Killing and Ricci-Hessian equations in (pseudo)Kähler geometry

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Results on two special types of holomorphic vector fields v on pseudo-Kähler manifolds (M, g) of complex dimensions $m > 1$ are presented. According to the first one, joint with Ivo Terek, holomorphicity of v follows from compactness of M when v is a Killing field and the $\partial\bar{\partial}$ lemma holds on M. The $\partial\bar{\partial}$ assumption may be relaxed if $m = 2$ or, obviously, if g is definite. The other two results, joint with Paolo Piccione, deal with functions τ on M satisfying a special Ricci-Hessian equation in the sense of Maschler (2008): $\alpha\nabla d\tau + \mathrm{Ric}$ equals a functional multiple of g, for some function α of the real variable τ, with g assumed positive definite, and $\alpha\nabla d\tau$ nonzero almost everywhere. Examples are provided by the non-Einstein cases of CEKM, GKRS and SKRP (conformally-Einstein Kähler metrics, gradient Kähler-Ricci solitons, and special Kähler-Ricci potentials). If, in addition, τ happens to be transnormal (that is, the integral curves of its holomorphic gradient $v = \nabla\tau$ are reparametrized geodesics), the triple (M, g, τ) must represent one of the well-understood types GKRS and SKRP. We show that, in the non-transnormal case, one necessarily has $m = 2$ and, up to normalizations, $\alpha/2$ equals 1, or $1/\tau$, or $\cot\tau$, or $\coth\tau$ or, finally, $\tanh\tau$. Furthermore, we prove, using the Cartan-Kähler theorem, that each of these five possibilities is actually realized by a non-transnormal function τ on a Kähler surface M. For 1 and $1/\tau$ this last fact is already known due to two classic existence theorems, with M equal to the two-point blow-up of $\mathbb{C}\mathrm{P}^2$ and g chosen to be the Wang-Zhu toric Kähler-Ricci soliton or, respectively, the Chen-LeBrun-Weber conformally-Einstein Kähler metric.