat{z}) - \xi^+(\widehat{z})$ and $\langle v_{p,l}, \widehat{z}\rangle = \xi_l(\widehat{z}) = \xi_l^+(\widehat{z}) - \xi^+(\widehat{z})$, therefore:

$$\langle v_{p,i}, \lambda\rangle = \sum_{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}} (\xi_i^+(\widehat{z}) - \xi^+(\widehat{z}))|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 = 0, \tag{3.13}$$

and

$$\langle v_{p,l}, \lambda\rangle = \sum_{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}} (\xi_l^+(\widehat{z}) - \xi^+(\widehat{z}))|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 \geq 0. \tag{3.14}$$

Recall that at least one of the characters $y_p, x_{p,1}, \ldots, x_{p,m_p}$ is equal to zero. If $y_p = 0$, then from equation (3.13) we get:

$$\langle v_{p,i}, \lambda\rangle = |x_{p,i}|^2 + \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}\setminus\{\widetilde{a}_{p,i}\}\\ \xi^+(\widehat{z})=0}} \xi_i^+(\widehat{z})|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 = 0,$$

so $x_{p,i} = 0$ as we want.
If $x_{p,i} = 0$, then from equation (3.13) we get:

$$\langle v_{p,i}, \lambda\rangle = -|y_p|^2 - \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}\setminus\{\widetilde{b}_p\}\\ \xi_i^+(\widehat{z})=0}} \xi^+(\widehat{z})|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 \geq 0,$$

so $y_p = 0$ as we want.
If $x_{p,l} = 0, l \neq i$, then from equation (3.14) we get:

$$\langle v_{p,l}, \lambda\rangle = -|y_p|^2 - \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}\setminus\{\widetilde{b}_p\}\\ \xi_l^+(\widehat{z})=0}} \xi^+(\widehat{z})|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 \geq 0,$$

so $y_p = 0$. From equation (3.13), we get $x_{p,i} = 0$ as above.

2. ($\Rightarrow$) If $x_{p,i} = x_{p,j} = 0, i \neq j$, we get:

$$\lambda = \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}\\ \xi_i^+(\widehat{z})=\xi_j^+(\widehat{z})=0}} \widehat{z}\cdot|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2.$$

As before, using equation (3.7), $V_pA_p = Id$, and $c_{p,l} \in \ker V_p$, we have $\langle v_{p,i} - v_{p,j}, \widehat{z}\rangle = \xi_i(\widehat{z}) - \xi_j(\widehat{z}) = \xi_i^+(\widehat{z}) - \xi_j^+(\widehat{z}) = 0$, $\langle v_{p,i}, \widehat{z}\rangle = \xi_i(\widehat{z}) = \xi_i^+(\widehat{z}) - \xi^+(\widehat{z}) \leq 0$, and $\langle v_{p,i} - v_{p,l}, \widehat{z}\rangle = \xi_i(\widehat{z}) - \xi_l(\widehat{z}) = \xi_i^+(\widehat{z}) - \xi_l^+(\widehat{z}) \leq 0$, when $\xi_i^+(\widehat{z}) = \xi_j^+(\widehat{z}) = 0$. Therefore, $\langle v_{p,i} - v_{p,j}, \lambda\rangle = 0$, $\langle v_{p,i}, \lambda\rangle \leq 0$, and $\langle v_{p,i} - v_{p,l}, \lambda\rangle \leq 0, \forall l \neq i, j$.

($\Leftarrow$) If $\langle v_{p,i} - v_{p,j}, \lambda\rangle = 0$, $\langle v_{p,i}, \lambda\rangle \leq 0$, and $\langle v_{p,i} - v_{p,l}, \lambda\rangle \leq 0, \forall l \neq i, j$. As above:

$$\langle v_{p,i} - v_{p,j}, \lambda\rangle = \sum_{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}} (\xi_i^+(\widehat{z}) - \xi_j^+(\widehat{z}))|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 = 0, \tag{3.15}$$

$$\langle v_{p,i}, \lambda \rangle = \sum_{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}} (\xi_i^+(\widehat{z}) - \xi^+(\widehat{z}))|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 \leq 0, \tag{3.16}$$

and,

$$\langle v_{p,i} - v_{p,l}, \lambda \rangle = \sum_{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}} (\xi_i^+(\widehat{z}) - \xi_l^+(\widehat{z}))|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 \leq 0. \tag{3.17}$$

Recall that at least one of the characters $y_p, x_{p,1}, \dots, x_{p,m_p}$ is equal to zero. If $x_{p,i} = 0$, then from equation (3.15) we get:

$$\langle v_{p,i} - v_{p,j}, \lambda \rangle = -|x_{p,j}|^2 - \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}\setminus\{\widetilde{a}_{p,j}\} \\ \xi_i^+(\widehat{z})=0}} \xi_j^+(\widehat{z})|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 = 0,$$

so $x_{p,j} = 0$ as we want.
If $x_{p,j} = 0$, then from equation (3.15) we get:

$$\langle v_{p,i} - v_{p,j}, \lambda \rangle = |x_{p,i}|^2 + \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}\setminus\{\widetilde{a}_{p,i}\} \\ \xi_j^+(\widehat{z})=0}} \xi_i^+(\widehat{z})|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 = 0,$$

so $x_{p,i} = 0$ as we want.
If $x_{p,l} = 0, l \neq i, j$, then from equation (3.17) we get:

$$\langle v_{p,i} - v_{p,l}, \lambda \rangle = |x_{p,i}|^2 + \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}\setminus\{\widetilde{b}_p\} \\ \xi_l^+(\widehat{z})=0}} \xi_i^+(\widehat{z})|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 \leq 0,$$

so $x_{p,i} = 0$. From equation (3.15), we get $x_{p,j} = 0$ as above.
If $y_p = 0$, then from equation (3.16) we get:

$$\langle v_{p,i}, \lambda \rangle = |x_{p,i}|^2 + \sum_{\substack{(\widehat{z},\phi(\widehat{z}))\in\mathscr{A}_{\mathscr{H}}\setminus\{\widetilde{a}_{p,i}\} \\ \xi_i^+(\widehat{z})=0}} \xi_i^+(\widehat{z})|\chi^{(\widehat{z},\phi(\widehat{z}))}|^2 \leq 0,$$

so $x_{p,i} = 0$. From equation (3.15), we get $x_{p,j} = 0$ as above.

$\square$

Convex base diagram

Now we will study the convex base diagram produced by this singular Lagrangian fibration. As in Section 3.1, we will consider codimension one cuts in the base of the Lagrangian fibration, in the complement of which we will have a toric structure. Let us now assume that $|\epsilon_{i+1} - 1| > |\epsilon_i - 1|$ for $i = 1, \dots, k-1$, so the circles $\gamma(r)$ pass first through ϵ_i and then through ϵ_{i+1} as r increases. If $r < 1$, $T_{\gamma(r),\delta}$ determine a toric structure and we can take action coordinates $(\lambda_1, \dots, \lambda_{n+1})$ given by the symplectic flux concerning a reference fibre $T_{\gamma(r_0),\mathbf{0}}$, as in the action-angle coordinates given in [23]. Considering the limit when $r_0 = 0$, we can interpret λ_{n+1} as the symplectic area of a disk with boundary in a cycle of $T_{\gamma(r),\delta}$ that collapses as $r \to 0$.

Our tori, $T_{\gamma(r),\lambda}$, have fixed values $\lambda = (\lambda_1, \ldots, \lambda_n)$. Therefore, when $r = |\epsilon_p - 1|$, we have singular fibres for values λ related with M_p and given by Lemma 3.2.9. To kill monodromies around singular fibres, we introduce codimension one cuts (i.e., disregard fibres) for which $r \geq |\epsilon_p - 1|$ and λ is related with M_p and given by Lemma 3.2.9, as above. (Recall Figure 3.1 on our toy model.)

We call the image B of $\pi : Y_{\tilde{\sigma},\epsilon} \to \mathbb{R}^{n+1}$ given by $(\lambda_1, \ldots, \lambda_{n+1})$, and denote by Σ the locus on B corresponding to singular Lagrangian fibres. With the same notations as in Section 2.3, let S_p be the image of the singular fibres corresponding to the case when $\epsilon_p \in \gamma(r)$, and let the cones C_p be defined with $v_p = (\mathbf{0}, 1), \mathbf{0} \in \mathbb{R}^n$, for $p = 1, \ldots, k$. Let

$$D_{p,0} = \{(\lambda_1, \ldots, \lambda_{n+1}) \in \mathrm{Im}(\pi) | \langle v_{p,l}, \lambda \rangle > 0, \forall l = 1, \ldots, m_p\}, \tag{3.18}$$

$$D_{p,j} = \{(\lambda_1, \ldots, \lambda_{n+1}) \in \mathrm{Im}(\pi) | \langle v_{p,j}, \lambda \rangle < 0, \langle v_{p,j} - v_{p,l}, \lambda \rangle < 0, \forall l \neq j\}, \tag{3.19}$$

for $j = 1, \ldots, m_p$, and define $\mathfrak{b}_{p,j}$, a circle in the convex base diagram going around the j-th codimension two strata of the set of singular fibres associated with M_p, as in Section 2.3.

In order to fix a basis for $H_1(T_{\gamma(r),\delta})$, recall the definition of θ_i in equation (3.10) and let θ_{n+1} be the collapsing class associated with λ_{n+1}. Fix $\{\theta_1, \ldots, \theta_{n+1}\}$ as the basis for $H_1(T_{\gamma(r),\delta})$. We will study the topological and affine monodromy of the circles $\mathfrak{b}_{p,j}$.

Each $\mathfrak{b}_{p,j}$ will cross the hyperplane $\langle v_{p,j}, \lambda \rangle = 0$, so the topological monodromy of $\mathfrak{b}_{p,j}$ is:

$$M_{\mathfrak{b}_{p,j}} = \begin{pmatrix} & & & | \\ & \mathrm{Id} & & v_{p,j} \\ & & & | \\ 0 & \ldots & 0 & 1 \end{pmatrix},$$

because the $(\theta_1, \ldots, \theta_n)$ cycles corresponding to the T^n-action remains invariant, while the θ_{n+1} cycle is twisted in the direction of the collapsing cycle of the monodromy.

Hence the affine monodromy, in dual coordinates of $(\theta_1, \ldots, \theta_{n+1})$, is the transpose inverse of $M_{\mathfrak{b}_{p,j}}$:

$$M^{\mathrm{af}}_{\mathfrak{b}_{p,j}} = \begin{pmatrix} & & & 0 \\ & \mathrm{Id} & & \vdots \\ & & & 0 \\ — & -v_{p,j} & — & 1 \end{pmatrix}$$

for $j = 1, \ldots, m_p$.

In this setting, we do the transferring the cut operation (Definition 2.3.14) concerning C_p for $p = 1, \dots, k$. We claim that applying the matrices $M^{\mathrm{af}}_{\mathfrak{b}_{p,j}}$ as above, we cancel the monodromies related to crossing the hyperplane $\langle v_{p,i} - v_{p,j}, \lambda \rangle = 0$. These affine monodromies are of the form:

$$M^{\mathrm{af}}_{v_{p,j}-v_{p,i}} = \begin{pmatrix} & & & 0 \\ & \mathrm{Id} & & \vdots \\ & & & 0 \\ - & v_{p,i} - v_{p,j} & - & 1 \end{pmatrix}$$

with $j \neq i$, and act in $\overline{D_{p,j}}$. The monodromy $M^{\mathrm{af}}_{v_{p,i}-v_{p,j}} = M_{\mathfrak{b}_{p,i}} \cdot (M_{\mathfrak{b}_{p,j}})^{-1}$ acts in $\overline{D_{p,i}}$. Note that in the transferring the cut operation, we are applying $M^{\mathrm{af}}_{\mathfrak{b}_{p,j}}$ in $\overline{D_{p,j}}$. We can conclude that the monodromies related to $v_{p,j} - v_{p,i}$ are canceled.

We conclude this section with a lemma that relates the convex base diagram obtained after applying all the transferring the cut operations and the cone $\sigma^\vee$.

Lemma 3.2.10. *The convex base diagram, obtained after applying all the transferring the cut operations to the convex base diagram B, has image $\sigma^\vee$.*

Proof. We will consider a description of a set of generators of $\sigma^\vee$ given in [6]. To each $c \in \mathbb{Z}^n$, we associate an integer by $\eta_0(c) := \max\{\langle c, -Q \rangle\}$. By the definition of η_0, we have

$$\partial \sigma^\vee \cap \mathbb{Z}^{n+1} = \{(c, \eta_0(c)) | c \in \mathbb{Z}^n\}.$$

Moreover, if $c_1, \dots, c_w \in \mathbb{Z}^n \setminus \mathbf{0}$ are those elements producing irreducible pairs $(c, \eta_0(c))$ (i.e. not allowing any non-trivial lattice decomposition $(c, \eta_0(c)) = (c', \eta_0(c'))$ $+(c'', \eta_0(c'')))$, then the elements

$$(c_1, \eta_0(c_1)), \dots, (c_w, \eta_0(c_w)), (\mathbf{0}, 1)$$

form a generator set for $\sigma^\vee \cap \mathbb{Z}^{n+1}$ as a semigroup.

We also have the following equality:

$$\begin{aligned} \eta_0(c) &= \max\{\langle c, -Q \rangle\} \\ &= \sum_{p=1}^{k} \max\{\langle c, -M_p \rangle\} \\ &= \sum_{p=1}^{k} \langle \phi(c), e_p \rangle \end{aligned}$$

This equality shows the relation between generators of $\sigma^\vee$ and our set $\mathscr{A}_{\mathscr{H}}$ of generators of $\widetilde{\sigma}^\vee$.

Let $(c,d) \in \mathbb{R}^n \times \mathbb{R}$. If $(c,d) \in \overline{D_{p,0}}$, by (3.18), we have that $\langle v_{p,l}, c\rangle \geq 0, \forall l = 1,\dots,m_p$, then $\max\{\langle c, -M_p\rangle\} = 0$. Note that this coincides with the fact that we do not apply any matrix to $\overline{D_{p,0}}$.

If $(c,d) \in \overline{D_{p,j}}$, by (3.19), we have that $\langle v_{p,j}, c\rangle \leq 0, \langle v_{p,j} - v_{p,l}, c\rangle \leq 0, \forall l \neq j$, then $\max\{\langle c, -M_p\rangle\} = \langle c, -v_{p,j}\rangle$. Recall that in the transferring the cut operation related with C_p, we apply the matrix $M^{\mathrm{af}}_{\mathfrak{b}_{p,j}}$ to $\overline{D_{p,j}}$, therefore we get:

$$\begin{pmatrix} & & & 0 \\ & \mathrm{Id} & & \vdots \\ & & & 0 \\ - & -v_{p,j} & - & 1 \end{pmatrix} \begin{pmatrix} | \\ c \\ | \\ d \end{pmatrix} = \begin{pmatrix} | \\ c \\ | \\ \max\{\langle c, -M_p\rangle\} + d \end{pmatrix}. \tag{3.20}$$

In equation (3.20), the additive property in the last coordinate of applying the transferring the cut operations is clear. After applying all the monodromies to $(c,0)$, we get:

$$\left(c, \sum_{p=1}^{k} \max\{\langle c, -M_p\rangle\}\right) = \left(c, \sum_{p=1}^{k} \langle \phi(c), e_p\rangle\right) = (c, \eta_0(c)).$$

Since the boundary of our initial diagram B is formed by $(c,0)$, $c \in \mathbb{R}^n$, we deduce the relation between $\sigma^\vee$ and the convex base diagram. $\square$

Monotone fibres

Considering the new diagram from Lemma 3.2.10, we are going to prove that the fibres over $\mathbb{R}e_{n+1} \cap \sigma^\vee$ outside of the cuts are monotone. Recall that the faces of the cone $\sigma^\vee$ are of the form $H_m \cap \sigma^\vee$ for some $m \in \sigma$, where

$$H_m := \{u \in \mathbb{R}^{n+1} | \langle u, m\rangle = 0\}.$$

Let $\mathfrak{L}_s$ to be the torus associated to $(\mathbf{0}, s) \in \sigma^\vee$ with $\mathbf{0} \in \mathbb{R}^n$ and $s \in \mathbb{R}_{\geq 0}$, that is the torus $T_{\gamma(r),0}$ for which $\lambda_{n+1} = s$. Recall that our cone σ is generated by elements of the form $(v,1), v \in \mathbb{R}^n$. Therefore, we have the following result.

Lemma 3.2.11. *The area of a disk with boundary on the torus $\mathfrak{L}_s$ and Maslov index 2 is s.*

Proof. From toric geometry, we know that the disk given by carrying the collapsing cycle $\mathfrak{v}$ associated to a facet along a path $c(t) \subset B^{\mathrm{reg}}$ from the reference fibre to the corresponding facet, has Maslov index two and area $\int_0^T \langle c'(t), \mathfrak{v}\rangle dt$. The normals to the faces of the cone $\sigma^\vee$ are of the form $(v,1), v \in \mathbb{R}^n$, then we consider

$$c(t) = L_v(t) = \begin{pmatrix} \mathbf{0} \\ s \end{pmatrix} - t \begin{pmatrix} v \\ 1 \end{pmatrix}$$

and the time t_0 such that $L_v(t_0) \in H_{(v,1)}$. We get the following equation:

$$\left\langle \begin{pmatrix} \mathbf{0} \\ s \end{pmatrix} - t_0 \begin{pmatrix} v \\ 1 \end{pmatrix}, \begin{pmatrix} v \\ 1 \end{pmatrix} \right\rangle = 0,$$

therefore:

$$t_0 = \frac{s}{|v|^2+1}.$$

Then the area of the disk is:

$$\int_0^{t_0} \langle c'(t), \mathfrak{v}\rangle dt = \int_0^{t_0} \langle (v,1),(v,1)\rangle dt = s.$$

Now note that the boundaries of the above disks generate $\pi_1(\mathfrak{L}_s)$. In particular, $\pi_1(\mathfrak{L}_s) \xrightarrow{0} \pi_1(Y_{\widetilde{\sigma},\epsilon})$ and hence $\pi_2(Y_{\widetilde{\sigma},\epsilon}, \mathfrak{L}_s) \cong \pi_1(\mathfrak{L}_s) \oplus \pi_2(Y_{\widetilde{\sigma},\epsilon})$. We know that $Y_{\widetilde{\sigma},\epsilon}$ is affine, hence Weinstein, so $c_1 = \omega = 0$ in $\pi_2(Y_{\widetilde{\sigma},\epsilon})$, therefore all Maslov index 2 disks with boundary on $\mathfrak{L}_s$ have area s, i.e., $\omega = \frac{s}{2}\mu$ in $\pi_2(Y_{\widetilde{\sigma},\epsilon}, \mathfrak{L}_s)$. □

3.2.3 Special Lagrangian condition

Let (X, ω, J) be a Kähler n-dimensional manifold, D a divisor of X, and Ω a non-vanishing holomorphic n-form on $X \setminus D$. Recall the definition of special Lagrangian submanifold from [8].

Definition 3.2.12 ([8, Definition 2.1])**.** A Lagrangian submanifold $L \subset X \setminus D$ is **special Lagrangian** with phase ϕ if Im $(e^{-i\phi}\Omega)_{|_L} = 0$.

In $Y_{\widetilde{\sigma},\epsilon} \setminus \{t_1 = 1\}$, consider the holomorphic $(n+1,0)$-form given by

$$\Omega = i^{n+1} \frac{dx_{1,1} \wedge \cdots \wedge dx_{1,m_1} \wedge dw^+_{1,1} \wedge \dots dw^+_{1,n-m_1} \wedge dt_1}{(t_1 - 1)\prod x_{1,j} \prod w^+_{1,l}}$$

Similar to in [8, Proposition 5.2], we can prove that the Lagrangians from Definition 3.2.8 are special.

Proposition 3.2.13. *The tori $T_{\gamma(r),\delta}$ are special Lagrangian with respect to Ω.*

Proof. Let $H : Y_{\widetilde{\sigma},\epsilon} \to \mathbb{R}$ given by $|t_1 - 1|^2$, and let X_H the vector field such that $\iota_{X_H}\omega = dH$. As in the proof of [8, Proposition 5.2], we have that X_H is tangent to $T_{\gamma(r),\delta}$.
The tangent space to $T_{\gamma(r),\delta}$ is spanned by X_H and by the vector fields generating the T^n-action. These vector fields are given by

$$X_{\theta_k} = i\left(\sum_{j=1}^{m_p} (a_{1,j})_k x_{1,j}\frac{\partial}{\partial x_{1,j}} + \sum_{l=1}^{n-m_p} (c_{1,l})_k w^+_{1,l}\frac{\partial}{\partial w^+_{1,l}}\right)$$

for $k = 1, \dots, n$, where $(a_{1,j})_k$ and $(c_{1,l})_k$ denotes the k-th coordinates of $a_{1,j}$ and $c_{1,l}$, respectively. Recalling that $\det Z_i = \pm 1$ (see (3.2))

$$\Omega(X_{\theta_1}, \dots, X_{\theta_n}, \cdot) = \pm i^{2n+1}\frac{dt_1}{t_1 - 1} = \pm i^{2n+1} d(\log(t_1 - 1))$$

Therefore, Im $\Omega(X_{\theta_1}, \dots, X_{\theta_n}, X_H) = \pm d(\log|t_1 - 1|)(X_H)$, which vanishes since X_H is tangent to the level sets of H.

□

3.3 Wall-crossing

In this section, we discuss a general formula for the potential by studying the wall-crossing phenomena. We are going to consider the monotone fibres $T_{\gamma(r),0}$; for $r > |\epsilon_k - 1|$ that arises as a series of "wall-crossing" mutations ([48]) from the monotone fibre $T_{\gamma(r),0}, r < |\epsilon_1 - 1|$.

The tori $T_{\gamma(r),0}$ is an oriented spin (taking the standard spin structure on the torus) monotone Lagrangian of dimension $n+1$ and, given a choice of basis for $H_1(T_{\gamma(r),0})$ and assuming $T_{\gamma(r),0}$ bounds no Maslov zero disks, its potential $W_{T_{\gamma(r),0}} \in \mathbb{C}[z_1^{\pm}, \dots, z_{n+1}^{\pm}]$, encoding information of Maslov index two holomorphic disks with boundary on $T_{\gamma(r),0}$, is given by [48, Definition 2.6]. Recall the basis $\{\theta_1, \dots, \theta_{n+1}\}$ of $H_1(T_{\gamma(r),0}, \mathbb{Z})$ defined in Section 3.2.2 and let z_i be the variable associated to a disk with boundary in θ_i. With this basis, the potential takes the form:

$$W_{T_{\gamma(r),0}} = \sum_{\substack{\beta \in H_2(Y_{\tilde{\sigma},\epsilon}, T_{\gamma(r),0}) \\ \mu(\beta)=2}} n_\beta \cdot \mathbf{z}^{\partial\beta} \tag{3.21}$$

where n_β is the count of disks in class β, we consider $\partial\beta \in H_1(T_{\gamma(r),0}) \cong \mathbb{Z}^{n+1}$, and denote $\mathbf{z}^l = z_1^{l_1} \cdots z_{n+1}^{l_{n+1}}$. We are interested in studying how this potential changes when we increase the value of r in $T_{\gamma(r),0}$. These changes in the potential occur whenever $T_{\gamma(r),0}$ passes through a wall formed by tori $T_{\gamma(r),\lambda}$ that bound Maslov zero disks. Hence the walls are formed by the tori corresponding to r equals to $|\epsilon_p - 1|$ for $p = 1, \dots, k$. By Theorem 3.2.4, our local model around the walls corresponds to the one studied in [48, Section 5.4]. Therefore we can use the mutation formula [48, Lemma 5.17] in the form:

Lemma 3.3.1 ([48, Lemma 5.17]). *Let $|\epsilon_{p-1} - 1| < r < |\epsilon_p - 1| < r' < |\epsilon_{p+1} - 1|$. The potentials $W_{T_{\gamma(r),0}}$ and $W_{T_{\gamma(r'),0}}$ are related by the mutation*

$$W_{T_{\gamma(r),0}} = \mu(W_{T_{\gamma(r),0}}),$$

where μ is given by

$$z_i \mapsto z_i, \quad i = 1, \dots, n,$$
$$z_{n+1} \mapsto z_{n+1}(1 + z^{v_{p,1}} + \cdots + z^{v_{p,m_p}}),$$

where we recall that $v_{p,l} \in \mathbb{Z}^n$ and consider $z^v = z_1^{v_1} \cdots z_n^{v_n}$.

From this, the potential function, $\mathfrak{PO} := W_{T_{\gamma(r),0}}$ for $r > |\epsilon_k - 1|$, associated to our construction based on an admissible Minkowski decomposition $Q = M_1 + \cdots + M_k$ is:

$$\mathfrak{PO}(z_1, \dots, z_{n+1}) = z_{n+1} \prod_{i=1}^{k} \left(1 + \sum_{j=1}^{m_i} z^{v_{i,j}}\right), \tag{3.22}$$

where $v_{i,j} \in \mathbb{Z}^n$ are the non-zero vertices of M_i and we use the notation $z^{(v_1,\dots,v_n)} = z_1^{v_1} \cdots z_n^{v_n}$.

Remark 3.3.2. *The potential* (3.22) *was found by Lau [42, Corollary 4.16] using a different method.*

Let $\rho \in (\mathbb{K}^*)^{n+1}$ encode a local system of $T_{\gamma(r),0}$, with respect to the given base of $H_1(T_{\gamma(r),0})$. Here we will take $K^* = \Lambda^*, \mathbb{C}^*$, or $\mathrm{U}(1)$ depending on the setting, where Λ denotes the Novikov field over $\mathbb{C}$. The self Floer cohomology of $(T_{\gamma(r),0}, \rho)$ is non-zero if and only if ρ is a critical point of $W_{T_{\gamma(r),0}}$ (see [48, Remark 2.2], [54, Corollary 2.8], [28, Theorem 2.3]). Let

$$P_i := \left(1 + \sum_{j=1}^{m_i} z^{v_{i,j}}\right)$$

for $i = 1, \dots, k$.

Lemma 3.3.3. *For $K^* = \Lambda^*, \mathbb{C}^*$, or $\mathrm{U}(1)$, the potential function (3.22) has a critical point in $(K^*)^{n+1}$ if and only if there exist $i, j \in \{1, \dots, k\}, i \neq j$, such that the system of equations $P_i = P_j = 0$ has a solution in $(K^*)^{n+1}$.*

Proof. First, suppose that there exist $i, j \in \{1, \dots, k\}, i \neq j$, such that the system of equations $P_i = P_j = 0$ has a solution in $(K^*)^{n+1}$, then:

$$\frac{\partial \mathfrak{PO}}{\partial z_{n+1}} = \prod_{l=1}^{k} P_l = 0,$$

and

$$\frac{\partial \mathfrak{PO}}{\partial z_m} = z_{n+1} \left(\sum_{l=1}^{k} \frac{\partial P_l}{\partial z_m} \cdot \prod_{q \neq l} P_q \right) = 0$$

for $m \neq n+1$. Therefore, the potential function has a critical point in $(K^*)^{n+1}$.

Now, suppose that the potential function has a critical point in $(K^*)^{n+1}$, then

$$\frac{\partial \mathfrak{PO}}{\partial z_{n+1}} = \prod_{l=1}^{k} P_l = 0$$

implies that there exists an $i \in \{1, \dots, k\}$, such that $P_i = 0$. Therefore, for $m \neq n+1$, we have:

$$\begin{aligned}\frac{\partial \mathfrak{PO}}{\partial z_m} &= z_{n+1} \left(\frac{\partial P_i}{\partial z_m} \cdot \prod_{q \neq i} P_q + P_i \left(\sum_{l \neq i} \frac{\partial P_l}{\partial z_m} \cdot \prod_{\substack{q \neq l \\ q \neq i}} P_q \right) \right) \\ &= z_{n+1} \frac{\partial P_i}{\partial z_m} \cdot \prod_{q \neq i} P_q = 0.\end{aligned}$$

Since the critical point is in $(K^*)^{n+1}$, ${\partial P_i}/{\partial z_m} = 0$ or there exists $j \neq i$ such that $P_j = 0$. For the sake of contradiction, suppose that there does not exist $j \neq i$ such that $P_j = 0$. In this case ${\partial P_i}/{\partial z_m} = 0, \forall m$, moreover:

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} z_1 \frac{\partial P_i}{\partial z_1} \\ \vdots \\ z_n \frac{\partial P_i}{\partial z_n} \end{pmatrix} = \begin{pmatrix} | & & | \\ v_{i,1} & \dots & v_{i,m_i} \\ | & & | \end{pmatrix} \begin{pmatrix} z^{v_{i,1}} \\ \vdots \\ z^{v_{i,m_i}} \end{pmatrix}.$$

The non-zero vertices $v_{i,1}, \dots, v_{i,m_i}$ of M_i are linearly independent because we are working with an admissible Mikowski decomposition. Therefore, $z^{v_{i,1}} = \dots = z^{v_{i,m_i}} = 0$, and it is a contradiction as our critical point is in $(K^*)^{n+1}$. We conclude that there exists $j \neq i$ such that $P_j = 0$ as we want. □

Example 3.3.4. The potential function related to our toy example in Section 3.1 is:

$$\mathfrak{PO}(z_1, z_2, z_3) = z_3(1 + z_1 z_2)(1 + z_1 + z_2).$$

Using Lemma 3.3.3, we conclude that this potential has two one-parameter families of critical points.

3.3.1 On the convex hull of the potential

We will analyze the monomials in $\mathfrak{PO}$.

Lemma 3.3.5. *The Newton polytope of the potential is $Q \times \{1\} \subset \mathbb{R}^{n+1}$.*

Proof. Set $v_{i,0} = \mathbf{0} \in \mathbb{Z}^n$ for $i = 1, \dots, k$. Observe that

$$\begin{aligned} \prod_{i=1}^{k} \left(1 + \sum_{j=1}^{m_i} z^{v_{i,j}}\right) &= \prod_{i=1}^{k} \left(\sum_{j=0}^{m_i} z^{v_{i,j}}\right) \\ &= \sum_{i_1, \dots, i_k} z^{v_{1,i_1} + \dots + v_{k,i_k}}. \end{aligned}$$

Let $\mathfrak{P} := \{v_{1,i_1} + \dots + v_{k,i_k} | i_j \in \{0, \dots, m_j\}, \forall j = 1, \dots, k\}$. From the definition of a Minkowski sum (Definition 2.2.24), we have $\partial Q \cap \mathbb{Z}^n \subseteq \mathfrak{P} \subseteq Q \cap \mathbb{Z}^n$. Therefore, using (3.22), we conclude that the Newton polytope of the potential is $Q \times \{1\} \subset \mathbb{R}^{n+1}$. □

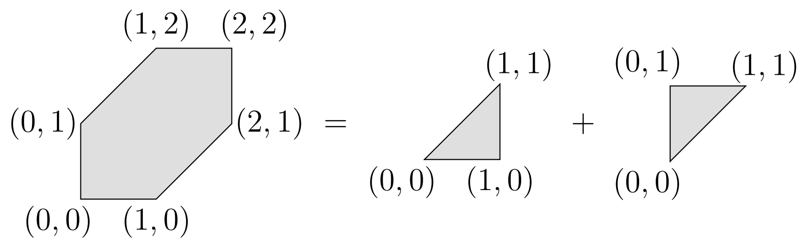
3.4 Other Examples

In this section, we will study some examples following the same steps as above. We will leave some details to the reader.

3.4.1 Cone of Q_6

Consider $Q_6 := \text{Conv}\{(0,0), (1,0), (0,1), (2,1), (1,2), (2,2)\} \subset \mathbb{R}^2$ and its two Minkowski decompositions:

First decomposition of Q_6



(3.23)

We write $Q_6 = M_1 + M_2$, where $M_1 = \mathrm{Conv}\{(0,0),(1,0),(1,1)\}$ and $M_2 = \mathrm{Conv}\{(0,0),(0,1),(1,1)\}$. We follow the same steps as in Section 3.2 to study the convex base diagram and the potential.

Recalling our restricted Lagrangian fibration construction, the walls are in $r = |\epsilon_i - 1|$. The collapsing classes corresponding to the wall $r = 1$ are $(1,0,0)$ and $(1,1,0)$, and the collapsing classes corresponding to the wall $r = |1-\epsilon|$ are $(0,1,0)$ and $(1,1,0)$ in the base $(\theta_1, \theta_2, \theta_3)$, as in (3.10). So, we obtain the convex base diagram presented on the left of Figure 3.4.

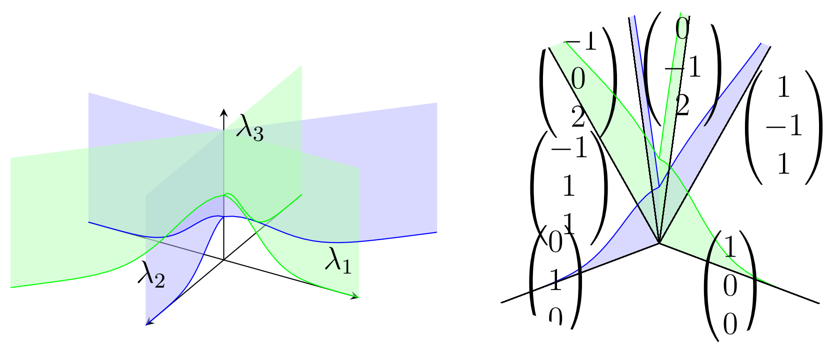


Figure 3.4: Convex base diagrams corresponding to the restricted Lagrangian fibration related with the first decomposition of Q_6.

With the same notations as in Section 2.3, let S_1 be the image of the singular fibres corresponding to the case when $0 \in \gamma(r)$, and S_2 be the image of the singular fibres corresponding to the case when $\epsilon \in \gamma(r)$. The cones C_i for $i = 1, 2$, are defined with $v_1 = v_2 = (0,0,1)$. Let

$$D_{1,0} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 > 0, \lambda_1 + \lambda_2 > 0\},$$

$$D_{1,1} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 < 0, \lambda_2 > 0\},$$

$$D_{1,2} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 + \lambda_2 < 0, \lambda_2 < 0\},$$

$$D_{2,0} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_2 > 0, \lambda_1 + \lambda_2 > 0\},$$

$$D_{2,1} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 < 0, \lambda_1 + \lambda_2 < 0\},$$

$$D_{2,2} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 > 0, \lambda_2 < 0\},$$

and define $\mathfrak{b}_{i,j}$ as in Section 2.3.

The topological monodromies $M_{\mathfrak{b}_{i,j}}$ around the circles $\mathfrak{b}_{i,j}$ are:

$$M_{\mathfrak{b}_{1,1}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_{\mathfrak{b}_{1,2}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, M_{\mathfrak{b}_{2,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{\mathfrak{b}_{2,2}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

respectively.

The corresponding affine monodromies are:

$$M^{af}_{\mathfrak{b}_{1,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, M^{af}_{\mathfrak{b}_{1,2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}, M^{af}_{\mathfrak{b}_{2,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix},$$

$$M^{af}_{\mathfrak{b}_{2,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

The result of applying two transferring the cut operations (Definition 2.3.14) is obtained by following the steps:

1. Apply the matrix $M^{af}_{\mathfrak{b}_{1,1}}$ to $\overline{D_{1,1}}$.
2. Apply the matrix $M^{af}_{\mathfrak{b}_{1,2}}$ to $\overline{D_{1,2}}$.
3. Apply the matrix $M^{af}_{\mathfrak{b}_{2,1}}$ to the image of $\overline{D_{2,1}}$ after applying step 1 and 2.
4. Apply the matrix $M^{af}_{\mathfrak{b}_{2,2}}$ to the image of $\overline{D_{2,2}}$ after applying step 1 and 2.

The resulting convex base diagram is presented on the right of Figure 3.4.

As we proved in Section 3.2, this cone coincides with:

$$\sigma^\vee = \mathrm{Cone}\left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

After applying the matrix

$$\mathscr{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in SL(3, \mathbb{Z})$$

to $\sigma^\vee$, we obtain:

$$\mathscr{M}\sigma^\vee = \text{Cone}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

In this new cone, we have $Q_6^\vee$ at height 1.

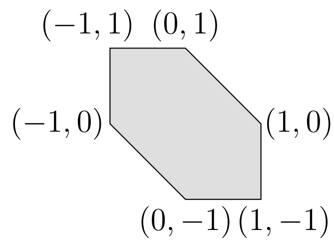


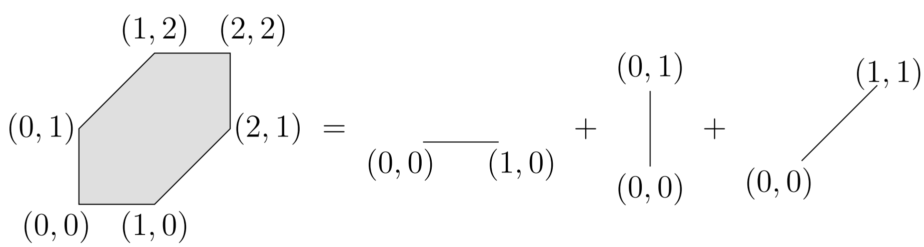
Figure 3.5: $Q_6^\vee$

The corresponding potential function for the monotone Lagrangian L_a is:

$$\mathfrak{PO}(z_1, z_2, z_3) = z_3(1 + z_1 + z_1 z_2)(1 + z_2 + z_1 z_2). \tag{3.24}$$

Using Lemma 3.3.3, we conclude that this potential has critical points for $K^* = \Lambda^*, \mathbb{C}^*$, or $\mathrm{U}(1)$, where $z_1 = z_2 = e^{\frac{2\pi}{3}ki}$ for $k = 1, 2$, i.e., are roots of $z^2 + z + 1$, and z_3 is any element of K^*.

Second decomposition of Q_6



(3.25)

We write $Q_6 = M_1 + M_2 + M_3$, where $M_1 = \text{Conv}\{(0,0),(1,0)\}$, $M_2 = \text{Conv}\{(0,0),(0,1)\}$, and $M_3 = \text{Conv}\{(0,0),(1,1)\}$. We follow the same steps as in Section 3.2 to study the convex base diagram and the potential.

Recalling our restricted Lagrangian fibration construction, the walls are in $r = |\epsilon_i - 1|$. The collapsing classes corresponding to the walls $r = 1, |1 - \epsilon_1|, |1 - \epsilon_2|$ are $(1,0,0)$, $(0,1,0)$, and $(1,1,0)$ in the base $(\theta_1, \theta_2, \theta_3)$, as in (3.10), respectively. So, we obtain the convex base diagram presented on the left of Figure 3.6.

As in the first decomposition, we can perform the transferring the cuts operations. The resulting convex base diagram is presented on the right of Figure 3.6.

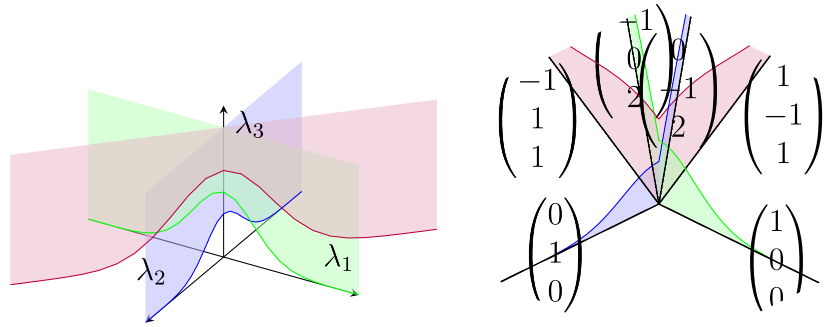


Figure 3.6: Convex base diagrams corresponding to the restricted Lagrangian fibration related with the second decomposition of Q_6.

We get the same cone as in Section 3.4.1. The corresponding potential function is:

$$\mathfrak{PO}(z_1, z_2, z_3) = z_3(1 + z_1)(1 + z_2)(1 + z_1 z_2). \tag{3.26}$$

Using Lemma 3.3.3, we conclude that this potential has critical points for $K^* = \Lambda^*, \mathbb{C}^*$, or $\mathrm{U}(1)$, where z_3 is free to be any element on K^*, and $(z_1, z_2) \in \{(-1,-1), (-1,1), (1,-1)\}$.

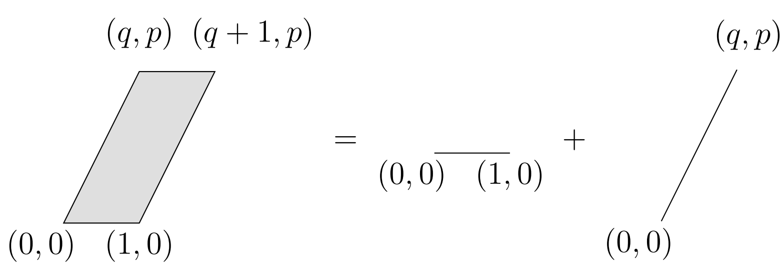
Remark 3.4.1. *These examples have been previously observed by Jonathan Evans and Renato Vianna.*

3.4.2 Symplectization of unit cosphere bundles of 3-dimensional lens spaces

As it was pointed out in [5], the symplectization of $S^*L^3_p(q)$ is the toric symplectic cone determined by the cone $C \subset \mathbb{R}^3$ with normals.

$$v_1 = (q+1, p, 1), v_2 = (0, 0, 1), v_3 = (1, 0, 1), \text{ and } v_4 = (q, p, 1).$$

If we let $Q := \mathrm{Conv}\{(0,0), (1,0), (q,p), (q+1,p)\} \subset \mathbb{R}^2$, and $\sigma = C(Q) \subset \mathbb{R}^3$, then we have that $C = \sigma^\vee$. For Q, we have the following Minkowski decomposition:



Remark 3.4.2. *For $p = q = 1$, we get the singular quadric studied in [19], which is equivalent to the cone on the unitary square.*

We write $Q = M_1 + M_2$, where $M_1 = \mathrm{Conv}\{(0,0),(1,0)\}$ and $M_2 = \mathrm{Conv}\{(0,0),(q,p)\}$, so we can use our construction to understand the convex base diagram and the superpotential of the symplectization of $T^*L_p^3(q)$.

Recalling our restricted Lagrangian fibration construction, the walls are in $r = |\epsilon_i - 1|$. The collapsing class corresponding to the walls $r = 1, |1-\epsilon|$ are $(1,0,0)$, $(q,p,0)$ in the base $(\theta_1,\theta_2,\theta_3)$, as in (3.10), respectively. Therefore, we obtain the convex base diagram at the left of Figure 3.7.

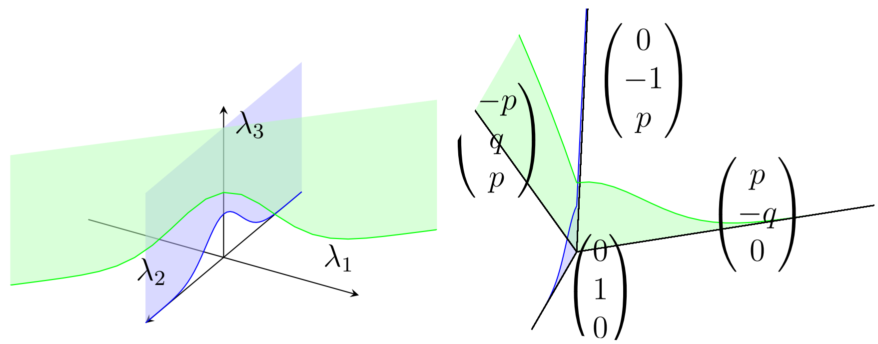


Figure 3.7: Convex base diagrams corresponding to the restricted Lagrangian fibration related with $S^*L_p^3(q)$).

With the same notations as in Section 2.3, let S_1 be the image of the singular fibres corresponding to the case when $0 \in \gamma(r)$, and S_2 be the image of the singular fibres corresponding to the case when $\epsilon \in \gamma(r)$. The cones C_i for $i = 1,2$, are defined with $v_1 = v_2 = (0,0,1)$. Let

$$\begin{aligned} D_{1,0} &= \{(\lambda_1,\lambda_2,\lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 > 0\}, \\ D_{1,1} &= \{(\lambda_1,\lambda_2,\lambda_3) \in \mathrm{Im}(\pi) | \lambda_1 < 0\}, \\ D_{2,0} &= \{(\lambda_1,\lambda_2,\lambda_3) \in \mathrm{Im}(\pi) | q\lambda_1 + p\lambda_2 > 0\}, \\ D_{2,1} &= \{(\lambda_1,\lambda_2,\lambda_3) \in \mathrm{Im}(\pi) | q\lambda_1 + p\lambda_2 < 0\}, \end{aligned}$$

and define $\mathfrak{b}_{i,j}$ as in Section 2.3.

The topological monodromies $M_{\mathfrak{b}_{i,j}}$ around the circles $\mathfrak{b}_{i,j}$ are:

$$M_{\mathfrak{b}_{1,1}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_{\mathfrak{b}_{2,1}} = \begin{pmatrix} 1 & 0 & q \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix}.$$

respectively.

The corresponding affine monodromies are:

$$M^{af}_{\mathfrak{b}_{1,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, M^{af}_{\mathfrak{b}_{2,1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -q & -p & 1 \end{pmatrix}.$$

The result of applying two transferring the cut operations (Definition 2.3.14) is obtained by following the steps:

1. Apply the matrix $M^{af}_{\mathfrak{b}_{1,1}}$ to $\overline{D_{1,1}}$.

2. Apply the matrix $M^{af}_{\mathfrak{b}_{2,1}}$ to the image of $\overline{D_{2,1}}$ after step 1.

The resulting almost toric base diagram is presented at the right of Figure 3.7.

The corresponding potential function is:

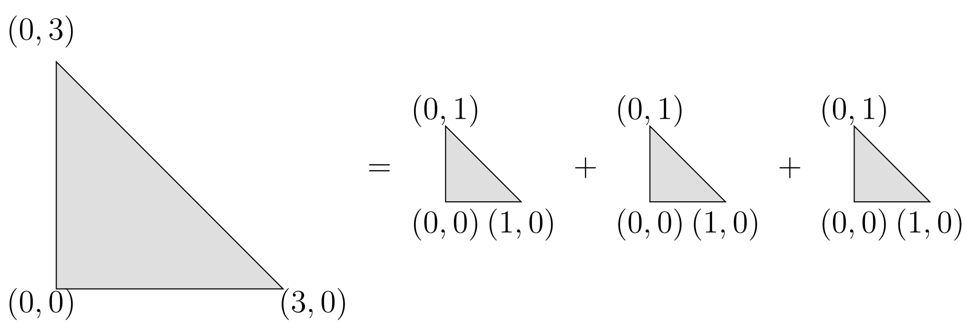
$$\mathfrak{PO}(z_1, z_2, z_3) = z_3(1 + z_1)(1 + z_1^q z_2^p).$$

Using Lemma 3.3.3, we conclude that this potential has critical points for $K^* = \Lambda^*, \mathbb{C}^*$, or $\mathrm{U}(1)$, where $z_1 = -1$, z_2 equals to one of the p roots of $(-1)^{q+1}$, and z_3 is free in K^*.

Remark 3.4.3. *We note that a Lagrangian Lens space $L^3_p(q)$ appears as the union of the T^2-orbits over a segment connecting the singularities at 0 and ϵ, for which $\lambda_1 = \lambda_2 = 0$. The smoothing $Y_{\tilde{\sigma},\epsilon} \cong T^* L^3_p(q)$.*

3.4.3 Cone over the cubic

Consider $Q_3 := \mathrm{Conv}\{(0,0),(3,0),(0,3)\} \subseteq \mathbb{Z}^2$ and its Minkowski decomposition:



We write $Q_3 = M_1 + M_2 + M_3$, where $M_1 = M_2 = M_3 = \mathrm{Conv}\{(0,0),(1,0),(0,1)\}$. This Minkowski decomposition is admissible. Let Y_σ the affine variety related to the cone $\sigma := C(Q_3)$ and $Y_{\tilde{\sigma},\epsilon}$ be the smoothing of Y_σ. The ideal $I(Y_{\tilde{\sigma}}, \epsilon) \subseteq \mathbb{C}[x_1, x_2, y, t]$ is generated by:

$$x_1 x_2 y = t(t - \epsilon_1)(t - \epsilon_2).$$

We obtain a complex fibration, given by the projection to the t character, with three singular fibres modeled as $x_1 x_2 y = 0 \subset \mathbb{C}^3_{(x_1,x_2,y)}$ over $t = 0, \epsilon_1, \epsilon_2$. We build our Lagrangian fibration with a convex base diagram as in the left picture of Figure 3.8. After transferring the cut operations, we get the right picture of Figure 3.8.

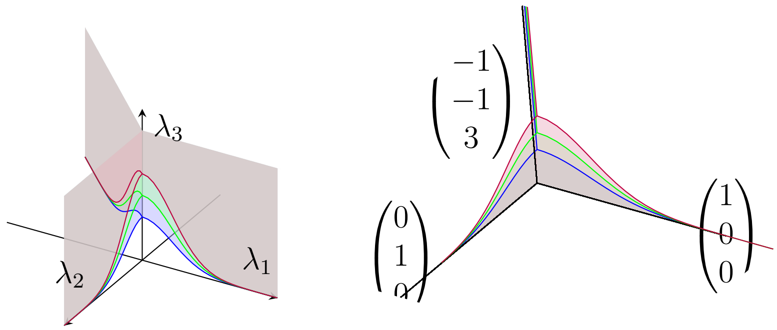


Figure 3.8: Convex base diagrams corresponding to the restricted Lagrangian fibration related to Minkowski decomposition of Q_3.

As we proved in Section 3.2, this cone coincides with:

$$\sigma^\vee = \text{Cone}\left\{\begin{pmatrix}-1\\-1\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}.$$

After applying the matrix

$$\mathscr{M} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & 1\end{pmatrix} \in SL(3,\mathbb{Z})$$

to $\sigma^\vee$, we obtain:

$$\mathscr{M}\sigma^\vee = \text{Cone}\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}.$$

In this new cone, we have $Q_3^\vee$ at height 1.

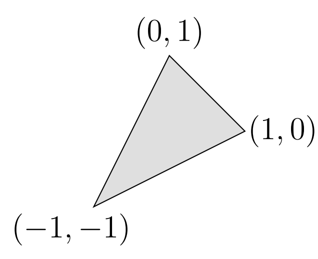


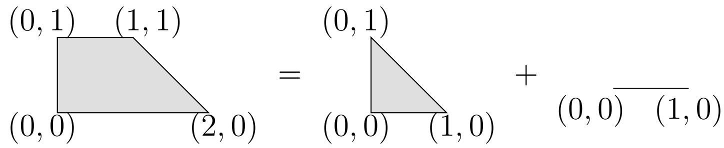
Figure 3.9: $Q_3^\vee$

By (3.22), the potential function is given by:

$$\mathfrak{PO} = z_3(1 + z_1 + z_2)^3, \tag{3.27}$$

and using Lemma 3.3.3, we get that the critical locus has $\dim_{\mathbb{C}} = 2$ when $K^* = \Lambda^*$ or $\mathbb{C}^*$, and a 2-$\dim_{\mathbb{R}}$ family of critical points, when $K^* = \mathrm{U}(1)$.

3.4.4 Example where $\mathfrak{PO}$ doesn't have critical points



Consider $M_1 = \mathrm{Conv}\{(0,0),(0,1),(1,0)\}$, $M_2 = \mathrm{Conv}\{(0,0),(1,0)\}$, and $Q = M_1 + M_2$. This Minkowski decomposition is admissible. Let Y_σ the affine variety related to the cone $\sigma := C(Q)$ and $Y_{\widetilde{\sigma},\epsilon}$ be the smoothing of Y_σ. The ideal $I(Y_{\widetilde{\sigma}},\epsilon) \subseteq \mathbb{C}[x_{1,1}, x_{1,2}, y_1, y_2, z, t]$ is generated by:

$$\begin{array}{lll} x_{1,1}y_1 - tz & x_{1,1}y_2 - t(t-\epsilon) & x_{1,2}z - (t-\epsilon) \\ x_{1,2}y_1 - y_2 & y_1(t-\epsilon) - zy_2 & \end{array}$$

We obtain a complex fibration, given by the projection to the t character, with two singular fibres modeled as $x_{1,1}x_{1,2}y_1 = 0 \subset \mathbb{C}^3_{(x_{1,1},x_{1,2},y_1)}$ over $t = 0$, and $x_{1,1}y_2 = 0 \subset \mathbb{C}^2_{(x_{1,1},y_2)} \times \mathbb{C}^*_{(x_{1,2})}$ over $t = \epsilon$. Out of that, we build our Lagrangian fibration with a convex base diagram as in the left picture of Figure 3.10 (analogous to the one depicted in Figure 3.1 for our toy example). After transferring the cut operations, we get the right picture of Figure 3.10.

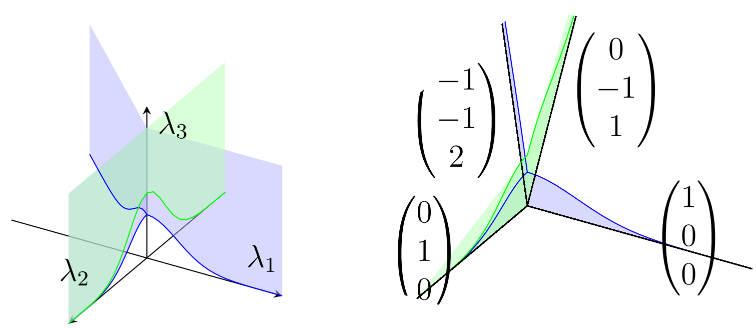


Figure 3.10: Convex base diagrams corresponding to the restricted Lagrangian fibration related to the polytope given by $\mathrm{Conv}\{(0,0),(0,1),(1,1),(2,0)\}$.

By (3.22), the potential function is given by:

$$\mathfrak{PO} = z_3(1 + z_1 + z_2)(1 + z_1), \tag{3.28}$$

and using Lemma 3.3.3, we get that (3.28) does not have critical points.

Remark 3.4.4. *In this case, we can not use McDuff's method of probes [45] to displace the monotone fibres over the line* $\{(0,0,l) | l \in \mathbb{R}_{\geq 0}\}$. *So it remains unclear whether the monotone fibres in this example are displaceable.*

3.5 Compactification

In this section, we will discuss the existence of a compactification of $Y_{\widetilde{\sigma},\epsilon}$ that is compatible with our construction. Under the assumption of Q giving rise to a smooth toric projective Fano variety F_Q, i.e., being the convex hull of the generators of its fan, we get a compactification of $Y_{\widetilde{\sigma},\epsilon}$ as a complete intersection of hyperplanes on a projective variety $X_{\mathscr{A}_{\mathscr{H}}}$. Moreover, this is well adapted with the complex fibration constructed in Theorem 3.2.4, in the sense that we get a 'pencil' in $\overline{Y_{\widetilde{\sigma},\epsilon}}$, well defined in $\overline{Y_{\widetilde{\sigma},\epsilon}} \setminus D \to \mathbb{P}^1$ where $Y_{\widetilde{\sigma},\epsilon} = \overline{Y_{\widetilde{\sigma},\epsilon}} \setminus \overline{F_\infty}$, $\overline{F_\infty} = F_\infty \cup D$ is the compactification of the fibre at infinity and is the toric projective manifold associated with the polytope Q. This pencil extends the fibration $Y_{\widetilde{\sigma},\epsilon} \to \mathbb{C}$ constructed in Theorem 3.2.4.

Symplectically, this corresponds to an infinite rescaling of the base diagram where the image is shrunk to height one, and the divisor $Q^\vee$ appears at height one. In particular, the singular fibres on $Y_{\widetilde{\sigma},\epsilon}$ are asymptotic to height one.

Example 3.5.1. Continuing with the cone over the cubic presented in Section 3.4.3, we discuss its compactification. In Figure 3.11, we show the base diagram of the algebraic compactification.

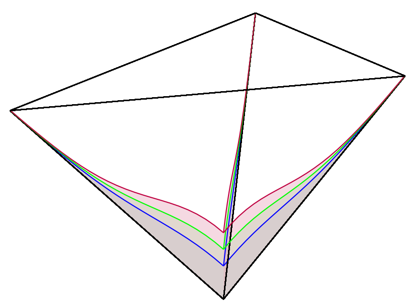


Figure 3.11: Compactifications of the cubic.

Let us now prove the following result that ensures the algebraic compactification when $Y_{\widetilde{\sigma}} = Y_{\mathscr{A}_{\mathscr{H}}}$ is the affine cone of the projective variety $X_{\mathscr{A}_{\mathscr{H}}}$ (see Proposition 2.2.9).

Theorem 3.5.2. *With the assumptions given in Theorem 3.2.4, suppose that $Q = M_1 + \cdots + M_k$ with $k \geq 2$, and that $Y_{\widetilde{\sigma}} = Y_{\mathscr{A}_{\mathscr{H}}}$ is the affine cone of the projective variety $X_{\mathscr{A}_{\mathscr{H}}}$. Then, the compactification of $Y_{\sigma,\epsilon}$ in $\mathbb{CP}^{|\mathscr{A}_{\mathscr{H}}|}$ is isomorphic to the intersection of $k-2$ different hyperplanes in $X_{\mathscr{A}_{\mathscr{H}}}$.*

Proof. The compactification of $Y_{\widetilde{\sigma},\epsilon}$ is the projective variety $\overline{Y_{\widetilde{\sigma},\epsilon}} = V(I^h(Y_{\widetilde{\sigma},\epsilon}))$, where $I^h(Y_{\widetilde{\sigma},\epsilon})$ is the homogenization of the ideal of $Y_{\widetilde{\sigma},\epsilon}$. Recall that

$$Y_{\widetilde{\sigma},\epsilon} = V(I(Y_{\mathscr{A}_{\mathscr{H}}})) \cap \left(\bigcap_{i=1}^{k-1} V(t_1 - t_{i+1} - \epsilon_i) \right) \subseteq \mathbb{C}^{|\mathscr{A}_{\mathscr{H}}|}_{(\ldots,t_1,\ldots,t_k)},$$

then

$$\overline{Y_{\widetilde{\sigma},\epsilon}} = V(I(Y_{\mathscr{A}_{\mathscr{H}}})) \cap \left(\bigcap_{i=1}^{k-1} V(t_1 - t_{i+1} - \epsilon_i z)\right) \subseteq \mathbb{CP}^{|\mathscr{A}_{\mathscr{H}}|}_{(\ldots,t_1,\ldots,t_k,z)}.$$

since by Proposition 2.2.9 the ideal $I(Y_{\mathscr{A}_{\mathscr{H}}})$ is homogeneous, and $t_1 - t_{i+1} - \epsilon_i z$ is the homogenization of $t_1 - t_{i+1} - \epsilon_i$.

Finally, we have the isomorphism:

$$\begin{aligned} F : X_{\mathscr{A}_{\mathscr{H}}} \cap \left(\bigcap_{i=2}^{k-1} V\left(t_1 - t_{i+1} - \epsilon_i \left(\frac{t_1 - t_2}{\epsilon_1}\right)\right)\right) &\longrightarrow \overline{Y_{\widetilde{\sigma},\epsilon}} \\ [\cdots : t_1 : \cdots : t_k] &\longmapsto \left[\cdots : t_1 : \cdots : t_k : \frac{t_1 - t_2}{\epsilon_1}\right] \end{aligned}$$

.

□

In this setting, we can consider the complex fibration

$$\begin{aligned} f : \overline{Y_{\widetilde{\sigma},\epsilon}} &\longrightarrow \mathbb{CP}^1 \\ [\cdots : t_1 : \cdots : t_k : z] &\longmapsto [t_1 : z]. \end{aligned}$$

Note that for $z \neq 0$, it coincides with the complex fibration we constructed before, and that $f^{-1}([1:0]) = X_\sigma \setminus V(t_1)$.

Our toy example in section 3.1 satisfies the conditions of Theorem 3.5.2. By examining the equations, we conclude that $\overline{Y_{\widetilde{\sigma},\epsilon}}$ is equal to the blow-up of $\mathbb{CP}^3$ in one point.

In the example discussed in Section 3.4.1, $Y_{\epsilon,\widetilde{\sigma}}$ compactifies to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (See Figure 3.12). Indeed, one recovers $Y_{\epsilon,\widetilde{\sigma}}$ by deleting a smooth fibre of the "pencil"

$$\begin{aligned} (\mathbb{P}^1)^3 &\to \mathbb{P}^1 \\ ([x_0 : x_1], [y_0 : y_1], [z_0, z_1]) &\mapsto [x_1 y_1 z_1 : x_0 y_0 z_0]. \end{aligned}$$

Here, the singular fibres are at 0 and ∞ instead of 0 and ϵ. We see in (3.26) that

$$\mathfrak{PO} = z_3(1 + z_1 + z_2 + z_1^2 z_2 + z_1 z_2^2 + z_1^2 z_2^2 + 3 z_1 z_2)$$

where $3z_1z_2$ is the term corresponding to the center of the polytope Q. According to [22], the Lagrangians $\mathcal{L}_a$ viewed in the compactification $Y_{\epsilon,\widetilde{\sigma}} \subseteq (\mathbb{P}^1)^3$ are lifts of the monotone Lagrangian $\mathcal{L}'$ on the compactification of the divisor F_∞, viewed as the added fibre at ∞ on the 'pencil' over the compactification, which is a toric manifold with moment polytope $Q^\vee$, i.e., $\mathrm{Bl}_3\mathbb{C}P^2$. Its potential is then given by $T^b\mathfrak{PO} + T^a(z_1z_2z_3)^{-1}$; where $(z_1z_2z_3)^{-1}$ corresponds to the fibre disk passing through the compactification of the fibre F_∞, with area a. $(\mathfrak{PO} - 3z_1z_2z_3)$ correspond the terms in the potential of $\mathcal{L}'$ and $3z_1z_2z_3$ is the extra term, where 3 is reinterpreted

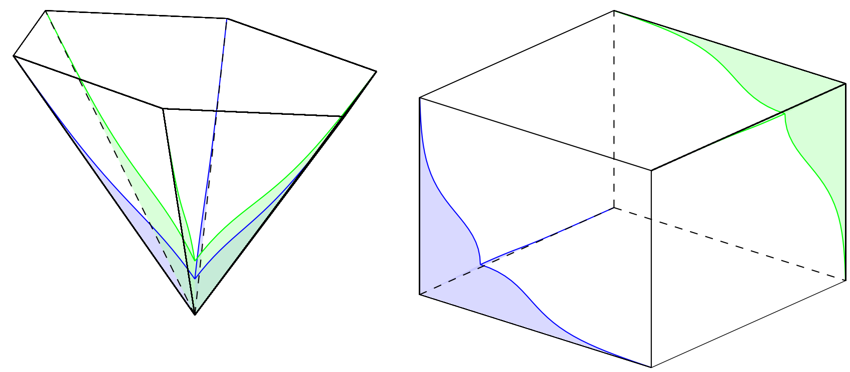


Figure 3.12: Transferring one set of trivalent cuts from the compactified cone over the hexagon, one gets the polytope of $(\mathbb{P}^1)^3$.

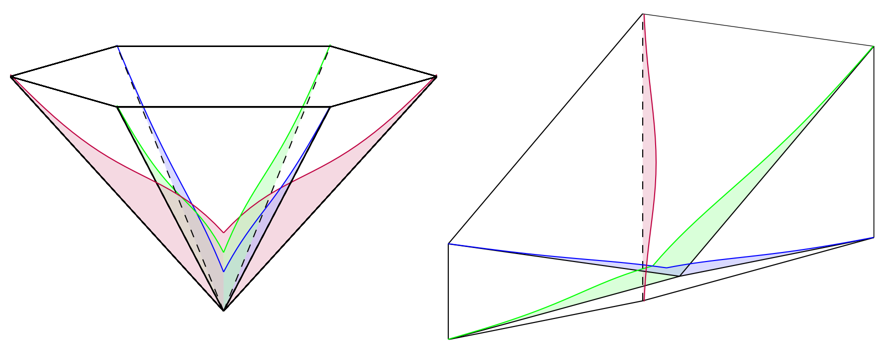


Figure 3.13: Transferring one set of cuts from the compactified cone over the hexagon, one gets a modification of the standard Gelfand-Cetlin polytope in the cases of $Fl(3)$.

in [22] as the number of $c_1 = 1$ rational curves in $(\mathbb{P}^1)^3$ passing transversally through a point in the divisor F_∞.

In the example discussed in Section 3.4.1, it can be seen that $Y_{\widetilde{\sigma},\epsilon}$ compactifies to $Fl(3)$ in this case. One can identify the constructed Lagrangian fibration as a modification of the standard Gelfand-Cetlin polytope in the cases of $Fl(3)$ as indicated in Figure 3.13.

Also for $Y_{\widetilde{\sigma},\epsilon}$ we get that the potential (3.26) is:

$$\mathfrak{PO} = z_3(1 + z_1 + z_2 + z_1^2 z_2 + z_1 z_2^2 + z_1^2 z_2^2 + 2z_1 z_2)$$

and for the corresponding Lagrangian $\mathcal{L}_a$ viewed in the compactification $\overline{Y_{\widetilde{\sigma},\epsilon}} \subseteq Fl(3)$ as a lift of the moment Lagrangian $\mathcal{L}'$ on the compactifying divisor F_∞, the potential is $T^b\mathfrak{PO} + T^a(z_1z_2z_3)^{-1}$ where $(z_1z_2z_3)^{-1}$ corresponds to the fibre disk, passing through the compactifying fibre F_∞, with area a. Similarly to the previous example, $(\mathfrak{PO} - 2z_1z_2z_3)$ corresponds the term in the potential of $\mathcal{L}'$ and $2z_1z_2z_3$ is the additional term, where 2 is reinterpreted in [22] as the number of $c_1 = 1$ rational curves in $Fl(3)$ passing transversally through a point in the divisor F_∞.

Chapter 4

Future Developments

In this section, we mention some developments that arise naturally from this work.

1. Lagrangian skeleton of $Y_{\widetilde{\sigma},\epsilon}$ and its relation to Gelfand-Cetlin fibrations.

   The affine smooth quadric $Y_{\widetilde{\sigma},\epsilon} = T^*S^3$ arise as the smoothing of the singular quadric Y_σ, where σ is the cone over the square with sides of length 1. The cone $\sigma^\vee$ describes a symplectic torus fibration of the singular quadric, but also the same cone describes a Gelfand-Cetlin fibration (in the sense of [50]) of $Y_{\widetilde{\sigma},\epsilon}$ with a Lagrangian S^3 in the vertex. This fibration can be regarded as a limit of a family of restricted almost toric fibrations constructed in Section 3.2.2. We expect to generalize this example to obtain Gelfand-Cetlin fibrations (in the sense of [50]) as limits of the fibrations in Section 3.2.2. These Gelfand-Cetlin fibrations have the cone $\sigma^\vee$ as a convex base diagram, and the singular non-toric fibres are in the codimension ≥ 2 faces. These varieties are affine, so they are Stein and Weinstein, and they have a Lagrangian Skeleton such that $Y_{\widetilde{\sigma},\epsilon}$ deforms to it. We expect to describe the Lagrangian Skeleton in terms of the Minkowski decomposition of Q, to describe a Weinstein structure of $Y_{\widetilde{\sigma},\epsilon}$ from the Gelfand-Cetlin fibration, and to compute symplectic homology and the wrapped Fukaya category for these spaces.

2. Homological mirror symmetry for $Y_{\widetilde{\sigma},\epsilon}$.

   We are interested in proving homological mirror symmetry results for Y_ϵ and $Y_\epsilon \backslash \pi^{-1}(1)$. This project can be divided into the following steps. First, construct the mirror following the SYZ approach in [8, 9], or following the construction in [3] (see also [16, 18, 20]). Second, study the Fukaya category and wrapped Fukaya category of Y_ϵ. We expect to use our complex fibration π in Theorem 3.2.4 and results about Lefschetz fibrations [49] to achieve this goal (see also [4]). Third, relate Lagrangians of the Fukaya category with holomorphic line bundles on the mirror using the results in [7, 43].

Bibliography

[1] Mohammed Abouzaid. The family Floer functor is faithful. *Journal of the European Mathematical Society*, 19(7):2139–2217, 2017.

[2] Mohammed Abouzaid. Homological mirror symmetry without correction. *Journal of the American Mathematical Society*, 34(4):1059–1173, 2021.

[3] Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov. Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces. *Publications mathématiques de l'IHÉS*, 123(1):199–282, 2016.

[4] Mohammed Abouzaid and Luís Diogo. Monotone lagrangians in cotangent bundles of spheres. *arXiv preprint arXiv:2011.13478*, 2020.

[5] Miguel Abreu, Leonardo Macarini, and Miguel Moreira. On contact invariants of non-simply connected Gorenstein toric contact manifolds. *arXiv preprint arXiv:1812.10361*, 2018.

[6] Klaus Altmann. The versal deformation of an isolated toric Gorenstein singularity. *Inventiones mathematicae*, 128(3):443–479, 1997.

[7] Dmitry Arinkin and Alexander Polishchuk. Fukaya category and Fourier transform. *arXiv preprint math/9811023*, 1998.

[8] Denis Auroux. Mirror symmetry and T-duality in the complement of an anticanonical divisor. *Journal of Gökova Geometry Topology*, 1:51–91, 2007.

[9] Denis Auroux. Special Lagrangian fibrations, wall-crossing, and mirror symmetry. *arXiv preprint arXiv:0902.1595*, 2009.

[10] Denis Auroux. A beginner's introduction to Fukaya categories. *Contact and symplectic topology*, pages 85–136, 2014.

[11] Paul Biran and Octav Cornea. A Lagrangian quantum homology. *New perspectives and challenges in symplectic field theory*, 49:1–44, 2009.

[12] W. Bruns, C. Söger B. Ichim, and U. von der Ohe. Normaliz. algorithms for rational cones and affine monoids. Available at `https://normaliz.uos.de`.

[13] Philip Candelas, Monika Lynker, and Rolf Schimmrigk. Calabi-Yau manifolds in weighted P4. *Nuclear Physics B*, 341(2):383–402, 1990.

[14] Philip Candelas, C Xenia, Paul S Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Physics B*, 359(1):21–74, 1991.

[15] Ricardo Castaño-Bernard and Diego Matessi. Lagrangian 3-torus fibrations. *Journal of Differential Geometry*, 81(3):483–573, 2009.

[16] Kwokwai Chan. Homological mirror symmetry for A_n-resolutions as a T-duality. *Journal of the London Mathematical Society*, 87(1):204–222, 2013.

[17] Kwokwai Chan, Cheol-Hyun Cho, Siu-Cheong Lau, and Hsian-Hua Tseng. Gross fibrations, SYZ mirror symmetry, and open Gromov–Witten invariants for toric Calabi–Yau orbifolds. *Journal of Differential Geometry*, 103(2):207 – 288, 2016.

[18] Kwokwai Chan, Siu-Cheong Lau, and Naichung Conan Leung. SYZ mirror symmetry for toric Calabi-Yau manifolds. *Journal of Differential Geometry*, 90(2):177–250, 2012.

[19] Kwokwai Chan, Daniel Pomerleano, and Kazushi Ueda. Lagrangian torus fibrations and homological mirror symmetry for the conifold. *arXiv preprint arXiv:1305.0968*, 2013.

[20] Kwokwai Chan, Daniel Pomerleano, and Kazushi Ueda. Lagrangian torus fibrations and homological mirror symmetry for the conifold. *Communications in Mathematical Physics*, 1(341):135–178, 2016.

[21] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Society, 2011.

[22] Luís Diogo, Dmitry Tonkonog, Renato Vianna, and Weiwei Wu. In preparation.

[23] Johannes J Duistermaat. On global action-angle coordinates. *Communications on pure and applied mathematics*, 33(6):687–706, 1980.

[24] Andreas Floer. Morse theory for Lagrangian intersections. *Journal of differential geometry*, 28(3):513–547, 1988.

[25] Kenji Fukaya. Morse homotopy, A^∞-category, and Floer homologies. In *Proc. of the GARC Workshop on Geometry and Topology'93, Seoul, 1993*, pages 1–102. Seoul Nat. Univ., 1993.

[26] Kenji Fukaya. Multivalued morse theory, asymptotic analysis, and mirror symmetry. In *Graphs and patterns in mathematics and theoretical physics, Proceedings of the Conference dedicated to Dennis Sullivan on his 60th birthday, Stony Brook Univ., 2001*, pages 205–278. Amer. Math. Soc., 2005.

[27] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian intersection Floer theory—anomaly and obstruction, parts I & II. *AMS/IP Studies in Advanced Mathematics*, 46(46.2), 2009.

[28] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Toric degeneration and nondisplaceable Lagrangian tori in $S^2 \times S^2$. *International Mathematics Research Notices*, 2012(13):2942–2993, 2012.

[29] Brian R Greene and M Ronen Plesser. Duality in Calabi-Yau moduli space. *Nuclear Physics B*, 338(1):15–37, 1990.

[30] Mark Gross. Examples of special Lagrangian fibrations. In *Symplectic geometry and mirror symmetry*, pages 81–109. World Scientific, 2001.

[31] Mark Gross. Topological mirror symmetry. *Inventiones mathematicae*, 1(144):75–137, 2001.

[32] Mark Gross, Paul Hacking, and Sean Keel. Mirror symmetry for log Calabi-Yau surfaces I. *Publications Mathématiques de l'IHES*, 122(1):65–168, 2015.

[33] Mark Gross and Bernd Siebert. Mirror symmetry via logarithmic degeneration data I. *Journal of Differential Geometry*, 72(2):169–338, 2006.

[34] Mark Gross and Bernd Siebert. From real affine geometry to complex geometry. *Annals of mathematics*, pages 1301–1428, 2011.

[35] Mark Gross and Bernd Siebert. Intrinsic mirror symmetry and punctured Gromov-Witten invariants. *Algebraic geometry: Salt Lake City 2015*, 97:199–230, 2018.

[36] Mark Gross and Bernd Siebert. Intrinsic mirror symmetry. *arXiv preprint arXiv:1909.07649*, 2019.

[37] Kentaro Hori. Linear models of supersymmetric D-branes. In *Symplectic geometry and mirror symmetry*, pages 111–186. World Scientific, 2001.

[38] Kentaro Hori and Cumrun Vafa. Mirror symmetry. *arXiv preprint hep-th/0002222*, 2000.

[39] J.E. Humphreys. *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer, 1975.

[40] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians: August 3–11, 1994 Zürich, Switzerland*, pages 120–139. Springer, 1995.

[41] Maxim Kontsevich and Yan Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic Geometry And Mirror Symmetry*, pages 203–263. World Scientific, 2001.

[42] Siu-Cheong Lau. Open Gromov–Witten invariants and SYZ under local conifold transitions. *Journal of the London Mathematical Society*, 90(2):413–435, 2014.

[43] Naichung Conan Leung, Shing Tung Yau, and Eric Zaslow. From special lagrangian to hermitian-yang-mills via fourier-mukai transform. *Advances in Theoretical and Mathematical Physics*, 4(6):1319–1341, November 2000.

[44] John Mather. Notes on topological stability. *Bulletin of the American Mathematical Society*, 49(4):475–506, 2012.

[45] Dusa McDuff. Displacing Lagrangian toric fibers via probes. *Low-dimensional and symplectic topology*, 82:131–160, 2011.

[46] Yong-Geun Oh. Addendum to "Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs, I". *Communications on Pure and Applied Mathematics*, 48(11):1299–1302, 1995.

[47] Yong-Geun Oh. Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings. *International Mathematics Research Notices*, 1996(7):305–346, 1996.

[48] James Pascaleff and Dmitry Tonkonog. The wall-crossing formula and Lagrangian mutations. *Advances in Mathematics*, 361:106850, 2020.

[49] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*, volume 10. European Mathematical Society, 2008.

[50] Egor Shelukhin, Dmitry Tonkonog, and Renato Vianna. Geometry of symplectic flux and Lagrangian torus fibrations. *arXiv preprint arXiv:1804.02044*, 2018.

[51] Margaret Symington. Four dimensions from two in symplectic topology. topology and geometry of manifolds,(athens, ga, 2001), 153–208. In *Proc. Sympos. Pure Math*, volume 71.

[52] Junwu Tu. On the reconstruction problem in mirror symmetry. *Advances in Mathematics*, 256:449–478, 2014.

[53] Renato Vianna. On exotic Lagrangian tori in $\mathbb{CP}^2$. *Geometry & Topology*, 18(4):2419–2476, 2014.

[54] Renato Vianna. Continuum families of non-displaceable Lagrangian tori in $(\mathbb{C}P^1)^{2m}$. *arXiv preprint arXiv:1603.02006*, 2016.

[55] Renato Vianna. Infinitely many exotic monotone Lagrangian tori in $\mathbb{CP}^2$. *Journal of topology*, 9(2):535–551, 2016.