Topological methods and the existence/non-existence problem of Einstein metrics on homogeneous spaces

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A Riemannian metric g in a manifold M is called an Einstein metric if it satisfies the Einstein equation

$$Ric(g) = \lambda \cdot g \tag{1}$$

It is important to note that, to solve the Einstein equation, it is necessary to solve a system of partial differential equations, which makes the problems of finding and/or determining Einstein metrics challenging.

In the context of invariant geometry, when we consider a homogeneous space G/H with G a Lie group and H a closed subgroup of G, we can search for G-invariant Einstein metrics metrics. In this work, G is always a compact and simple Lie group. In this situation, the Einstein equation can be seen as a non-linear algebraic system of equations and the the problem of classification of G-invariant Einstein metrics becomes more manageable in some families of homogeneous spaces.

The main topic of this project is an other approach to the problem of existence/non-existence of Einstein metrics in homogeneous spaces. Such approach uses topological methods associated to Lie's theory. With this approach, we can obtain information about invariant Einstein metrics without needing to solve the Einstein system of equations.

We will discuss one of the results of this project that can be found in "Christoph Böhm. «Homogeneous Einstein metrics and simplicial complexes». In: J. Differential Geom. 67.1 (2004), pp. 79–165.". Let G be a connected Lie group and H a connected subgroup of G such that G/H is a compact homogeneous space with finite fundamental group. We can construct an simplical complex $\Delta_{G/H}$ as follows: let T be a maximal torus in the compact complement of H in $N_G(H)$, so $\Delta_{G/TH}$ is homeomorphic to the simplical complex of flags of intermediary connected subgroups K with $TH < K < G$. We have that $\Delta_{G/TH}$ is independent of T, so we can define $\Delta_{G/H} \coloneqq \Delta_{G/TH}$ and we have this theorem:

Theorem 1. *Let G be a connected Lie group and H a connected subgroup of G such that G/H is a compact homogeneous space with finite fundamental group. If the simplical complex $\Delta_{G/H}$ is not contractible, then G/H admits a G-invariant Eintein metric.*