Poster Proposal: A special class of k-harmonic maps inducing calibrated fibrations

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1 Introduction

This is the poster proposal for the IMPA workshop on Special holonomy. It is based on my paper "A special class of k-harmonic maps inducing calibrated fibrations" [1] with my supervisor Spiro Karigiannis. I will quickly highlight the importance of such maps, list some results and discuss numerous future questions. I understand that it currently has too much information for the poster itself but if you aprove it, I will make everything more compact. Thank you for the consideration!

2 Abstract

We consider two special classes of k-harmonic maps between Riemannian manifolds which are related to calibrated geometry, satisfying a first order fully nonlinear PDE. The first is a special type of weakly conformal map $u\colon (L^k, g) \to (M^n, h)$ where $k \leqslant n$ and α is a calibration k-form on M. Away from the critical set, the image is an α-calibrated submanifold of M. These were previously studied by Cheng–Karigiannis–Madnick when α was associated to a vector cross product, but we clarify that such a restriction is unnecessary. The second, which is new, is a special type of weakly horizontally conformal map $u\colon (M^n, h) \to (L^k, g)$ where $n \geqslant k$ and α is a calibration $(n-k)$-form on M. Away from the critical set, the fibres $u^{-1}\{u(x)\}$ are α-calibrated submanifolds of M.

We also review some previously established analytic results for the first class; we exhibit some explicit noncompact examples of the second class, where (M, h) are the Bryant–Salamon manifolds with exceptional holonomy; we remark on the relevance of this new PDE to the Strominger–Yau–Zaslow conjecture for mirror symmetry in terms of special Lagrangian fibrations and to the G_2 version by Gukov–Yau–Zaslow in terms of coassociative fibrations; and we present several open questions for future study.

3 Importance

The natural partial differential equations with arise in Riemannian geometry are usually second order. Some important examples are:

(i) an Einstein metric [$\mathrm{Ric}_g = \lambda g$, where Ric is the Ricci curvature]

(ii) a minimal submanifold [$H = 0$, where H is the mean curvature]

(iii) a Yang–Mills connection ∇ on a vector bundle [$(d^\nabla)^* F^\nabla = 0$, where F^∇ is the curvature]

(iv) a k-harmonic map $u\colon (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds [$\mathrm{div}(|du|^{k-2} du) = 0$]

All of the above geometric objects are also *variational.* That is, the PDEs are Euler–Lagrange equations for some natural geometric functional or “energy”, and hence such objects are *critical points* of these functionals, but may not in general be (local) minima.

A common feature is that when there is additional geometric structure present, one can identify a natural *special class of solutions* which:

- satisfy a (usually fully nonlinear) *first order* PDE, and
- are actually global minimizers of the functional within a particular class of variations.

With respect to the particular examples above, these special first order solutions are:

(i) a *special holonomy metric*: Calabi–Yau, hyperkähler, quaternionic-Kähler, G_2, or Spin(7). These are all Einstein, and most are Ricci-flat.

(ii) a *calibrated submanifold* of a special holonomy manifold. These are all minimal. The calibrated condition is a first order condition on the immersion. They are global minimizers of the volume functional in a given homology class.

(iii) an *instanton* on a vector bundle over a special holonomy manifold. These are all Yang–Mills. The instanton condition is a first order condition on the connection, being an algebraic condition on the curvature. In many cases, a characteristic class argument shows that they are global minimizers of the Yang–Mills energy.

Here, we discuss two classes of special first order solutions to (iv) above, called *Smith maps.* They are special types of k-harmonic maps $u\colon (M_1, g_1) \to (M_2, g_2)$ between pairs of Riemannian manifolds, which are intimately related to both *calibrated geometry* and *conformal geometry*:

- For $u\colon (L^k, g) \to (M^n, h)$, with $k \leqslant n$ and $\alpha \in \Omega^k(M)$ a closed calibration, we define a *Smith immersion*, which is a special type of weakly conformal k-harmonic map. If L^0 is the open subset on which $du \neq 0$, then $u\colon L^0 \to M$ is an immersion, whose image $u(L^0)$ is k-dimensional α-calibrated submanifold of (M, h). Moreover, this notion is invariant under *conformal change* of the domain metric g.
- For $u\colon (M^n, h) \to (L^k, g)$, with $n \geqslant k$ and $\alpha \in \Omega^{n-k}(M)$ a closed calibration, we define a *Smith submersion*, which is a special type of weakly horizontally conformal k-harmonic map. If M^0 is the open subset on which $du \neq 0$, then the fibres $u^{-1}\{u(x)\}$ of $u\colon M^0 \to L$ are $(n-k)$-dimensional α-calibrated submanifolds of (M, h). Moreover, this notion is invariant under *horizontally conformal change* of the domain metric h.

The notion of Smith immersions was previously studied by Cheng–Karigiannis–Madnick, inspired by an unpublished preprint of Smith. The notion of Smith submersions is *new.*

The two constructions should also be viewed as special first order versions of the following particular classical results from harmonic map theory:

- a Riemannian immersion $u\colon (L, g) \to (M, h)$ is harmonic $\iff$ the image is minimal,
- a Riemannian submersion $u\colon (M, h) \to (L, g)$ is harmonic $\iff$ the fibres are minimal.

4 Definitions and Notation

Definition 4.1. Let $\alpha \in \Omega^k$ on (M^n, h). We say that α is a *calibration* if

$$\alpha(v_1 \wedge \cdots \wedge v_k) \leqslant |v_1 \wedge \cdots \wedge v_k| \quad \text{for all } v_1, \ldots, v_k \in T_x M \text{ and all } x \in M. \tag{4.2}$$

Let L^k be an oriented submanifold of M. We say L is *calibrated* with respect to α if $\alpha|_L = \mathsf{vol}_L$. (That is, if equality in (4.2) is attained on each oriented tangent space $T_x L$ of L.) ▲

Definition 4.3. Let $u\colon (M_1, g_1) \to (M_2, g_2)$ be a smooth map. If M_1 is compact, then the *p-energy* of u is defined to be

$$E_p(u) := \frac{1}{(\sqrt{p})^p} \int_{M_1} |du|^p \mathsf{vol}_{M_1} = \frac{1}{(\sqrt{p})^p} \int_{M_1} \Big(\mathrm{tr}_{g_1}(u^* g_2) \Big)^{\frac{p}{2}} \mathsf{vol}_{M_1}.$$

We say that a map u is *p-harmonic* if it is a critical point of the functional E_p. That is, a p-harmonic map is a solution to the Euler–Lagrange equation for the p-energy functional. This equation is

$$\operatorname{div}(|du|^{p-2} du) = 0 \in \Gamma(u^* TM_2), \tag{4.4}$$

and is called the *p-harmonic map equation*. When $p = 2$, this reduces to the classical elliptic harmonic map equation $\operatorname{div}(du) = 0$, and a 2-harmonic map is just called a harmonic map. But for $p > 2$ this equation is a *degenerate* elliptic equation.

More generally, the section of $u^* TM_2$ given by

$$\tau_p(u) := \operatorname{div}(|du|^{p-2} du) \tag{4.5}$$

is called the *p-tension* of u, so a map u is p-harmonic if and only if it has vanishing p-tension.
Note that if M_1 is not compact we can still take equation (4.4) as the definition of p-harmonic.
Note that p-energy of a map $u\colon (M_1^{n_1}, g_1) \to (M_2^{n_2}, g_2)$ is *conformally invariant.* ▲

5 Smith immersions

In this section, $u\colon (L^k, g) \to (M^n, h)$ is a smooth map between Riemannian manifolds, with $k \leqslant n$.

Definition 5.1. A smooth map $u\colon (L^k, g) \to (M^n, h)$ is called *(weakly) conformal* if

$$u^* h = \lambda^2 g$$

for some smooth function $\lambda \geqslant 0$ which is continuous (and smooth away from 0) on L. It then follows that necessarily $\lambda^2 = \frac{1}{k}|du|^2$.

Let $L^0 \subseteq L$ be the open set where $|du| \neq 0$. From $u^* h = \frac{1}{k}|du|^2 g$, we deduce that $u|_{L^0}\colon L^0 \to M$ is an immersion. ▲

Theorem 5.2. *Let $u\colon (L^k, g) \to (M^n, h)$ be a smooth map. Let $\alpha \in \Omega^k(M)$ be a calibration. Then*

$$u^* \alpha \leqslant \lambda^k \mathsf{vol}_L, \quad \textit{where } \lambda = \tfrac{1}{\sqrt{k}}|du|. \tag{5.3}$$

Moreover, equality holds if and only if:

- $u^* h = \lambda^2 g$ *(so u is a weakly conformal immersion), and*
- *the image $u(L^0)$ is calibrated with respect to α.*

Definition 5.4. If equality holds in (5.3), we say that u is a **Smith immersion** with respect to α. That is, a Smith immersion with respect to α is a smooth map $u\colon (L^k, g) \to (M^n, h)$ such that

$$u^* \alpha = \frac{1}{(\sqrt{k})^k} |du|^k \mathsf{vol}_L, \qquad u^* h = \frac{1}{k}|du|^2 g, \tag{5.5}$$

at all points on L. [However, recall that the first equation automatically implies the second equation.] ▲

Proposition 5.6. *Let $u\colon (L^k,g)\to(M^n,h)$ be a smooth map and $\alpha\in\Omega^k$ on M be a calibration form. Then u is a Smith immersion iff*

$$P_\alpha\circ\Lambda^{k-1}(du)\circ\star_L=\frac{(-1)^{k-1}}{(\sqrt{k})^{k-2}}|du|^{k-2}du. \tag{5.7}$$

Note that when L^2 is a Rieamann surface, M^n with $\alpha=\omega$ is an almost complex manifold, then 5.7 gives a J-holomorphic map equation.
Also, the benefit of this equation is that we can differentiate it, as differentiating (5.7) just gives $0=0$.

Theorem 5.8 (Energy Inequality)**.** *Let $\alpha\in\Omega^k(M)$ be a* closed *calibration. Let $u\colon (L^k,g)\to(M^n,h)$ be a Smith immersion with respect to α. Suppose L is compact. Then u is k-harmonic in the sense that it is a critical point of E_k.*

Proof. For any smooth map $u\colon (L^k,g)\to(M^n,h)$, let $\lambda=\frac{1}{\sqrt{k}}|du|$. Using (5.3) we have

$$E_k(u)=\frac{1}{(\sqrt{k})^k}\int_L|du|^k\mathsf{vol}_L=\int_L\lambda^k\mathsf{vol}_L\geqslant\int_L u^*\alpha=[\alpha]\cdot u_*[L],$$

which is a topological quantit. Moreover, by Theorem 5.2, equality holds if and only if u is a Smith immersion. This shows that such maps are local minimizers of E_k and thus are k-harmonic. □

Theorem 5.9. *Let $u\colon (L^k,g)\to(M^n,h)$ be a Smith immersion with respect to the calibration form $\alpha\in\Omega^k$. If $d\alpha=0$, then u is k-harmonic in the sense that $\tau_k(u)=0$.*

6 Smith submersions

In this section, $u\colon (M^n,h)\to(L^k,g)$ is a *surjective* smooth map between Riemannian manifolds, with $n\geqslant k$.

Definition 6.1. Let $u\colon (M^n,h)\to(L^k,g)$ be a smooth surjection. Let $M^0\subseteq M$ be the open set where $|du|\neq0$. Suppose that the restriction $u|_{M^0}\colon M^0\to L$ is a submersion, so that $\operatorname{rank}(du_x)=k$ for all $x\in M^0$. Then the tangent bundle TM^0 of M^0 decomposes as

$$TM^0=(\ker du)\oplus_\perp(\ker du)^\perp,$$

where $\ker du=VM^0$ is the *vertical* subbundle, which has rank $n-k$, and $(\ker du)^\perp=HM^0$ is the *horizontal* subbundle, which has rank k.

It follows that an m-tensor $\alpha\in\mathcal{T}^m$ on M^0 is a smooth section of

$$\bigoplus_{p+q=m}(\ker du)^{\otimes p}\otimes((\ker du)^\perp)^{\otimes q},$$

with $p\leqslant n-k, q\leqslant k$. We denote by $\alpha^{(p,q)}$ the component of α which lies in

$$\mathcal{T}^{(p,q)}:=\Gamma((\ker du)^{\otimes p}\otimes((\ker du)^\perp)^{\otimes q})$$

and we say that $\alpha^{(p,q)}$ is of *type* (p,q).

It follows that the metric h on M^0 decomposes as $h=h^{2,0}+h^{0,2}$, where $h^{2,0}$ is the metric on the vertical subbundle $\ker du$, and $h^{0,2}$ is the metric on the horizontal subbundle $(\ker du)^\perp$. In particular, we have

$$\operatorname{tr}_h(h^{0,2})=k. \tag{6.2}$$

Finally, we use $\Omega^{(p,q)}$ to denote the totally skew-symmetric elements of $\mathcal{T}^{(p,q)}$. ▲

Definition 6.3. A smooth surjection $u\colon (M^n, h) \to (L^k, g)$ is called *(weakly) horizontally conformal* if for every point $x \in M$, we either have $du_x = 0$, or if $du_x \neq 0$, then $\operatorname{rank}(du_x) = k$ is maximal and

$$u^*g = \lambda^2 h^{(0,2)}$$

for some smooth function $\lambda > 0$ on M^0. We can extend λ^2 by zero to obtain a continuous non-negative function on M. It then follows that necessarily $\lambda^2 = \frac{1}{k}|du|^2$. ▲

Theorem 6.4. *Let $u\colon (M^n, h) \to (L^k, g)$ be a smooth surjection. Let $\alpha \in \Omega^{n-k}(M)$ be a calibration. Then*

$$\alpha \wedge u^*\mathsf{vol}_L \leqslant \lambda^k \mathsf{vol}_M, \quad \textit{where } \lambda = \frac{|du|}{\sqrt{k}}. \tag{6.5}$$

Moreover, equality holds if and only if:

- $u^*g = \lambda^2 h^{(0,2)}$ *(so u is a weakly horizontally conformal submersion) and,*
- *the fibres of the restriction of u to M^0 are calibrated with respect to α.*

Definition 6.6. If equality holds in (6.5), we say that u is a **Smith submersion** with respect to α. That is, a Smith submersion with respect to α is a smooth map $u\colon (M^n, h) \to (L^k, g)$ such that

$$\alpha \wedge u^*\mathsf{vol}_L = \frac{1}{(\sqrt{k})^k}|du|^k \mathsf{vol}_M, \qquad u^*g = \frac{1}{k}|du|^2 h^{(0,2)}, \tag{6.7}$$

at all points on M. [However, recall that the first equation automatically implies the second equation.] ▲

Proposition 6.8. *Let $u\colon (M^n, h) \to (L^k, g)$ be a smooth map and $\alpha \in \Omega^{n-k}$ on M is a calibration form. Then u is a Smith submersion iff*

$$\star_L \Lambda^{k-1}(du)(\cdot \lrcorner \star\alpha) = \frac{(-1)^{k-1}}{(\sqrt{k})^{k-2}}|du|^{k-2} du. \tag{6.9}$$

The benefit of this equation is that we can differentiate it, as differentiating (6.7) just gives $0 = 0$.

Theorem 6.10 (Energy Inequality)**.** *Let $u\colon (M^n, h) \to (L^k, g)$ be a Smith map. Let $\alpha \in \Omega^{n-k}$ be a* closed *calibration form on M. Then u is k-harmonic in the sense that it is a critical point of E_k.*

Proof. For any smooth map $u\colon (M^n, h) \to (L^k, g)$, let $\lambda = \frac{1}{\sqrt{k}}|du|$. Using (6.5) we have

$$E_k(u) = \frac{1}{(\sqrt{k})^k}\int_M |du|^k \mathsf{vol}_M = \int_M \lambda^k \mathsf{vol}_M \geqslant \int_M \alpha \wedge u^*\mathsf{vol}_L = ([\alpha] \cup u^*[\mathsf{vol}_L]) \cdot [M],$$

which is a topological quantity. Moreover, by Theorem 6.4, equality holds if and only if u is a Smith submersion. This shows that such maps are local minimizers of E_k and thus are k-harmonic. □

Theorem 6.11. *Let $u\colon (M^n, h) \to (L^k, g)$ be a Smith submersion with respect to the calibration form $\alpha \in \Omega^{n-k}$. If $d\alpha = 0$, then u is k-harmonic in the sense that $\tau_k(u) = 0$.*

7 Future Questions

7.1 Analytic results for Smith immersions

Numerous analytic results for Smith immersions were established in Cheng–Karigiannis–Madnick. In that paper the authors assumed that the calibration form $\alpha \in \Omega^k(M)$ was associated to a vector cross product (VCP), but this assumption was not necessary. Here is the list of the results:

Removable singularities. If u is a C^1_{loc} Smith immersion on a punctured open ball in $\mathbb{R}^k$ with finite k-energy, then u extends to a C^1 Smith immersion across the puncture.

Energy gap. There exists a "threshold energy" $\varepsilon_0 > 0$ such that every Smith immersion $u\colon S^k \to M$ with k-energy less than ε_0 is *constant.* (That is, any nontrivial solution has a *minimum* k-energy.) This is used to show that there are only a finite number of "bubbles".

Compactness modulo bubbling. Let $W \subseteq L$ be open, and let $\{W_m\}_{m\in\mathbb{N}}$ an increasing sequence of open sets exhausting W, and g_m a sequence of metrics on W_m such that $g_m \to g$ in C^∞_{loc} on W. Let $u_m\colon (W_m, [g_m]) \to (M, h)$ be a sequence of Smith immersions with *uniformly bounded k-energy.*

Then there exists a Smith immersion $u_\infty\colon (L, g) \to (M, h)$ and a (possibly empty) finite subset $\mathcal{B} = \{x_1, \ldots, x_N\}$ of L such that (after passing to a subsequence) the following three properties hold:

(a) $u_m \to u_\infty$ in C^1_{loc} on $U\backslash\mathcal{B}$ uniformly on compact subsets of $U\backslash\mathcal{B}$,

(b) as Radon measures on L, we have $|du_m|^k \mathsf{vol}_L \to |du_\infty|^k \mathsf{vol}_L + \sum_{i=1}^N c_i \delta(x_i)$, where $\delta(x_i)$ is a Dirac measure at x_i, and each $c_i \geqslant \frac{1}{2}\varepsilon_0$, where ε_0 is the "threshold energy". This says that the energy density can concentrate at points, where a minimum amount of energy is lost.

(c) If the u_m have uniformly bounded p-energy for some $p \in (k, \infty]$, then $\mathcal{B} = \varnothing$. (There is no bubbling.)

This result can be applied to a sequence $u_m\colon L \to M$ of Smith immersions representing *the same homology class in $H_k(M)$*, as they have a uniform k-energy bound. For each x_i, by rescaling about x_i and using *conformal invariance*, and reapplying this result, we obtain a "bubbled off" Smith immersion $\tilde{u}_{\infty,i}\colon S^k \to M$. This process stops after a *finite number of iterations* due to the energy gap.

No energy loss. We have $\lim_{m\to\infty} E_k(u_m) = E_k(u_\infty) + \sum_i E_k(\tilde{u}_{\infty,i})$. This says that the limiting k-energy is the sum of the k-energy of u_∞ plus the k-energy of each of the bubble maps.

Zero neck length. We have $u_\infty(x_i) = \tilde{u}_{\infty,i}(p^-)$. This says that for $m >> 0$, then u_m is *homotopic* to the connect sum $u_\infty \#(\underset{i}{\#}\tilde{u}_{\infty,i})$.

It would of course be very interesting to establish analogous analytic results for Smith submersions. However, the *conformal invariance* of Smith immersions was used crucially to establish the above analytic results. By contrast, Smith submersions are only *horizontally conformally invariant.* But perhaps this is indeed the right notion that is needed in this context.

7.2 Calibrated fibrations and the SYZ and GYZ "conjectures"

Strominger–Yau–Zaslow argue that one should expect (at least for certain types of points near the boundary of the moduli space) that a compact Calabi–Yau 6-manifold should admit a fibration over a 3-dimensional base, necessarily with singular fibres. The generic (smooth) fibre should be a special Lagrangian torus. The mathematical inspiration comes from the deformation theory of McLean, which shows that a compact special Lagrangian 3-manifold L^3 in a Calabi–Yau 6-manifold locally smoothly deforms in a family of dimension $b^1(L^3)$. One then expects to construct the "mirror Calabi–Yau manifold" by dualizing smooth fibres and then somehow compactifying.

Similarly, Gukov–Yau–Zaslow explain in that, again under certain conditions, a compact torsion-free G_2-manifold should admit a fibration over a 3-dimensional base, again with singular fibres. The generic (smooth) fibre should be a coassocative submanifold with is topologically either T^4 or $K3$. Again, this is inspired by McLean's result in that a compact coassociative 4-manifold L^4 in a torsion-free G_2-manifold locally smoothly deforms in a family of dimension $b^2_+(L^4)$, modulo orientations.

7.3 Questions for future study

Deformation theory of Smith maps. Currently we have shown that the only deformations of Smith submersions are the ones induced by a flow of a conformally horizontal Killing vector field. For the Smith immersions, the only purely vertical deformations are, again, the conformally Killing ones. We wish to investigate this further and see if other deformations are obstructed or not. The work of McClean studied the deformation theory of calibrated submanifolds, and depending on the calibration some cases such as Special Lagrangian and coassociative submanifolds deform smoothly, while complex, associative, and Cayley submanifolds in general have obstructed deformations. It would also be interesting to see if the deformation theory of Smith immersions is "better behaved". This is not completely unreasonable, given the added freedom of precomposing by an orientation-preserving conformal diffeomorphism. For example, start with a (compact) associative or Cayley submanifold, and describe it by a Smith immersion. Can we always deform it *as a Smith immersion* in a smooth family? We would expect that the images of these deformations would immediately acquire singular points. That is, perhaps the reason that associative or Cayley submanifolds do not deform easily is because "they generically want to be singular", but a singular calibrated submanifold could still be described as the image of a *smooth* Smith immersion.

Stability. We have seen from the energy inequalities that Smith immersions and Smith submersions are global minimizers of the k-energy in a particular class of maps. Suppose that u is a k-harmonic map, which is *stable* in the sense that the second variation of the k-energy at u is nonnegative, so u is a local minimum of the k-energy. Under what additional assumptions on the geometry of the source and target could we ensure that such a stable k-harmonic map is necessarily a Smith map? The classical example of such a stability theorem is the demonstration by Siu–Yau that a stable harmonic map from $S^2 = \mathbb{CP}^1$ into a compact Kähler manifold (M, h, ω) with positive holomorphic bisectional curvature is necessarily $\pm$-holomorphic. Generalizing such a result should involve finding analogues of "positive holomorphic bisectional curvature" in Riemannian manifolds with special holonomy.

Constructing Smith maps via flows. If a general stability theorem as described in the previous paragraph could be established, then one could use this to attempt to construct examples of Smith immersions or Smith submersions by running the k-harmonic map heat flow. This is the negative gradient flow of the k-energy. One would have to show that (under certain assumptions on the geometries of the source and target) that the flow exists for all time and converges to a k-harmonic map. Then one would hope to argue that the limit must in fact be a Smith map.

Local properties. A Smith immersion equation is a generalization of a J-holomorphic map equation between a Rieamann surface and an almost complex manifold. Many local properties hold for J-holomorphic maps, such as: unique continuation, which says that two maps with the same ∞-jet at a point must be equal; preimages of critical values are isolated; existence of injective points, etc. We want to investigate if these also hold for Smith maps.

References

[1] A. Iliashenko, S. Karigiannis, "A special class of k-harmonic maps inducing calibrated fibrations". arxiv.org/abs/2311.14074 (2023)