A brief tour through the theory of Zoll surfaces

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1 Introduction

It is a remarkable fact, although standard, that the images of the maximal geodesics of the Euclidean sphere are embedded circles, all of them having the same length ($2\pi r$ for a sphere of radius r). Of course, this is not only true for $\mathbb{S}^2$: in addition to higher-dimensional Euclidean spheres, other manifolds such as the projective spaces ($\mathbb{RP}^n$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$) and the Cayley projective plane $\mathrm{Ca}\mathbb{P}^2$, with their respective canonical metrics, share this property.

At first glance, requiring all geodesics to be simple closed curves of the same length seems too restrictive. In fact, the examples given above are rather special for their rich geometrical structure, and they are known as the *Compact Rank One Symmetric Spaces* (CROSS). Hence, what may have been a plain observation raises an interesting question. Is a Riemannian manifold all of whose geodesics are simply closed and of the same length isometric to a CROSS?

As surprisingly as it might sound, the answer is no. Even more astonishing, at the beginning of the last century, Otto Zoll [Zoll] found the first nontrivial examples of two-dimensional spheres of revolution with this property. Due to his contributions, we now call a Riemannian manifold a *Zoll Manifold* when all its geodesics are simple closed curves of the same length.

Zoll's discovery opened the door to further explorations. Not long after his examples, Funk [Fun] tried to construct one-parameter families $g(t)$ of Zoll metrics starting at the canonical metric $\mathrm{can} = g(0)$ on the sphere, and found a necessary condition for such a family to exist. His method, however, was based on the computation of Taylor series expansions, and he could not prove that his condition was also sufficient, since there was no guarantee that the series would converge. More than fifty years passed until Guillemin [Gui] was finally able to answer affirmatively Funk's sufficiency problem. His proof relied on an implicit function theorem of Nash-Moser type, a result not available at the moment of Funk's works.

Both constructions given by Zoll and Guillemin have a crucial aspect in common: they are intimately related to odd functions. As we are going to see in Chapter 3, in cylindrical coordinates $(r, \theta) \in [-1, 1] \times [0, 2\pi]$, a metric of the form

$$g = [1 + h(\cos r)]^2 dr^2 + \sin^2 r d\theta^2 \tag{1.1}$$

is a Zoll metric on $\mathbb{S}^2$ if and only if $h : [-1, 1] \to (-1, 1)$ is a smooth odd function that maps 1 to 0. Moreover, any Zoll metric of revolution can be written as in the formula above. On the other hand, Guillemin's theorem states that, for any smooth odd function $f : \mathbb{S}^2 \to \mathbb{R}$, there exists a smooth one parameter family of C^∞-functions ρ_t such that $\rho_0 = 0$, $(d\rho_t/dt)|_{t=0} = f$ and $e^{\rho_t} \cdot \mathrm{can}$ is a Zoll metric for all t sufficiently small.

As a consequence, none of those examples gives us nontrivial Zoll metrics on the real projective plane, since they are not invariant under the antipodal map. In fact, although there is a great variety of Zoll spheres (many of them with trivial isometry group — see [Bes], Chapter 4, Corollary 4.71), no nontrivial example exists on $\mathbb{RP}^2$. This was first shown by L. Green [Gre].

Green's proof relied on a kind of "area comparison". The idea is the following: One proves that, if g is a Zoll metric on $\mathbb{RP}^2$ all of whose geodesics have length π, then $\mathrm{Area}(\mathbb{RP}^2, g) \geq 2\pi$, with equality holding if and only if $g = \mathrm{can}$. Then the argument ends by showing that we must have $\mathrm{Area}(\mathbb{RP}^2, g) = 2\pi$.

Actually, Green did not work with $(\mathbb{RP}^2, g)$, but with the Riemannian covering $(\mathbb{S}^2, \widetilde{g}) \to (\mathbb{RP}^2, g)$ instead — the only difference is that we change 2π by 4π. When we do not consider the normalized case, in which the length of the geodesics is π, the result can be stated as

Green's Theorem. *Any Zoll metric on $\mathbb{RP}^2$ is isometric to a constant multiple of the canonical metric.*

In the early 2000s, C. LeBrun and L. Mason [LM1] introduced new ideas to the study of Zoll surfaces by applying methods from Twistor Theory. Their approach yielded another proof of Green's theorem, which relies on the rigidity of a duality between points and lines — something that actually precedes the given metric. What they observed is that a Zoll metric on $\mathbb{RP}^2$ has a special property: through any two distinct points passes a unique geodesic, and any two distinct geodesics intersect at exactly one point. This is quite similar to what one has when working with plane projective geometry. In fact, LeBrun and Mason were able to prove Green's Theorem by exploring this analogy.

The main aim of this work is to explain this new approach in detail.

We now summarise what will be discussed throughout the text. In Chapter 2, we define Zoll metrics and Zoll projective structures — the latter being a generalization of the former. These two notions are closely related and impose important topological restrictions on the manifold. One could, in fact, spend many pages discussing the topological implications of the existence of such structures (see [Bes], Chapter 7), but we content ourselves with two simple results. First, we describe the fundamental group of such manifolds: a striking property of Zoll manifolds (and manifolds equipped with Zoll projective structures, more generaly) is that their fundamental groups cannot have order greater than two. Then we move on to the two-dimensional case, and give a fairly precise description of the number of intersections between any two distinct geodesics.

All the main results we present in this text are about Zoll surfaces. The emphasis on dimension two starts at the end of Chapter 2, and prevails through all Chapter 3. There we give the complete characterization of the Zoll metrics of revolution on the sphere. In particular, formula (1.1) is derived.

The heart of this monograph is Chapter 4. In it, we give a complete proof of Green's Theorem from the perspective of LeBrun and Mason's ideas. The argument will be divided in three parts, each explained in a section. In our presentation, all the important calculations were done in detail, for we believe that, by doing so, the construction becomes clearer and more concrete.

We conclude the text with a more informal discussion in Chapter 5. There, we briefly talk about possible further directions one can go, and state some results not proved here that complement the exposition.

Notations and conventions

Throughout this monograph, some conventions were made. For example, *manifolds* are always connected and of class C^∞, except when explicitly stated otherwise. The word *smooth* is to be understood as C^∞, and we will assume smoothness whenever possible. For the absence of *a priori* differentiability (and even of continuity), we use the adjective *rough*. This becomes clearer in the context of a vector bundle $\pi : E \to M$. In this case, a rough section $s : M \to E$ is a not necessarily continuous function $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

Geodesics will be denoted, in most cases, by the letter γ, while c will be used for other parametrized curves. The image of a maximal geodesic $\gamma : \mathbb{R} \to M$ will often be written as $\mathfrak{C} = \gamma(\mathbb{R})$.

Finally, Einstein's summation convention is used in many parts of the text. This is to say that when we have equal upper and lower indices together, they are to be implicitly summed. For example, if $v_1, ..., v_n$ is a basis for a real vector space V and $a^1, ..., a^n$ are real numbers, then $a^i v_i$ is written instead of $\sum_{i=1}^n a^i v_i$.

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2 Different notions of Zoll structures

2.1 Zoll manifolds as special types of Riemannian manifolds

Let (M, g) be a Riemannian manifold. A curve γ in M is a *periodic geodesic with period l* provided that γ is a geodesic of M that is periodic as a map from $\mathbb{R}$ to M (parametrized by arc length), with least period l. The number l is the length of the periodic geodesic, and we use the terminologies "periodic geodesic with period l", "periodic geodesic of length l", and "l-periodic geodesic" interchangeably. For our purposes, we will say that a geodesic γ is *simply closed* if it is l-periodic for some period $l > 0$ and if the function $\gamma : [0, l] \to M$ is a simple closed parametrized curve, in the sense that if $0 \leq t_1 < t_2 \leq l$ satisfy $\gamma(t_1) = \gamma(t_2)$, then $t_1 = 0$ and $t_2 = l$. In other words, γ is simply closed when $\gamma(\mathbb{R})$ is an embedded circle.

Definition 2.1. A *Zoll metric* is a Riemannian metric all of whose geodesics are simply closed and have the same length. A *Zoll manifold* is a Riemannian manifold whose metric is Zoll.

For simplicity, we will say that a Riemannian manifold (M, g) is a $\mathcal{Z}_l$-manifold whenever it is a Zoll manifold and its geodesics are l-periodic. In this case, we write $g \in \mathcal{Z}_l$, or $g \in \mathcal{Z}(M, l)$ if we want to emphasize the manifold and the length. We will also denote by $\mathcal{Z}(M) = \bigcup_{l>0} \mathcal{Z}(M, l)$ the set of all Zoll metrics on a given manifold M.

As to be expected, the existence of a Zoll metric on a manifold is quite restrictive. For example, if (M, g) is a $\mathcal{Z}_l$-manifold, then since all geodesics are periodic, they are defined for all values of time, *i.e.* (M, g) is geodesically complete. Thus, by the Hopf-Rinow Theorem, (M, g) is complete as a metric space, and through any two points $p, q \in M$ passes some minimizing geodesic γ. In particular, $\operatorname{diam}(M, g) \leq l/2$ and M is compact.

Way more can be said about the topology of M (see Chapter 7 of [Bes]), but fairly simple methods are sufficient to give us the following result.

Lemma 2.2. *Suppose (M, g) is a $\mathcal{Z}_l$-manifold. Then $\operatorname{diam}(M, g) \leq l/2$, both M and its universal cover $\widetilde{M}$ are compact, and the pull-back metric $\widetilde{g} = \pi^* g$ of g by the cover map $\pi : \widetilde{M} \to M$ is a Zoll metric on $\widetilde{M}$. Moreover, $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is either an isometry or a Riemannian double cover, and $\widetilde{g} \in \mathcal{Z}(\widetilde{M}, 2l)$ in the second case. In particular, $\pi_1(M)$ is either trivial or isomorphic to $\mathbb{Z}_2$.*

Proof. We already proved that M is compact, and that $\operatorname{diam}(M, g) \leq l/2$. Let us now present another argument that does not rely on the Hopf-Rinow Theorem.

Fix a point $p \in M$, let $\overline{B}(0_p, l/2) = \{v \in T_pM : |v| \leq l/2\}$, and consider the restriction $\exp_p : \overline{B}(0_p, l/2) \to M$ of the exponential map to this ball. This map is not injective because all geodesics passing through p are periodic with period l, hence $\exp_p(v) = \exp_p(-v)$ for all $v \in S(0_p, l/2) = \{u \in T_pM : |u| = l/2\}$. One way to overcome this problem is to define $X := \overline{B}(0_p, l/2)/\sim$, where $u \sim v$ if and only if $u, v \in S(0_p, l/2)$ and $u = \pm v$. Then $X \approx \mathbb{RP}^n$, where $n = \dim M$, and there is a smooth map $\rho : X \to M$ that factors $\exp_p : \overline{B}(0_p, l/2) \to M$ through the canonical projection $\overline{B}(0_p, l/2) \to X$.

It is important to note, however, that this construction does not solve the lack of injectivity of $\exp_p : \overline{B}(0_p, l/2) \to M$. Indeed, there is no guarantee that the map $\rho : X \to M$ is one-to-one, since there could be two distinct geodesics starting at p that meet at a point $q \neq p$. This is the case of $(\mathbb{S}^2, \mathrm{can})$, for any two linearly independent vectors $u, v \in S(0_p, \pi)$ represent distinct points of X, and are both mapped to the antipodal point $-p$ via the exponential map.

Nonetheless, we argue that this possible lack of injectivity causes no harm, because the pair (X, ρ) satisfies the following properties:

(i) X is a closed manifold (*i.e.* compact and without boundary);

(ii) $\rho^{-1}(p) = \{0_p\}$ (here we think of 0_p as the image of $0_p \in T_pM$ under the projection $\overline{B}(0_p, l/2) \to X$), because l is the least period of the geodesics passing through p; and

(iii) $\rho_{*,0_p} : T_{0_p}X \to T_pM$ is an isomorphism, because ρ is modeled by the exponential map near 0_p.

Thus the proper map $\rho : X \to M$ has mod-2 degree $1 \in \mathbb{Z}_2$, which implies that ρ is onto. Since X is compact, we conclude that M is also compact. In particular, for any point $q \in M$, there is a geodesic starting at p that passes through q. Furthermore, because p was chosen arbitrarily, through any two points of (M, g) passes some geodesic.

Assume we are given any two normalized geodesics γ_0 and γ_1 in M with $\gamma_0(0) = p_0$, $\gamma_1(0) = p_1$, $\gamma_0'(0) = v_0$ and $\gamma_1'(0) = v_1$. We can then take a smooth curve $c : [0,1] \to M$ which goes from p_0 to p_1, and write $P_t : T_pM \to T_{c(t)}M$ for the parallel transport along c from $c(0) = p_0$ to $c(t)$. Since the parallel transport is an isometry, $|P_t(v_0)| = 1$ for all $t \in [0,1]$, so that there is a smooth one-parameter family $O(t)$ in $SO(T_{p_1}M, g_{p_1})$ such that $O(0) = \mathrm{Id}$ and $O(1)P_1(v_0) = v_1$. After concatenating $t \mapsto P_t(v_0)$ with $t \mapsto O(t)P_1(v_0)$, we get a path $v(t) : [0,1] \to UM$ along the unit tangent bundle $UM = \{w \in TM : |w|_g = 1\}$ from $v(0) = v_0$ to $v(1) = v_1$. This induces a (free) homotopy $H : [0,1] \times \mathbb{R}/l\mathbb{Z} \to M$, given by

$$H(s,t) = \gamma_{v(s)}(t) = \exp(tv(s)),$$

between $\gamma_0 = H(0, \cdot)$ and $\gamma_1 = H(1, \cdot)$.

In other words, in a Zoll manifold M, all geodesics are freely homotopic. Even more, the homotopy can be given along a fixed point p if both $p_0 = p_1 = p$ — for we can take c to be constant in this case. In particular, any geodesic is homotopic to the reverse orientation of itself, and this implies that the homotopy class of any geodesic of M has order less than or equal to two. Moreover, this order must be the same for every geodesic. Therefore, if we denote by $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ the universal Riemannian cover of M, then the restriction

$\pi : \widetilde{\gamma}(\mathbb{R}) \to \gamma(\mathbb{R})$ is either a diffeomorphism for the lift $\widetilde{\gamma}$ in $\widetilde{M}$ of every geodesic γ of M, or a double cover for all γ. In both cases, $(\widetilde{M}, \widetilde{g})$ is also a Zoll manifold.

Now fix $p \in M$. It is well known that any homotopy class $\alpha \in \pi_1(M,p)$ can be represented by a geodesic loop $\gamma : [0,1] \to M$ at p — this is true for any complete Riemannian manifold, but $\gamma'(0) \neq \gamma'(1)$ in general, *i.e.* γ does not need to be periodic. (See, for instance, the argument presented in the proof of Theorem 2.2 of Chapter XII in [doC], which can easily be changed to give the desired result.) But since M is Zoll, γ must extend periodically, so that $\gamma'(0) = \gamma'(1)$. Thus either $\pi_1(M,p)$ is trivial, or a non-trivial element $\alpha \in \pi_1(M,p)$ is the homotopy class of a geodesic of M passing through p. This second case implies $\pi_1(M,p) = \mathbb{Z}_2$, for all geodesics passing through p are homotopic. □

In the proofs of most of the assertions in Lemma 2.2, we did not use the hypothesis that (M,g) is Zoll to all its extent. Indeed, the arguments assumed only that all the geodesics passing through *some point* $p \in M$ were periodic and of the same length. More generally, we can consider the following:

Definition 2.3. Let (M,g) be a Riemannian manifold and let p be some fixed point of M. We say that (M,g) is a *Zoll manifold at* p (or a $\mathcal{Z}^p$*-manifold*) if all normalized geodesics passing through p are simply closed with the same length. When it happens, and the length of those geodesics is l, we say that (M,g) is a $\mathcal{Z}^p_l$*-manifold*, and write $g \in \mathcal{Z}(M,l,p)$.

Example 2.4. An ellipsoid of revolution is a $\mathcal{Z}^p$-manifold, where p is one of its poles. By the symmetries of its construction, this induces a metric on $\mathbb{RP}^2$ which is $\mathcal{Z}^{[p]}$, where $[p]$ is the class containg the poles. It is not true, however, that both the ellipsoid, and the induced $(\mathbb{RP}^2, g)$ are $\mathcal{Z}$-manifolds in general.

We point out that such manifolds have most of the properties stated in Lemma 2.2.

Lemma 2.5. *Suppose* (M,g) *is a* $\mathcal{Z}^p_l$*-manifold for some point* $p \in M$*. Then both* M *and its universal cover* $\widetilde{M}$ *are compact, and* $\operatorname{diam}(M,g) \leq l$*. Moreover, the pull-back metric* $\widetilde{g} = \pi^* g$ *by the cover map* $\pi : \widetilde{M} \to M$ *is a* $\mathcal{Z}^{\widetilde{p}}$*-metric for any* $\widetilde{p} \in \pi^{-1}(p)$ *on* $\widetilde{M}$*, and* $\pi : (\widetilde{M}, \widetilde{g}) \to (M,g)$ *is either an isometry or a Riemannian double cover. When* π *is 2-1,* $\widetilde{g} \in \mathcal{Z}(\widetilde{M}, 2l, \widetilde{p})$*. In particular,* $\pi_1(M,p)$ *is trivial or isomorphic to* $\mathbb{Z}_2$*.*

Proof. Everything was already proved except for the estimate on the diameter of (M,g). But this is a simple use of the triangle inequality after noting that, for every point $q \in M$, there is a geodesic passing through p and q. Since all geodesics passing through p are periodic of length l, $\operatorname{dist}(p,q) \leq l/2$ for all $q \in M$. □

2.2 Zoll projective structures

The Hopf-Rinow Theorem is about Riemannian metrics g and the distance functions induced by them on the underlying space (see Chapter VII of [doC], or Chapter 5, Section 5.7.1 of [Pe]). On the other hand, we did not need to use this result in the proof of Lemma 2.2.

Our argument was based on the existence of the exponential map (which is determined by the Levi-Civita connection ∇^g of g), the periodicity assumption on the geodesics, and degree theory. The metric had a more subtle influence on the proof: it was needed when we considered balls $B(0_p, r) \subset T_pM$, and on the construction of the homotopy between geodesics (but only to make sure that $\gamma_{v(t)}$, viewed as a map from $\mathbb{R}$ to M, had the same period for all t). If not pointed out explicitly, the metric g could almost be forgotten and overshadowed by the rest of the argument.

The topological techniques used in Lemma 2.2 are not only a choice of different methods, but also a hint that it might be possible to work with Zoll manifolds in some kind of generalized setting. As it is well known, there are two distinct approaches when dealing with geodesics. The first is to consider them as solutions of a variational problem: they are the paths that locally minimize length. The second is a dynamical one: geodesics are curves with zero acceleration. Hence, if we look at our problem only through the lens of dynamical systems, we need only a connection to be able to determine the geodesics — not a Riemannian metric. It then makes sense to work with the following:

Definition 2.6. A *Zoll connection* on a manifold M is a connection ∇ for which the images of all its maximal geodesics are embedded circles.

The point, however, is that we are not concerned with geodesics viewed as maps from an interval to the manifold, but rather as embedded circles $\mathfrak{C} \subset M$, where $\mathfrak{C} = \gamma(\mathbb{R})$ for some maximal geodesic $\gamma : \mathbb{R} \to M$. In particular, we should also not be concerned with their parametrization. Therefore, a specific Zoll connection contains, in a sense, more information than what we actually need. Different connections that have the same geodesics — viewed as unparametrized curves — should not be distinguishable for our purposes.

Definition 2.7. Two connections ∇^1 and ∇^2 on a manifold M are *projectively equivalent* — written as $\nabla^1 \sim \nabla^2$ — when all their geodesics are the same, as unparametrized curves. A *projective structure* $[\nabla]$ on a manifold M is an equivalent class of connections on M for the relation of being projectively equivalent.

This is to say that two connections ∇^1 and ∇^2 are projectively equivalent if and only if, for every geodesic $\gamma : I \to M$ of ∇^1, there is an interval $J \subset \mathbb{R}$ and a diffeomorphism $\phi : J \to I$ such that $\gamma \circ \phi : J \to M$ is a geodesic of ∇^2.

In this context, Definition 2.6 can be replaced by:

Definition 2.8. A *Zoll projective structure* on a manifold M is a projective structure $[\nabla]$ of Zoll connections on M.

Whenever $[\nabla]$ is a projective structure represented by a connection ∇, there is no loss to assume that ∇ is torsion-free. Indeed, in any coordinate chart $(U; x^1, ..., x^n)$ of M, the connection ∇ is completely characterized by its Christoffel symbols $\Gamma^k_{ij} = dx^k(\nabla_{\partial_i}\partial_j)$, and any geodesic $\gamma : I \to M$ satisfies the system of equations

$$\ddot{\gamma}^k + \dot{\gamma}^i\dot{\gamma}^j\Gamma^k_{ij} = 0, \ k = 1, ..., n.$$

We can replace Γ^k_{ij} by their symmetrization $\frac{1}{2}(\Gamma^k_{ij}+\Gamma^k_{ji})$ and obtain a torsion-free connection $\hat{\nabla}$ such that

$$\ddot{\gamma}^k + \dot{\gamma}^i\dot{\gamma}^j\frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji}) = \ddot{\gamma}^k + \dot{\gamma}^i\dot{\gamma}^j\Gamma^k_{ij} = 0$$

for all geodesics γ of ∇. Hence both ∇ and $\hat{\nabla}$ have the same parametrized geodesics, so they are projectively equivalent.

For some purposes, when dealing with a manifold equipped with a projective structure $[\nabla]$, it is easier to fix a connection $\nabla \in [\nabla]$ so that computations can be carried over. If we want to return to the projective structure, however, it is important to keep in mind what is invariant under changes of a representative ∇. The following result tells us how it can be done, and will be important later on.

Lemma 2.9. *Two torsion-free connections ∇^1 and ∇^2 are projectively equivalent if and only if there is a 1-form $\omega \in \Omega^1(M)$ such that*

$$\nabla^1_X Y = \nabla^2_X Y + \omega(X)Y + \omega(Y)X \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Suppose ∇^1 and ∇^2 are projectively equivalent. Since both connections are torsion-free, the difference $\nabla^1 - \nabla^2$ is a symmetric (1,2)-tensor, so we can think of $\nabla^1 - \nabla^2$ as a symmetric bilinear function $TM \oplus TM \to TM$.

Now fix $p \in M$, a nonzero tangent vector $X_p \in T_pM$, and let $\gamma : I \to M$, $I = (-\varepsilon, \varepsilon)$, be the ∇^1-geodesic passing through p at time zero with velocity X_p (*i.e.* $\gamma(0) = p$, $\gamma'(0) = X_p$, and $\nabla^1_{\frac{d}{dt}}\gamma' \equiv 0$). Since ∇^2 has the same unparametrized geodesics of ∇^1, there is some interval $J = (-\delta, \delta)$ and an embedding $\phi : J \to I$ such that $\gamma \circ \phi$ is a ∇^2-geodesic with initial values $\gamma \circ \phi(0) = p$ and $(\gamma \circ \phi)'(0) = X_p$. (In particular, $\phi(0) = 0$ and $\phi'(0) = 1$.) If we denote by s the standard coordinate on J, and by t the standard coordinate on I, then we get

$$\begin{aligned}
0 = \nabla^2_{\frac{d}{ds}}(\gamma \circ \phi)' &= \nabla^2_{\frac{d}{ds}}\left[\phi' \cdot (\gamma' \circ \phi)\right] \\
&= \phi'' \cdot (\gamma' \circ \phi) + \phi' \cdot \nabla^2_{\frac{d}{ds}}(\gamma' \circ \phi) \\
&= \phi'' \cdot (\gamma' \circ \phi) + \phi' \cdot \left[\left(\nabla^2_{\phi' \cdot \frac{d}{dt}}\gamma'\right) \circ \phi\right] \\
&= \phi'' \cdot (\gamma' \circ \phi) + (\phi')^2 \cdot \left[\left(\nabla^2_{\frac{d}{dt}}\gamma'\right) \circ \phi\right],
\end{aligned}$$

hence

$$\nabla^1_{X_p}X_p - \nabla^2_{X_p}X_p = \phi''(0)X_p.$$

Of course the expression $\nabla^i_{X_p}X_p$, $i = 1, 2$, does not make any sense, but we think of $\nabla^1_{X_p}X_p - \nabla^2_{X_p}X_p$ as the tensor $\nabla^1 - \nabla^2$ evaluated on the pair (X_p, X_p).

By the uniqueness and smooth dependence of solutions of ordinary differential equations, the function $\phi = \phi_{p,X_p}$ is uniquely determined by, and depends smoothly on p and X_p. This means that, for any $(p, X_p) \in TM$, there is a neighborhood U of (p, X_p) and an open interval $J = (-\delta, \delta)$ such that ϕ_{q,Y_q} is defined on J, and the map $(q, Y_q, t) \in U \times J \mapsto \phi_{q,Y_q}(t) \in \mathbb{R}$ is smooth. Moreover, since the geodesic equation is homogeneous, $\phi_{p,X_p}(\lambda t) = \lambda \phi_{p,\lambda X_p}(t)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and all $|t|$ sufficiently small. (The only case remaining is when $X_p = 0$, which is trivial.)

Thus there is a well defined 1-form $\omega \in \Omega^1(M)$ given by $\omega(X)(p) = \frac{1}{2}\phi''_{p,X_p}(0)$ if $X_p \neq 0$, and $\omega(X)(p) = 0$ if $X_p = 0$, for all $X \in \mathfrak{X}(M)$. This form satisfies the identity

$$\nabla^1_X X - \nabla^2_X X = 2\omega(X)X, \;\; X \in \mathfrak{X}(M),$$

and the bilinearity of $\nabla^1 - \nabla^2$ then implies the desired result:

$$\begin{aligned}\nabla^1_X Y - \nabla^2_X Y &= \frac{1}{2}\Big[\Big(\nabla^1_{X+Y}(X+Y) - \nabla^2_{X+Y}(X+Y)\Big) \\ &\qquad - \Big(\nabla^1_X X - \nabla^2_X X\Big) - \Big(\nabla^1_Y Y - \nabla^2_Y Y\Big)\Big] \\ &= \omega(X)Y + \omega(Y)X.\end{aligned}$$

Conversely, assume that ∇^1 and ∇^2 are related by formula (2.1). Then a geodesic γ of ∇^1 defined on an open interval $I = (-\varepsilon, \varepsilon)$ satisfies the equation

$$\nabla^2_{\frac{d}{ds}}(\gamma \circ \phi)' = [2(\phi')^2\omega(\gamma' \circ \phi) - \phi''] \cdot (\gamma' \circ \phi)$$

for any reparametrization ϕ. By the existence and uniqueness of solutions of ordinary differential equations, there is an interval $J = (-\delta, \delta)$ and a unique smooth function $\phi : J \to I$ that solves the initial value problem

$$\begin{cases}\phi'' = 2(\phi')^2\omega(\gamma' \circ \phi), \\ \phi(0) = 0, \;\; \phi'(0) = 1.\end{cases}$$

Since $\phi'(0) > 0$, after restricting I and J if necessary, we may assume that $\phi : J \to I$ is a diffeomorphism — hence a reparametrization. By construction, the curve $\gamma \circ \phi$ is a geodesic for the connection ∇^2 that passes through $\gamma(0)$ at time zero with velocity $\gamma'(0)$. This finishes the proof, since γ was chosen arbitrarily. □

In order to avoid an ambiguity, a *Zoll manifold* will always be understood as a Riemannian manifold (M, g) whose metric is Zoll. Whenever $[\nabla]$ is a Zoll projective structure on a manifold M, we will call the pair $(M, [\nabla])$ a *manifold equipped with a Zoll projective structure.*

Before we move on, there is still one problem to deal with. A Zoll projective structure is far from the notion of Zoll metrics — at least further than is to be desired for our purposes. Indeed, there are many examples of Zoll projective structures that do not quite fit in what we expect. All the maximal geodesics of a lens space $(\mathbb{S}^3/\mathbb{Z}_p, \mathrm{can})$ are simply closed, but not all of them have the same length when $p \geq 3$. Thus, except for $(\mathbb{RP}^3, \mathrm{can})$, lens spaces are examples of a Riemannian manifolds whose metrics are not Zoll, but have canonical Zoll projective structures (see [Bes] for more examples). Moreover, $\mathbb{S}^3/\mathbb{Z}_p$ does not admit a Zoll metric when $p \geq 3$, because its fundamental group has order greater than two (see Lemma 2.2).

What we want is a structure slightly more general than a Zoll metric, but not so much. For example, we would like to impose a restriction on the Zoll projective structures here considered so that Lemma 2.2 is still valid. This is what we do next.

Given an immersed curve $c : [a, b] \looparrowright M$, the class $[c'(t)]$ is a well defined element of

$$\mathbb{P}TM := (TM - 0_M)/\mathbb{R}^\times,$$

since $c'(t) \in T_{c(t)}M$ does not vanish at any time $t \in [a, b]$. The *canonical lift* of c along the canonical projection $\mu : \mathbb{P}TM \to M$ is the map $\hat{c} : t \mapsto [c'(t)]$. Observe that the image $\hat{c}([a, b]) \subset \mathbb{P}TM$ does not depend on the parametrization of the curve. Hence any geodesic $\mathfrak{C}$ of a given Zoll projective structure $[\nabla]$ can be canonically lifted to an embedded circle $\hat{\mathfrak{C}} \subset \mathbb{P}TM$, and the set of all these lifted circles is a foliation of $\mathbb{P}TM$, which will be denoted by $\mathcal{F}$.

The next lemma suggests what is the extra piece of structure we are looking for.

Lemma 2.10. *If (M, g) is a n-dimensional Zoll manifold, then the induced foliation of $\mathbb{P}TM$ by lifted geodesics is locally trivial, in the sense that each leaf has a neighborhood which is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{2n-2}$ in such a way that every leaf corresponds to a circle of the form $\mathbb{S}^1 \times \{\mathrm{pt}\}$.*

Proof. Assume first that M is orientable. Let $\gamma : [0, 1] \to M$ be a geodesic loop of length $l \equiv |\gamma'(t)|$, and denote by $\mathfrak{C} = \gamma([0, 1])$ its image. The orientability of M and $\mathfrak{C}$ tells us that the normal bundle $T^\perp\mathfrak{C}$ is trivial, *i.e.* there is an orthonormal frame $e_1(t), ..., e_{n-1}(t)$ of $T^\perp\mathfrak{C}$ along γ such that $e_i(0) = e_i(1)$ for all $i = 1, ..., n-1$. Moreover for some $\varepsilon > 0$ the map

$$\begin{aligned} \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{n-1} &\to M \\ (t, r^1, ..., r^{n-1}) &\mapsto \exp_{\gamma(t)}(r^i e_i(t)) \end{aligned} \tag{2.2}$$

induces a diffeomorphism between the cylinder $\mathbb{R}/\mathbb{Z} \times B(0, \varepsilon)$, $B(0, \varepsilon) \subset \mathbb{R}^{n-1}$, and a tubular neighbourhood $U = B(\mathfrak{C}, \varepsilon)$ of $\mathfrak{C}$ — this allows us to denote any point $p \in U$ by $(t, r^1, ..., r^{n-1})$. After taking $\varepsilon > 0$ sufficiently small, we may assume that any curve in U of the form $t \in [0, 1] \mapsto (t \mod 1, r(t))$ has length greater than $3l/4$.

By the smooth dependence of solutions of ordinary differential equations, we can get an open set $V \subset TM$ satisfying:

(i) $\gamma'(s) \in V$ for all $t \in [0, 1]$;

(ii) $|v| \in (l/2, 3l/2)$ for all $v \in V$; and

(iii) for any $v \in V$, the geodesic γ_v with initial condition $\gamma_v'(0) = v$ stays inside $U = B(\mathfrak{C}, \varepsilon)$ for any time $|s| \leq 3/2$, *i.e.* $\gamma_v(s) \in U$ whenever $s \in [-3/2, 3/2]$.

Properties (i), (ii) and (iii) together with the fact that all geodesics have the same length imply that all $\gamma_v : \mathbb{R} \to M$, $v \in V$, stay inside U for all $s \in \mathbb{R}$. Hence any geodesic γ_v can be written as $s \in \mathbb{R} \mapsto (t_v(s), r_v(s))$ in U. Furthermore, the derivative dt_v/ds cannot vanish at any time: on the contrary, γ_v would pass through $\mathfrak{C}$ with velocity normal to $T\mathfrak{C}$, and so would eventually move out of U.

What we have so far is that any geodesic $\mathfrak{C}_{[v]} = \gamma_v(\mathbb{R})$, $v \in V$, is completely inside of U, and can be locally parametrized by $s \in \mathbb{R} \mapsto (t_v(s), r_v(s))$ with $dt_v/ds > 0$. The retraction $(t, r) \mapsto (t, 0)$ induces a local diffeomorphism $\mathfrak{C}_{[v]} \to \mathfrak{C}$, which is a covering map, since $\mathfrak{C}_{[v]}$ is compact. The cover $\mathfrak{C}_{[v]} \to \mathfrak{C}$ must be a diffeomorphism: otherwise $\mathfrak{C}_{[v]}$ could be parametrized as $s \in [0, k] \mapsto (s \mod 1, r(s))$ for some integer $k > 1$, and some smooth periodic function r with least period k. But this implies that the length of $\mathfrak{C}_{[v]}$ is greater than $3kl/4 > l$, which contradicts the assumption that (M, g) is a Zoll manifold.

Now let W be the image of V under the canonical projection $TM - 0_M \to \mathbb{P}TM$, and consider the set $A = \{(t, r) \in U : t \equiv 0 \mod 1\}$. Since any geodesic $\mathfrak{C}_{[v]}$, $[v] \in W$, passes through A at some point $p_{[v]} = (0, r([v]))$ with direction

$$T_{p_{[v]}}\mathfrak{C}_{[v]} = \left[\left(\frac{\partial}{\partial t} + \xi^i([v]) \frac{\partial}{\partial r^i} \right) \Bigg|_{p_{[v]}} \right], \tag{2.3}$$

any $[v] \in W$ is uniquely described as $(t([v]), r^1([v]), ..., r^{n-1}([v]), \xi^1([v]), ..., \xi^{n-1}([v]))$, where $r([v])$ and $\xi([v])$ are the ones described in formula (2.3), while $t([v]) = t(\mu([v]))$ is given by the cylindrical coordinates in (2.2). This identifies W with an open set of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n-2}$, and induces the desired diffeomorphism by restricting W, if necessary.

Finally, assume M is not orientable. In this case, Lemma 2.2 tells us that $\pi_1(M) = \mathbb{Z}_2$ and that the homotopy class of a geodesic loop $\gamma : [0, 1] \to M$ is not the identity element. Hence the normal bundle $T^\perp\mathfrak{C}$ of $\mathfrak{C} = \gamma([0, 1])$ is nontrivial, and we cannot construct global cylindrical coordinates as in (2.2). There is still a sufficiently small $\varepsilon > 0$ for which $U = B(\mathfrak{C}, \varepsilon) \approx T^\perp\mathfrak{C}$ is a tubular neighborhood of $\mathfrak{C} = \gamma([0, 1])$, though, and we have a well defined map $t : U \to \mathbb{R}/\mathbb{Z}$ given by the composition of the retraction $U \to \mathfrak{C}$ with the diffeomorphism $\mathfrak{C} \approx \mathbb{R}/\mathbb{Z}$ that identifies a point $p \in \mathfrak{C}$ with $t \in \mathbb{R}/\mathbb{Z}$ if $p = \gamma(t)$.

By choosing ε sufficiently small, we may assume that any curve $c : (a, b) \to U$ for which the composition $t \circ c : (a, b) \to \mathbb{R}/\mathbb{Z}$ is onto has length greater than $3l/4$. We can then obtain an open set $V \subset TM$ with properties (i), (ii) and (iii) as before, and consider W as its image via $TM - 0_M \to \mathbb{P}TM$. Similarly to the orientable case, (i)—(iii) imply that any geodesic $\mathfrak{C}_{[v]}$, $[v] \in W$, is contained in U.

Even though global cylindrical coordinates do not exist, we can take local coordinates on U of the form $(t, r) \in (a, b) \times B(0, \varepsilon)$, for $b - a < 1$, induced by the map

$$\begin{aligned} (a, b) \times \mathbb{R}^{n-1} &\to M \\ (t, r^1, ..., r^{n-1}) &\mapsto \exp_{\gamma(t)}(r^i e_i(t)) \end{aligned} \tag{2.4}$$

(here we think of γ as a 1-periodic function defined for all values in $\mathbb{R}$). Just like in the orientable case, the geodesics $\mathfrak{C}_{[v]}$, for $[v] \in W$, are contained in U, and the restriction of t to $\mathfrak{C}_{[v]}$ induces a diffeomorphism $\mathfrak{C}_{[v]} \to \mathfrak{C}$. Indeed, in the coordinates (2.4) any local parametrization $\gamma_v(s) = (t(s), r(s))$, $v \in V$, of $\mathfrak{C}_{[v]}$ has $dt/ds > 0$: on the contrary, $\mathfrak{C}_{[v]}$ would pass through $\mathfrak{C}$ with direction normal to $T\mathfrak{C}$, and so would eventually move out of U. Since $\mathfrak{C}_{[v]} \to \mathfrak{C}$ is an immersion and both $\mathfrak{C}$ and $\mathfrak{C}_{[v]}$ are circles, $\mathfrak{C}_{[v]} \to \mathfrak{C}$ is a covering map. But $\mathfrak{C}_{[v]} \to \mathfrak{C}$ is, in fact, a diffeomorphism: otherwise the length of $\mathfrak{C}_{[v]}$ would be greater than l, which contradicts the assumption that (M, g) is Zoll. By taking $-1/2 < a < 0 < b < 1/2$ in (2.4), we can define $A = \{(0, r) \in U\}$ and put coordinates (t, r, ξ) on W by $t([v]) = t(\mu([v]))$ and $r([v])$, $\xi([v])$ as in (2.3). □

Definition 2.11 ([LM1], Definition 2.5)**.** Let $[\nabla]$ be a Zoll projective structure on a n-dimensional manifold M. We say that $[\nabla]$ is *tame* if the induced foliation of $\mathbb{P}TM$ by lifted geodesics is locally trivial, in the sense that each leaf has a neighbourhood which is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{2n-2}$ in such a way that every leaf corresponds to a circle of the form $\mathbb{S}^1 \times \{\text{pt}\}$.

Lemma 2.10 tells us that a Zoll metric induces a tame Zoll projective structure. The next result goes the other way around. This, in a sense, justifies the hypothesis of tameness as the right generalization.

Lemma 2.12. *Let M be a manifold equipped with a tame Zoll projective structure* $[\nabla]$. *If* $[\nabla]$ *is represented by the Levi-Civita connection of some metric g, then g is Zoll.*

Proof. The geodesics of g are, by assumption, simply closed, and it only remains to prove that every one of them has the same length.

For a fixed unparametrized geodesic $\mathfrak{C}$ of M, there is a trivializing neighborhood V of its canonical lift $\hat{\mathfrak{C}} \subset \mathbb{P}TM$, in the sense that there is a diffeomorphism $\phi : \mathbb{S}^1 \times \mathbb{R}^{2n-2} \xrightarrow{\approx} V$, where the lifted geodesics in V are identified with $\mathbb{S}^1 \times \{\text{pt}\}$. In particular,

$$\begin{aligned} c_r : \mathbb{S}^1 &\to M \\ t &\mapsto \mu(\phi(t, r)) \end{aligned}$$

is a smooth $(2n-2)$-parameter family of loops on M, so that the length

$$L(c_r) = \int_{\mathbb{S}^1} |c_r'(t)| dt$$

depends smoothly on $r \in \mathbb{R}^{2n-2}$. Thus we can construct a smooth $(2n-2)$-parameter family of affinely parametrized geodesic loops $\gamma_r : \mathbb{S}^1 \to M$ by requiring that γ_r is the unique geodesic that passes through $c_r(0)$ at time zero with velocity $(L(c_r)/|c_r'(0)|) \cdot c_r'(0)$ (here we think of $t \in \mathbb{S}^1$ as a number in $[0, 1)$). Of course γ_r is a reparametrization of c_r, but the point is that now each γ_r is a critical point for the energy functional (see [doC] or [Pe])

$$E(\gamma_r) = \int_{\mathbb{S}^1} |\gamma_r'(t)|^2 dt.$$

Since $r \mapsto \gamma_r$ is a smooth $(2n-2)$-parameter family of critical points, the energy does not depend on r, *i.e.* the map $r \mapsto E(\gamma_r)$ is constant. This implies that the length $L(\gamma_r) = L(c_r)$ is also constant as a function of r.

Finally, for any two directions $[v_0], [v_1] \in \mathbb{P}TM$, there is a path $\alpha : [0,1] \to \mathbb{P}TM$, $\alpha(t) = [v_t]$, that starts at $[v_0]$ and ends at $[v_1]$. Denote by $\mathfrak{C}_t$ the geodesic of M whose lift $\hat{\mathfrak{C}}_t \subset \mathbb{P}TM$ passes through $[v_t]$. Then the previous argument showed that the length $L(t) = L(\mathfrak{C}_t)$ of $\mathfrak{C}_t$ is a smooth locally constant function of $t \in [0,1]$. Hence $L(t)$ is constant, and $L(\mathfrak{C}_0) = L(\mathfrak{C}_1)$. Since $[v_0], [v_1] \in \mathbb{P}TM$ where chosen arbitrarily, this proves that all geodesics have the same length. $\square$

As a consequence of Definition 2.11, if $[\nabla]$ is a tame Zoll projective structure on a manifold M^n, then the leaf space $N := \mathbb{P}TM/\mathcal{F}$, called the *space of unparametrized geodesics* of $(M,[\nabla])$, is a connected manifold of dimension $2n-2$, and the canonical projection $\nu : \mathbb{P}TM \to N$ is an $\mathbb{S}^1$-bundle. Moreover, as seen in the argument used in the proof of the previous lemma, all geodesics of $(M[\nabla])$ are freely homotopic to each other. The picture can then be put together to form the following double fibration:

$$\begin{tikzcd} & \mathbb{P}TM \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ M & & N, \end{tikzcd} \tag{2.5}$$

and the tangent spaces to the fibers of μ and ν are linear independent everywhere, *i.e.* $(\ker \mu_*) \cap (\ker \nu_*) = 0$.

Another space of importance is the sphere tangent bundle:

$$\mathbb{S}TM := (TM - 0_M)/\mathbb{R}^+.$$

When $c : [a,b] \looparrowright M$ is an immersed curve, its class $\mathbb{R}^+ c'(t) = \{\lambda c'(t) : \lambda > 0\}$ is a well defined element for all $t \in [a,b]$, and $\mathbb{R}^+ c' : t \mapsto \mathbb{R}^+ c'(t)$ lifts c to $\mathbb{S}TM$ along the canonical projection $\mathbb{S}TM \to M$. The image $\mathbb{R}^+ c'([a,b])$ is invariant under orientation-preserving reparametrizations, in the sense that, if $\phi : [a',b'] \to [a,b]$ is a orientation-preserving diffeomorphism, then $\mathbb{R}^+ c'([a,b]) = \mathbb{R}^+ (c \circ \phi)'([a',b'])$. When ϕ reverses orientation, however, $\mathbb{R}^+ c'(\phi(t)) \neq \mathbb{R}^+[c'(\phi(t))\phi'(t)]$, so that $\mathbb{R}^+ c'([a,b]) \cap \mathbb{R}^+ (c \circ \phi)'([a',b']) = \emptyset$.

As a consequence, a geodesic $\mathfrak{C}$ of $(M,[\nabla])$ lifts to $\mathbb{S}TM$ in two possible ways $\hat{\mathfrak{C}}_+$ and $\hat{\mathfrak{C}}_-$. These lifts give us a foliation $\widetilde{\mathcal{F}}$ on $\mathbb{S}TM$, which is 'locally trivial' in the sense of Definition 2.11 when $[\nabla]$ is tame, for there is a canonical double cover $\mathbb{S}TM \to \mathbb{P}TM$. The leaf space $\widetilde{N} = \mathbb{S}TM/\widetilde{\mathcal{F}}$ is then a connected manifold, called the *space of directed geodesics* of $(M,[\nabla])$, and there is a canonical nontrivial double cover $\widetilde{N} \to N$.

Our goal now is to understand the topological restrictions shared by manifolds equipped with a tame Zoll projective structure. The approach is similar to that of Section 2.1.

Lemma 2.13. *Let $(M,[\nabla])$ be a n-dimensional manifold equipped with a tame Zoll projective structure, and let $\pi : \widetilde{M} \to M$ be its universal cover. Then $[\pi^*\nabla]$ is a tame Zoll projective structure on $\widetilde{M}$.*

Proof. There is nothing to prove when M is simply connected. So we can assume that this is not the case.

Since any two geodesics of M that pass through a given point are homotopic, every parametrized geodesic of M is homotopic to its reverse parametrization. Hence the homotopy class of any geodesic of M has order no greater than two. Moreover, because any two geodesics of M are freely homotopic, this order is the same for all homotopy classes of geodesics, independent of a chosen fixed point. Thus the restriction $\pi : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ is either a double cover for any component $\widetilde{\mathfrak{C}}$ of $\pi^{-1}(\mathfrak{C})$ of any geodesic $\mathfrak{C} \subset M$, or a diffeomorphism for every $\widetilde{\mathfrak{C}}$ and every $\mathfrak{C}$. This shows that the geodesics of $(\widetilde{M}, [\pi^*\nabla])$ are embedded circles, since they are the components of the pre-images of geodesics of M. Equivalently, $[\pi^*\nabla]$ is a Zoll projective structure on $\widetilde{M}$.

The tameness is then proved in a simple manner. Assume $[\nabla]$ is tame. Fix any geodesic $\mathfrak{C}$ of M, and let $\widetilde{\mathfrak{C}} \subset \pi^{-1}(\mathfrak{C})$ be a geodesic of $\widetilde{M}$. Then, the tameness of $[\nabla]$ tells us that there is a neighbourhood $U \subset \mathbb{P}TM$ that contains the canonical lift of $\mathfrak{C}$, and is diffeomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n-2}$ in such a way that the lifted geodesics of M correspond to the circles $\mathbb{R}/\mathbb{Z} \times \{\mathrm{pt}\}$. At the same time, $\pi : \widetilde{M} \to M$ induces a covering map $\hat{\pi} : \mathbb{P}T\widetilde{M} \to \mathbb{P}TM$ such that the components of the pre-images of lifted geodesics of M are lifted geodesics of $\widetilde{M}$. Hence the component $\widetilde{U} \subset \hat{\pi}^{-1}(U)$ containing the lift of $\widetilde{\mathfrak{C}}$ is diffeomorphic to $\mathbb{R}/k\mathbb{Z} \times \mathbb{R}^{2n-2}$, for $k \in \{1, 2\}$ independent of the choice of $\widetilde{\mathfrak{C}}$ and $\mathfrak{C}$, in such a way that the lifted geodesics of M correspond to the circles $\mathbb{R}/k\mathbb{Z} \times \{\mathrm{pt}\}$. The arbitrary choice of $\mathfrak{C}$ and $\widetilde{\mathfrak{C}}$ then concludes the proof of the tameness of $[\pi^*\nabla]$. □

Lemma 2.14. *If $(M, [\nabla])$ is a manifold equipped with a tame Zoll projective structure, then M is compact, and has finite fundamental group. Furthermore any two points of M are connected by a geodesic.*

Proof. Fix a point p in M. Since $(\ker \mu_*) \cap (\ker \nu_*) = 0$, the set

$$\hat{X} := \nu^{-1}[\nu(\mu^{-1}(p))]$$

is a closed submanifold of $\mathbb{P}TM$, and the projection $\mu : \mathbb{P}TM \to M$ restricts to a smooth map $\mu : \hat{X} \to M$. Observe that $\hat{X}$ can be identified with the set of pairs $(q, T_q\mathfrak{C})$, where q is a point of a geodesic $\mathfrak{C}$ that passes through p. With this identification, we see that $\mu^{-1}(p)$ is diffeomorphic to an $\mathbb{RP}^{n-1}$ whose normal bundle is the tautological line bundle. Hence we can blow-down $\hat{X}$ at $\mu^{-1}(p)$ to obtain a manifold X, and a smooth map $\rho : X \to M$. (This is the analogous construction to that in the proof of Lemma 2.2.) Let $x \in X$ be the image of $\mu^{-1}(p)$ by the blowing-down map $\hat{X} \to X$. The pair (X, ρ) has the following properties:

(i) X is a closed manifold;

(ii) $\rho^{-1}(p) = \{x\}$; and

(iii) $\rho_{*,x} : T_xX \to T_pM$ is an isomorphism, because ρ is modeled by the exponential map near x.

Thus p is a regular value of ρ, and the proper map $\rho : X \to M$ has mod-2 degree $1 \in \mathbb{Z}_2$, which implies that ρ is surjective. This proves that M is compact, and that any two points of M can be joined by a geodesic, since p was arbitrary.

For the finiteness of the fundamental group, observe that Lemma 2.13 tells us that the universal cover $\widetilde{M}$ has a tame Zoll projective structure. Hence $\widetilde{M}$ is compact by the above argument, which implies that the fundamental group of M is finite. □

Remark. It can be shown that X, defined in the above proof, is diffeomorphic to $\mathbb{RP}^n$. To see this, let g be a Riemannian metric on M. By the same reasoning as in the proof of Lemma 2.12, there is a smooth function $L_g : N \to (0, \infty)$ that assigns to each $y \in N$ the length of the geodesic $\mathfrak{C}_y = \mu(\nu^{-1}(y))$ (in terms of the metric g). Pulling back L_g via the composition $U^g M \to \mathbb{P}TM \xrightarrow{\nu} N$, where $U^g M = \{u \in TM : g(u,u) = 1\}$ and $U^g M \to \mathbb{P}TM$ is the projection $u \mapsto [u]$, we get a smooth function $L_g : U^g M \to (0, \infty)$.

Now let

$$B_g := \{\lambda u \in T_p M : u \in U^g_p M,\ 0 \leq \lambda \leq L_g(u)/2\},$$

and define a smooth application $f : B_g \to M$ in the following way. For each $u \in U^g_p M$, there is a unique parametrization $c_u : \mathbb{R}/\mathbb{Z} \xrightarrow{\approx} \mathfrak{C}_{\nu([u])}$ such that $c'_u(0) = L_g(u)u$ and $|c'_u(t)| \equiv L_g(u)$, and c_u depends smoothly on u. We then put $f(\lambda u) = c_u(\lambda / L_g(u))$ for $\lambda > 0$, and $f(0_p) = p$. Since $L_g(u) = L_g(\nu([u])) = L_g(-u)$, and since $c_u(t) = c_{-u}(1-t)$, the map f factors through the quotient $B_g \to Y = B_g/\sim$, where $v \sim w$ if and only if $v = -w = (L(u)/2)u$ for some $u \in U^g_p M$.

Of course $Y \approx \mathbb{RP}^n$, but we also claim that Y is canonically identified with X. Indeed, f induces a smooth function $F : B_g \to X$, given by $F(v) = (f(v), T_{f(v)}\mathfrak{C}_{\nu([v])})$ when $v \neq 0$, and $F(0_p) = x$. Moreover, since $F((L_g(u)/2)u) = F(-(L_g(-u)/2)u)$, it factors through $B_g \to Y$, so that we get a smooth map $G : Y \to X$. Observe that G is injective because $f(v) = f(w)$ if and only if $v = w$ or $v = -w = (L_g(u)/2)u$ for some $u \in U^g_p M$ — this is because $c_u(\lambda_1) \neq c_{-u}(\lambda_2)$ for all $\lambda_1, \lambda_2 \in (0, 1/2)$, and all $u \in U^g_p M$. Furthermore, any point of X not equal to x can be written as a pair $(q, T_q\mathfrak{C})$ for some point q contained in a geodesic $\mathfrak{C}$ that passes through p, so that G is also onto. In particular, G is a homemorphism, for it is a bijective continuous map between compact manifolds.

In fact, G is actually a diffeomorphism. This follows from the tameness of $[\nabla]$ and the very construction of c_u, which imply that, for any $u_0 \in U^g_p M$, there is a neighborhood U of u_0 such that the map $U \times \mathbb{R}/\mathbb{Z} \ni (u,t) \mapsto [c'_u(t)] \in \mathbb{P}TM$ is a diffeomorphism onto its image, which lies in $\hat{X}$. Since $c_u(t) = c_{-u}(1-t)$, this implies that G is a local diffeomorphism, and hence a diffeomorphism. The picture we get is the commutative diagram drawn below.

$$\begin{tikzcd}
 & X \arrow[dr, "\rho"] & \\
 & & M \\
T_pM \supset B_g \arrow[uur, "F"] \arrow[r, two heads] \arrow[urr, bend right, "f"'] & Y \arrow[uu, "G", "\approx"'] \arrow[ur] &
\end{tikzcd}$$

Corollary 2.15. *If $[\nabla]$ is a tame Zoll projective structure on a manifold M, then $\pi_1(M)$ is either trivial or isomorphic to $\mathbb{Z}_2$. In the second case, the homotopy class of any geodesic is the nontrivial element of $\pi_1(M)$.*

Proof. We may assume that M has nontrivial fundamental group. We already know that the order of the homotopy class of a geodesic does not depend on the choice of geodesic and of the fixed point for the homotopy, since all geodesics are freely homotopic to each other. Moreover, this order is no greater than two, for a parametrized geodesic is homotopic to itself with the reverse parametrization.

Now fix p in M, let $\pi : \widetilde{M} \to M$ be the universal cover, and consider the induced tame Zoll projective structure $[\pi^*\nabla]$ by Lemma 2.13. Choose a nontrivial homotopy class $\alpha \in \pi_1(M,p)$, represented by a loop $c : [0,1] \to M$ at p, and take a lift $\widetilde{c} : [0,1] \to \widetilde{M}$ of c to $\widetilde{M}$. Since $\alpha \neq 1$, we have $\widetilde{c}(0) \neq \widetilde{c}(1)$. By Lemma 2.14, there is a geodesic $\widetilde{\mathfrak{C}} \subset \widetilde{M}$ that passes through $q_0 = \widetilde{c}(0)$ and $q_1 = \widetilde{c}(1)$. Hence its image $\mathfrak{C} = \pi(\widetilde{\mathfrak{C}})$, viewed as an embedded circle in M, is then a geodesic that represents a nontrivial homotopy class $\beta \in \pi_1(M,p)$. Also, since $\pi^{-1}(p) \cap \widetilde{\mathfrak{C}}$ contains q_0 and q_1, the induced projection $\pi : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ is a k-sheeted covering for some integer $k > 1$. On the other hand, $\beta^2 = 1$, so $k \leq 2$ and $\pi : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ is, in fact, a double cover. But then a parametrization $\gamma : \mathbb{R}/\mathbb{Z} \xrightarrow{\approx} \mathfrak{C}$ can be lifted to a curve $\widetilde{\gamma} : [0,1] \to \widetilde{M}$ in a unique way such that $\widetilde{\gamma}(0) = q_0$ and $\widetilde{\gamma}(1) = q_1$. This shows that $\alpha = \beta$, and thus $\pi_1(M,p) = \{1, \alpha\}$, where α is a nontrivial element of order two represented by any geodesic of M that passes through p. $\square$

Remark. From Lemmas 2.13 and 2.14 together with Corollary 2.15, we see that a manifold equipped with a tame Zoll projective structure also has the same topological restrictions as those stated in Lemma 2.2 for Zoll manifolds.

2.3 The two-dimensional case

We now turn to the study of the topological properties of Zoll structures on surfaces. In this case, there are only two possible manifolds for which a Zoll metric or a tame Zoll projective structure can exist: $\mathbb{S}^2$ and $\mathbb{RP}^2$. All the other compact surfaces have nontrivial fundamental groups of order greater than two, so they are ruled out by Corollary 2.15.

Let us start by unraveling the spaces present in the double fibration (2.5). The discussion here follows the one in [LM1].

Lemma 2.16. *The order of $\pi_1(\mathbb{P}TM)$ is 4 when $M \approx \mathbb{S}^2$, and is 8 when $M \approx \mathbb{RP}^2$. In particular, $\mathbb{P}TM$ has finite fundamental group whenever $(M^2, [\nabla])$ is a surface equipped with a tame Zoll projective structure.*

Proof. Viewing $\mathbb{S}^2 \subset \mathbb{R}^3$, we can identify its unit bundle $U\mathbb{S}^2 = \{(p,v) \in T\mathbb{S} : |v| = 1\}$ with $SO(3)$ in the following way. Any element $(p,v) \in U\mathbb{S}^2$ (where $p \in \mathbb{S}^2$ and $v \in U_p\mathbb{S}^2$) corresponds to a unique orthogonal matrix $O = O(p,v)$ whose first and second columns are p and v, respectively, and whose third column is the unique unit vector $w = w(p,v)$

perpendicular to both p and v, and satisfying the condition $\det(O) = 1$. This let us identify $\mathbb{P}T\mathbb{S}^2$ with $SO(3)/\mathbb{Z}_2$, where the nontrivial element of $\mathbb{Z}_2$ acts on $SO(3)$ by $[p\ v\ w] \mapsto [p\ -v\ -w]$. Thus $|\pi_1(\mathbb{P}T\mathbb{S}^2)| = 2|\pi_1(SO(3))| = 4$ (see [Bre]).

When $M \approx \mathbb{RP}^2$ one only needs to notice that the double cover $\mathbb{S}^2 \to \mathbb{RP}^2$ induces a double cover $SO(3) = UT\mathbb{S}^2 \to UT\mathbb{RP}^2 = SO(3)/\mathbb{Z}_2$, where the action of the nontrivial element of $\mathbb{Z}_2$ maps $[p\ v\ w]$ to $[-p\ -v\ w]$ (this is the action induced by the antipodal map). From this we get a double cover $\mathbb{P}T\mathbb{S}^2 \to \mathbb{P}T\mathbb{RP}^2$, and we conclude that $|\pi_1(\mathbb{P}T\mathbb{RP}^2)| = 2|\pi_1(\mathbb{P}T\mathbb{S}^2)| = 8$. $\square$

Lemma 2.17. *If $(M, [\nabla])$ is a compact surface with a tame Zoll projective structure, then the space N of its uparametrized geodesics is diffeomorphic to $\mathbb{RP}^2$.*

Proof. Since M is compact and has dimension two, both $\mathbb{P}TM$ and N are compact, and have dimensions three and two, respectively. Moreover, the induced homomorphism

$$\nu_{\#} : \pi_1(\mathbb{P}TM) \to \pi_1(N)$$

is onto, since each fiber of ν is path connected. Thus N is a compact surface with finite fundamental group, so it must be diffeomorphic either to $\mathbb{S}^2$ or to $\mathbb{RP}^2$. On the other hand, we know that N has a double cover $\widetilde{N}$, the space of directed geodesics of M. Hence N is diffeomorphic to $\mathbb{RP}^2$. $\square$

Now our goal is to study the following problem: Given a surface M^2 equipped with a tame Zoll projective structure $[\nabla]$, how many times two distinct geodesics intersect each other? It is not hard to obtain rough estimates, as we will see in the following two results.

Lemma 2.18. *Assume $(M, [\nabla])$ is a orientable surface, equipped with a tame Zoll projective structure. Then the number of intersections of any two distinct geodesics is even.*

Proof. Since M is orientable and $[\nabla]$ is tame, $M \approx \mathbb{S}^2$. Hence a geodesic $\mathfrak{C}$, being a smooth simple closed curve, divides M in two regions diffeomorphic to discs. The desired result then follows, because any other geodesic $\mathfrak{C}'$ is simply closed, and any possible intersection between $\mathfrak{C}$ and $\mathfrak{C}'$ is transversal. $\square$

Lemma 2.19. *Assume $(M, [\nabla])$ is a nonorientable surface, equipped with a tame Zoll projective structure. Then any two distinct geodesics intersect at least once.*

Proof. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two distinct geodesics of M. Since $[\nabla]$, we know that the homotopy class of $\mathfrak{C}_1$ is nontrivial and has order equal to two. In particular, the normal bundle $N\mathfrak{C}_1 = TM|_{\mathfrak{C}_1}/T\mathfrak{C}_1$ is a Möbius band and a tubular neighborhood $V \approx N\mathfrak{C}_1$ of $\mathfrak{C}_1$ divides $M \approx \mathbb{RP}^2$ in two parts: the open set V, and the closed set $M - V$, which is diffeomorphic to a disc. If $\mathfrak{C}_1$ and $\mathfrak{C}_2$ did not intersect, then $V \cap \mathfrak{C}_2 = \emptyset$ for some tubular neighborhood V of $\mathfrak{C}_1$, so that $\mathfrak{C}_2 \subset M - V$. But this would imply that $\mathfrak{C}_2$ is homotopic to a constant. However, $\mathfrak{C}_2$ is homotopically nontrivial, so that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ intersect each other at least once. $\square$

If we want to obtain finer results, however, we will need to combine these observations with a local analysis. In the Riemannian case, we have an answer for an infinitesimal version of the question: Given two points $p, q \in M$, and given a geodesic $\gamma : I \looparrowright M$ passing through p and q, if p and q are conjugate along γ — *i.e.* if all Jacobi fields along γ that vanish at p also vanish at q —, then all geodesics infinitesimally close to γ pass through p and q. We can also give a similar answer to the case in which we have only a tame Zoll projective structure $[\nabla]$, but it will require some generalizations.

Given a manifold M equipped with a torsion-free connection ∇, a *variation of geodesics by geodesics* is a map $F : I \times (-\varepsilon, \varepsilon) \to M$ such that the curves $\gamma_s : t \mapsto F(t, s)$ are geodesics for each $s \in (-\varepsilon, \varepsilon)$. The variational field of F (at zero) is the vector field along γ_0 given by

$$J(t) := \left.\frac{d}{ds}\right|_{s=0} F(t, s),$$

and is called a *Jacobi field.* As in the Riemannian setting, a Jacobi field J along a geodesic γ of M satisfies the *Jacobi equation*:

$$J'' + R(J, \gamma')\gamma' = 0,$$

where $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, for $X, Y, Z \in \mathfrak{X}(M)$. More generally, a Jacobi field is any vector field along some geodesic of M that satisfies the above equation. It can be shown that, if J is a Jacobi field along a geodesic $\gamma : (a, b) \looparrowright M$, then for any $t_0 \in (a, b)$ there are $a < \alpha < t_0 < \beta < b$, and a variation of geodesics by geodesics $F : [\alpha, \beta] \times (-\varepsilon, \varepsilon) \to M$ such that $F(\cdot, 0) = \gamma|_{[\alpha,\beta]}$, and $(dF/ds)(\cdot, 0) = J|_{[\alpha,\beta]}$.

The problem with the definition of Jacobi fields given above is that it depends on the given connection ∇, since we considered geodesics as parametrized curves. On the other hand, if $\hat{\nabla}$ is a torsion-free connection projectively equivalent to ∇, then a variation $F : I \times (-\varepsilon, \varepsilon) \to M$ of geodesics by geodesics with respect to ∇ can be changed into a variation of geodesics by geodesics with respect to $\hat{\nabla}$ by taking a reparametrization of the form $F(\phi_s(t), s)$. In other words, the map $(t, s) \mapsto F(\phi_s(t), s)$ is such that the curves $t \mapsto F(\phi_s(t), s)$ are geodesics of $\hat{\nabla}$ for all s. A careful analysis of the proof of Lemma 2.9 shows that the family of reparametrizations ϕ_s can be taken by varying smoothly with respect to s, in the sense that $(t, s) \mapsto \phi_s(t)$ is smooth. We may further assume that, for a fixed point $t_0 \in I$, $\phi_s(t_0) = t_0$ and $\phi_s'(t_0) = 1$ for all s. With this, we know that $J(t) = (dF/ds)(t, 0)$ is a Jacobi field along $\gamma(t) = F(t, 0)$ with respect to ∇, while

$$\hat{J}(t) := \left.\frac{d}{ds}\right|_{s=0} [F(\phi_s(t), s)]$$

is a Jacobi field along $\hat{\gamma} := \gamma \circ \phi_0$ for $\hat{\nabla}$. By observing that

$$\hat{J} = \left(\left.\frac{d}{ds}\right|_{s=0} \phi_s \right) \gamma' \circ \phi_0 + J \circ \phi_0,$$

we see that $J \circ \phi_0$ and $\hat{J}$ represent the same class in the normal bundle $N\mathfrak{C} = TM|_{\mathfrak{C}}/T\mathfrak{C}$, where $\mathfrak{C} = \gamma(I)$ is the unparametrized geodesic.

Definition 2.20. Let M be a manifold equipped with a projective structure $[\nabla]$, and let $\mathfrak{C} \looparrowright M$ be an unparametrized geodesic of M. A *Jacobi class* along $\mathfrak{C}$ is a section $\mathfrak{J}$ of the normal bundle $N\mathfrak{C} = TM|_{\mathfrak{C}}/T\mathfrak{C}$ that locally is the class of a Jacobi field along $\mathfrak{C}$.

In other words, for any point $p \in \mathfrak{C}$, and for any chosen connection $\nabla \in [\nabla]$ there is an affine parametrization $\gamma : (-\varepsilon, \varepsilon) \to \mathfrak{C}$ passing through p at time zero, and a Jacobi field J along γ such that

$$\mathfrak{J} \equiv J \mod T\mathfrak{C}.$$

Definition 2.21. Let M be a manifold equipped with a projective structure $[\nabla]$, and let $\mathfrak{C} \looparrowright M$ be an unparametrized geodesic of M. Two points $p, q \in \mathfrak{C}$ are *conjugate along* $\mathfrak{C}$ if and only if there exists some nontrivial Jacobi class along $\mathfrak{C}$ that vanishes at both p and q.

Let us move back to our case of interest: a surface M^2 equipped with a tame Zoll projective structure. Fix a geodesic $\mathfrak{C} \subset M$ and a connection $\nabla \in [\nabla]$, so that we can get a local affine parametrization $\gamma : (a, b) \hookrightarrow \mathfrak{C}$. Since the pullback bundle $\gamma^*(TM|_{\mathfrak{C}})$ is trivial, and since ∇ induces a section on $\gamma^*(TM|_{\mathfrak{C}})$, there is a parallel section $e \in \Gamma(\gamma^*(TM|_{\mathfrak{C}}))$ along γ such that $\{\gamma', e\}$ is a frame of $\gamma^*(TM|_{\mathfrak{C}})$, *i.e.* $\{\gamma'(t), e(t)\}$ is a basis for $T_{\gamma(t)}M$ for each $t \in (a, b)$. This implies that the class $[e] \in \Gamma(\gamma^* N\mathfrak{C})$ trivializes $\gamma^* N\mathfrak{C}$. A Jacobi field J along γ can then be written as $J = x\gamma' + ye$, for some $x, y : (a, b) \to \mathbb{R}$, and the Jacobi equation becomes

$$0 = \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} J + R(J, \gamma')\gamma' = \frac{d^2 x}{dt^2}\gamma' + \frac{d^2 y}{dt^2} e + yR(e, \gamma')\gamma'.$$

Writing $R(e, \gamma')\gamma' = \alpha\gamma' + \kappa e$, we see from the equation above that a section $\mathfrak{J} \in \Gamma(N\mathfrak{C})$ is a Jacobi class if and only if it can be written locally as $\mathfrak{J} = [ye]$ for $y : (a, b) \to \mathbb{R}$ a solution of

$$\frac{d^2 y}{dt^2} + \kappa y = 0, \tag{2.6}$$

and for every local affine parametrization $\gamma : (a, b) \hookrightarrow \mathfrak{C}$.

Observe that equation (2.6) is a linear ordinary differential equation of second order, so its solutions form a two-dimensional vector space, and we can fix a basis $y_1, y_2 : (a, b) \to \mathbb{R}$ for this space. Now consider the Wronskian of $\{y_1, y_2\}$, which is the function

$$W(t) = \begin{vmatrix} y_1(t) & y_1'(t) \\ y_2(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Since

$$\begin{aligned} W' &= y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' \\ &= -y_1\kappa y_2 + \kappa y_1 y_2 \\ &= 0, \end{aligned}$$

we see that W is constant, and since (y_1, y_1') and (y_2, y_2') are linearly independent (by the fact that y_1 and y_2 form a basis of solutions of equation (2.6), $W \neq 0$.

Thus there is a well defined map $\phi : (a, b) \to \mathbb{RP}^1$ given by $\phi(t) = [y_1(t) : y_2(t)]$, which is an immersion, since

$$\frac{d}{dt}\left(\frac{y_1}{y_2}\right) = \frac{W}{y_2^2} \quad \text{and} \quad \frac{d}{dt}\left(\frac{y_2}{y_1}\right) = -\frac{W}{y_1^2}$$

are never zero. The idea is that we interpret $\phi(t)$ as the set of Jacobi classes that vanish at $\gamma(t) \in \mathfrak{C}$. Indeed, if $\phi(t) = [\lambda_1 : \lambda_2]$, the Jacobi class that is locally written as $t \mapsto [(\lambda_2 y_1(t) - \lambda_1 y_2(t))e(t)]$ is a nontrivial Jacobi class that vanishes at $\gamma(t)$. Since the set of Jacobi classes that vanish at $\gamma(t)$ is a one-dimensional vector space, any such Jacobi class is locally written as $t \mapsto [\lambda(\lambda_2 y_1(t) - \lambda_1 y_2(t))e(t)]$ near the point $\gamma(t)$ for some $\lambda \in \mathbb{R}$.

Consequently, two points $p = \gamma(t_1)$ and $q = \gamma(t_2)$ are conjugate along $\mathfrak{C}$ if and only if $\phi(t_1) = \phi(t_2)$. Moreover, since a Jacobi class depends only on the projective structure $[\nabla]$, but not on a connection $\nabla \in [\nabla]$, so does the map ϕ. This means that the value $\phi(t)$ depends only on the point $\gamma(t) = p$ and the geodesic $\mathfrak{C}$, but not on the specific local parametrization $\gamma : (a, b) \hookrightarrow \mathfrak{C}$, so that we can glue the functions $\phi : (a, b) \to \mathbb{RP}^1$ obtained from different choices of parametrizations γ, and get a smooth map $\phi : \mathfrak{C} \to \mathbb{RP}^1$. This new $\phi : \mathfrak{C} \to \mathbb{RP}^1$ is a covering map because it is a local diffeomorphism — for it is an immersion between curves —, and because $\mathfrak{C}$ is compact. The order of the cover ϕ is called the *conjugacy number of the geodesic* $\mathfrak{C}$.

We can now start to answer the question posed about the number of intersections of two geodesics. When $(M^2, [\nabla])$ is a surface equipped with a tame Zoll projective structure we have a double fibration of the form

$$\begin{tikzcd} & \mathbb{P}TM \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ M & & N, \end{tikzcd}$$

and such that $\ker \mu_* \cap \ker \nu_* = 0$. For any point p and any geodesic $\mathfrak{C}$ passing through p, a Jacobi class $\mathfrak{J}$ along $\mathfrak{C}$ that vanishes at p can be obtained by a variation of $\mathfrak{C}$ by geodesics in such a way that all geodesics of the variation contain p. But since a geodesic of M is identified with an element of N, a variation of geodesics by geodesics can be viewed as a curve $c : (-\varepsilon, \varepsilon) \to N$, and the Jacobi class of the variation corresponds to the tangent vector $c'(0)$. In other words, a point $y \in N$ is thought of as a geodesic $\mathfrak{C}_y \hookrightarrow M$ in such a way that the tangent vectors to y become the Jacobi classes along $\mathfrak{C}_y$.

With this identification, the set of geodesics passing through a fixed point $p \in M$ corresponds to the circle $\ell_p := \nu(\mu^{-1}(p))$ in N, and the tangent space $T_y\ell_p$ of an element $y \in \ell_p$ may be viewed as the set of Jacobi classes along $\mathfrak{C}_y = \mu[\nu^{-1}(y)]$. A curve $c : (-\varepsilon, \varepsilon) \to \ell_p$ starting at y induces a variation $\mathfrak{C}_{c(s)}$ of $\mathfrak{C}_y$ by geodesics, and by taking their canonical lifts $\hat{\mathfrak{C}}_{c(s)} \subset \mathbb{P}TM$ we obtain a curve $\alpha : s \mapsto (p, T_p\hat{\mathfrak{C}}_{c(s)}) \subset \mathbb{P}TM$. Observe that $\alpha'(0) \in \ker \mu_{*,z}$ where $z = (p, T_p\mathfrak{C})$ because $\mu(\alpha(s)) = p$ for all $s \in (-\varepsilon, \varepsilon)$. Moreover, $c'(0) = \nu_{*,z}(\alpha'(0))$ since $\nu(\alpha(s)) = c(s)$ by construction.

Thus $\nu_*(\ker \mu_{*,z}) = T_y \ell_p$ is identified with the set of all Jacobi classes along $\mathfrak{C}_{\nu(z)}$ that vanish at the point $p = \mu(z)$. That is, the map

$$\begin{aligned} \varphi : \mathbb{P}TM &\to \mathbb{P}TN \\ z &\mapsto \nu_{*,z}(\ker \mu_{*,z}), \end{aligned} \tag{2.7}$$

when restricted to a lifted geodesic $\hat{\mathfrak{C}} \subset \mathbb{P}TM$, is modeled as the map $\phi : \mathfrak{C} \to \mathbb{RP}^1$ defined above. Furthermore, there is a commutative diagram

$$\begin{tikzcd} \mathbb{P}TM \arrow[r, "\varphi"] \arrow[d, "\mu"'] \arrow[dr, "\nu"] & \mathbb{P}TN \arrow[d, "\pi"] \\ M & N, \end{tikzcd} \tag{2.8}$$

where $\pi : \mathbb{P}TN \to N$ is the canonical projection.

Proposition 2.22 ([LM1], Proposition 2.14)**.** *Let $(M^2, [\nabla])$ be a surface equipped with a tame Zoll projective structure. Then the map $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ defined as in (2.7) is a covering map, and its order is the conjugacy number of any geodesic of M. In particular, all geodesics of M have the same conjugacy number.*

Proof. We first prove that φ is an immersion. Observe that the tangent space of a fixed point $z \in \mathbb{P}TM$ can be written as $T_z\mathbb{P}TM = \ker \mu_{*,z} \oplus \ker \nu_{*,z} \oplus L$ for some one-dimensional subspace $L \subset T_z\mathbb{P}TM$, so that the commutativity of diagram (2.8) implies that the image $\varphi_{*,z}(\ker \mu_{*,z} \oplus L)$ is a two-dimensional subspace of $T_{\varphi(z)}\mathbb{P}TN$. This is true because $\pi_{*,\varphi(z)}(\varphi_{*,z}(\ker \mu_{*,z} \oplus L)) = \nu_{*,z}(\ker \mu_{*,z} \oplus L_z) = T_{\nu(z)}N$. On the other hand, there is a unique lifted geodesic $\hat{\mathfrak{C}}_z$ that passes through z, and the map φ restricted to $\hat{\mathfrak{C}}_z$ is modeled by $\phi : \mathfrak{C}_z \to \mathbb{RP}^1$, which is an immersion. Hence $\varphi_{*,z}(v)$ is a nonzero vector in $\ker \pi_{*,\varphi(z)}$ for any $0 \neq v \in \ker \nu_{*,z}$. Since $z \in \mathbb{P}TM$ was chosen arbitrarily, this proves that φ is an immersion. In particular, φ is a local diffeomorphism. But because $[\nabla]$ is tame, M is compact, so that $\mathbb{P}TM$ is also compact. Thus φ is a covering map.

Let k be the order of the covering φ, fix $z \in \mathbb{P}TM$ and write $\varphi^{-1}(\varphi(z)) = \{z_1, ..., z_k\}$, where $z = z_1$. Since all z_i are mapped to the same image $\varphi(z) \in \mathbb{P}TN$, and since $\pi \circ \varphi = \nu$, they are all points contained in the unique lifted geodesic $\hat{\mathfrak{C}}_z$ passing through z. Restricted to $\hat{\mathfrak{C}}_z$, the map φ is modeled by the cover map $\phi : \mathfrak{C}_z \to \mathbb{RP}^1$ — because $\varphi(z)$ is identified with the set of Jacobi classes along $\mathfrak{C}_z$ that vanish at $\mu(z)$. Hence k is precisely equal to the order of the covering ϕ, which is the conjugacy number of $\mathfrak{C}_z$. $\square$

Definition 2.23. Let $(M^2, [\nabla])$ be a surface equipped with a tame Zoll projective structure. The order of the covering φ defined in (2.7) is called the *conjugacy number* of $(M, [\nabla])$.

We are now ready to prove the main results of this section.

Theorem 2.24 ([LM1], Theorem 2.15)**.** *If $[\nabla]$ is a tame Zoll projective structure on a surface M^2 diffeomorphic to $\mathbb{S}^2$, then the conjugacy number of $(M,[\nabla])$ is 2, and the cover $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ can be lifted to a diffeomorphism $\hat{\varphi} : \mathbb{P}TM \to \mathbb{S}TN$ in such a way that $\nu : \mathbb{P}TM \to N$ is the composition of $\hat{\varphi}$ with the canonical projection $\mathbb{S}TN \to N$. Furthermore, the real line bundle $\ker \mu_* \to \mathbb{P}TM$ is trivial.*

Proof. Lemma 2.16 tells us that the covering φ has order

$$\frac{|\pi_1(\mathbb{P}TN)|}{|\pi_1(\mathbb{P}TM)|} = \frac{|\pi_1(\mathbb{P}T\mathbb{R}\mathbb{P}^2)|}{|\pi_1(\mathbb{P}T\mathbb{S}^2)|} = \frac{8}{4} = 2,$$

i.e. φ is a double cover. This proves that $(M,[\nabla])$ has conjugacy number equal to two.

Remember that, viewing $\mathbb{S}^2 \subset \mathbb{R}^3$, the unit bundle $U\mathbb{S} = \{(p,u) \in T\mathbb{S}^2 : |u| = 1\}$ is identified with $\mathbb{S}T\mathbb{S}^2$, and that $SO(3)$ acts freely and transitively on $U\mathbb{S}^2$ by $O \cdot (p,u) = (Op, Ou)$ — here p and u are orthonormal vectors on $\mathbb{R}^3$. This action preserves the line bundle $\ker \tilde{\mu}_* \to U\mathbb{S}^2$ induced by the canonical projection $\tilde{\mu} : U\mathbb{S}^2 \to \mathbb{S}^2$, in the sense that $O_* w \in \ker \tilde{\mu}_{*,(Op,Ou)}$ whenever $w \in \ker \tilde{\mu}_{*,(p,u)}$. Indeed, any curve $t \mapsto (p, u(t))$ in $\tilde{\mu}^{-1}(p)$ is mapped to a curve $t \mapsto (Op, Ou(t))$ in $\tilde{\mu}^{-1}(Op)$. Hence, if we start with some element $(p_0,u_0) \in U\mathbb{S}^2$ and take a nonzero vector $w_0 \in \ker \tilde{\mu}_{*,(p_0,u_0)}$, we obtain a nonvanishing section $W \in \Gamma(\ker \tilde{\mu}_*)$ defined by $W_{(Op_0,Ou_0)} := O_* w_0$. This proves that the line bundle $\ker \tilde{\mu}_* \to U\mathbb{S}^2$ is trivial.

Now $\mathbb{P}T\mathbb{S}^2 = U\mathbb{S}^2/\langle\sigma\rangle$, where σ acts on $U\mathbb{S}^2$ by $\sigma(p,u) = (p,-u)$, and the action of $SO(3)$ on $U\mathbb{S}^2$ commutes with σ, in the sense that $O(\sigma(p,u)) = (Op, -Ou) = \sigma(O(p,u))$. Hence the action of $SO(3)$ on $U\mathbb{S}^2$ induces an action on $\mathbb{P}T\mathbb{S}^2$. Moreover, given $(p,u) = (Op_0, Ou_0) \in U\mathbb{S}^2$, there is a unique element $T \in SO(3)$ such that $(Tp, Tv) = (p, -v) = \sigma(p,v)$ for all $(p,v) \in \tilde{\mu}^{-1}(p)$. Writing $w_0 = c'(0)$ for some curve $c : t \mapsto (p_0, u(t)) \in \tilde{\mu}^{-1}(p_0)$, we obtain

$$\begin{aligned}
\sigma_{*,(p,u)} W_{(p,u)} &= \sigma_* O_* c'(0) \\
&= (\sigma \circ O)_* c'(0) \\
&= \left.\frac{d}{dt}\right|_{t=0} (p, -Ou(t)) \\
&= \left.\frac{d}{dt}\right|_{t=0} (TOp_0, TOu(t)) \\
&= (TO)_* c'(0) \\
&= (TO)_* w_0 \\
&= W_{(p,-u)}.
\end{aligned}$$

Thus W induces a nonvanishing section $V \in \Gamma(\ker \mu_*)$, *i.e.* the line bundle $\ker \mu_* \to \mathbb{P}T\mathbb{S}^2$ is trivial.

Finally, the double cover $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ can be lifted via the projection $\mathbb{S}TN \to \mathbb{P}TN$ to a diffeomorphism $\tilde{\varphi} : \mathbb{P}TM \to \mathbb{S}TN$ by taking a nonvanishing section $V \in \Gamma(\ker \mu_*)$, and defining $\tilde{\varphi}(z) = \mathbb{R}^+ \nu_{*,z}(V_z) = \{\lambda \nu_{*,z}(V_z) : \lambda > 0\}$. Indeed, the fact that the composition

$\mathbb{P}TM \xrightarrow{\tilde{\varphi}} \mathbb{S}TN \to \mathbb{P}TN$ equals φ implies that $\tilde{\varphi}$ is a local diffeomorphism, so that it is a covering map since $\mathbb{P}TM$ is compact. Furthermore, the fundamental groups of $\mathbb{P}TM \approx \mathbb{P}T\mathbb{S}^2$ and $\mathbb{S}TN \approx \mathbb{S}T\mathbb{RP}^2$ both have the same order, so that $\tilde{\varphi}$ is actually a diffeomorphism. $\square$

Theorem 2.25 ([LM1], Theorem 2.17)**.** *If $[\nabla]$ is a tame Zoll projective structure on a surface M^2 diffeomorphic to $\mathbb{RP}^2$, then the conjugacy number of $(M, [\nabla])$ is 1, and the cover $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ is a diffeomorphism such that $\nu : \mathbb{P}TM \to N$ is the composition of φ with the canonical projection $\pi : \mathbb{P}TN \to N$, and the line bundle $\ker \mu_* \to \mathbb{P}TM$ is isomorphically mapped to the 'tautological' real line bundle $L \to \mathbb{P}TN$.*

Proof. Lemma 2.16 tells us that the covering φ has order

$$\frac{|\pi_1(\mathbb{P}TN)|}{|\pi_1(\mathbb{P}TM)|} = \frac{|\pi_1(\mathbb{P}T\mathbb{RP}^2)|}{|\pi_1(\mathbb{P}T\mathbb{RP}^2)|} = 1,$$

i.e. φ is a diffeomorphism. The 'tautological' line bundle $L \to \mathbb{P}TN$ is the bundle whose points can be represented as a pair (y, v), where $y \in \mathbb{P}TN$, and $v \in y$. Similarly, we can represent an element of the bundle $\ker \mu_* \to \mathbb{P}TM$ as a pair (z, u), where $z \in \mathbb{P}TM$, and $u \in \ker \mu_{*,z}$. With these identifications, we see that the map φ induces a vector bundle isomorphism $\tilde{\varphi} : \ker \mu_* \to L$ given by $\tilde{\varphi}(z, u) = (\nu_{*,z}(\ker \mu_{*,z}), \nu_{*,z}(u))$ — and this implies that $\varphi^* L = \ker \mu_*$, as desired. $\square$

Corollary 2.26. *If $[\nabla]$ is a tame Zoll projective structure on $\mathbb{RP}^2$, then any two distinct geodesics of $(\mathbb{RP}^2, [\nabla])$ intersect at exactly one point.*

Proof. Fix a point $p \in \mathbb{RP}^2$, and let X be defined as in Lemma 2.14. In other words, X is obtained blowing down the manifold $\hat{X} = \nu^{-1}[\nu(\mu^{-1}(p))]$ at $\mu^{-1}(p)$. The remark following Lemma 2.14 shows that X is diffeomorphic to $\mathbb{RP}^2$, and Theorem 2.25 tells us that, for any geodesic $\mathfrak{C}$ containing p, a nonconstant Jacobi class along $\mathfrak{C}$ that is zero at p cannot vanish at any other point. This implies that the map $\rho : X \to \mathbb{RP}^2$ is an immersion by our construction of X and ρ. In particular, ρ is a covering map, and since $\rho^{-1}(p) = \{x\}$, it is actually a diffeomorphism. But this is equivalent to saying that through any point $q \neq p$ of $\mathbb{RP}^2$ passes a unique geodesic containing p. Since p was arbitrary, this finishes the proof. $\square$

In every part of this section, the assumption of tameness was always made. This does not seem to cause much trouble, since, as argued in Lemma 2.10, the projective class of the Levi-Civita connection of a Zoll metric is always a tame Zoll projective structure. In this sense, we are not losing much if we assume the extra tameness condition. On the other hand, we do loose something making this choice. As previously observed, lens spaces are examples of manifolds whose canonical metrics are not Zoll in general, but have Zoll projective structures. As a consequence of Lemma 2.12, their Zoll projective structures are not tame, at least for most of them. This raises the question if it is possible to find examples of non-tame Zoll projective structures in dimension two. The answer turns out to be negative, and so the tameness assumption here considered is innocuous.

Theorem 2.27 ([LM1], Theorem 2.16). *Any Zoll projective structure on a compact surface is tame.*

We will not prove this theorem here. The proof for the case of a compact orientable manifold uses results from the theory of foliations, specially a result in [Eps]. The nonorientable case, however, is proved with the results on the conjugacy number here studied. The complete proof can be found in [LM1], Proposition 2.6 and Theorem 2.16.

Throughout this chapter, we have encountered different notions of Zoll structures. Zoll manifolds are special types of Riemannian manifolds, while Zoll projective structures do not depend on a metric. In the first two sections, we showed that many basic topological properties are the same in both cases, and the last section proved some consequences for the two-dimensional case.

At first, it might be unreasonable to consider Zoll projective structures instead of Zoll metrics. What would be the need for a generalization for the sake of generalization? However, our argument is that to highlight this distinction makes things more transparent.

In the next chapter, we construct nontrivial examples of Zoll metrics on the sphere, and classify all of the Zoll spheres of revolution. This is done exploring heavily the properties of the metric itself, specially its representation on certain cylindrical coordinates. On the other hand, Zoll projective structures will play a prominent role in the proof of Green's theorem presented in Chapter 4. The argument will rely on a kind of point-line duality, and almost nothing specific to a Riemannian metric will be used.

3 Examples of Zoll Manifolds

In this chapter, we construct examples of Zoll surfaces which were first discovered by [Zoll] at the beginning of last century. The presentation here follows that done in Section 4.B of [Bes].

As shown in the previous chapter (see Lemma 2.2), the fundamental group of a Zoll surface (M, g) is either 1 or $\mathbb{Z}_2$, hence it must be diffeomorphic to $\mathbb{S}^2$ or to $\mathbb{RP}^2$ by the classification of compact surfaces. Furthermore, we know that any Zoll metric on $\mathbb{RP}^2$ determines a Zoll metric on $\mathbb{S}^2$, so there is no loss if we assume that $M \approx \mathbb{S}^2$ when studying examples of Zoll metrics.

One further simplification we impose is that the metric g on $\mathbb{S}^2$ is a *metric of revolution*. This is to say there is an effective action of $\mathbb{S}^1$ on $(\mathbb{S}^2, g)$ by isometries. The reason we consider this type of metric is because its geodesic flow is integrable. Moreover, it already has a family of simply closed geodesics: the ones passing through the poles (see the discussion below), called *meridians*. In fact, a rotationally symmetric sphere $(\mathbb{S}^2, g)$ is a $\mathscr{Z}_l^p$-manifold for at least two points $p \in \mathbb{S}^2$, and some $l > 0$ (see Definition 2.3). Our goal is to determine when g is Zoll.

Let us explain this in more detail. It is a well known fact from topology (see [Bre], ch. IV, Corollary 6.14) that any continuous vector field on the sphere vanishes at some point. As a consequence, if we denote by $\theta \in \mathbb{S}^1 \mapsto F_\theta \in \mathrm{Isom}(\mathbb{S}^2, g)$ the group action, and let $X = dF_\theta/d\theta$ be its infinitesimal isometry, then $X_p = 0$ for at least one $p \in \mathbb{S}^2$. Any such point necessarily is a fixed point for all isometries F_θ. From this we obtain an $\mathbb{S}^1$-action on $(T_p\mathbb{S}^2, g_p)$ by isometries, given by $\theta \mapsto DF_\theta(p)$, and all of them are orientation preserving, since $DF_0(p) = \mathrm{Id}$. (Here we think of θ as a number in $[0, 2\pi)$.) If $DF_\theta(p)Y_p = Y_p$ for some nonzero vector $Y_p \in T_p\mathbb{S}^2$, then $DF_\theta(p)$ would be an orientation preserving isometry of $T_p\mathbb{S}^2 \simeq \mathbb{R}^2$ that fixes the line $\mathrm{span}\{Y_p\}$ — hence $DF_\theta(p) = \mathrm{Id}$. But this implies that $F_\theta = \mathrm{Id}$, and so $\theta = 0$, for the action is effective by assumption. In particular, $\theta \mapsto DF_\theta(p)$ is a Lie group isomorphism between $\mathbb{S}^1$ and $SO(2)$, and p is an isolated singularity of X of index 1.

We are then in position to use a theorem by Hopf (see [Bre], ch. VI, Theorem 12.11 and Proposition 12.12) which asserts that the Euler characteristic of a compact manifold can be written as the sum of the indices at the zeros of any vector field with isolated singularities. Since $\chi(\mathbb{S}^2) = 2$, we conclude that X vanishes at precisely two distinct points of $\mathbb{S}^2$, called the *north* and *south poles* of g, and denoted by N and S, respectively.

Now fix a normalized geodesic segment $\gamma_0 : [0, L] \to \mathbb{S}^2$ from N to S (*i.e.* $\mathrm{dist}(N, S) = L$), and denote by $\gamma_\theta := F_\theta \circ \gamma_0$, for $\theta \in \mathbb{S}^1$. Thanks to the effectiveness of the action, we know that all points $p \in M$ can be written uniquely as $p = \gamma_\theta(t)$ for some $t \in [0, L]$ and $\theta \in [0, 2\pi)$, except for $p = N$ or S. In other words, after identifying θ with a number in the interval

$[0, 2\pi)$, we get cylindrical coordinates (t, θ) on $U = M \setminus \{N, S\}$. We will also consider the open sets with coordinates

$$\begin{aligned} U_N :=& \{N\} \cup \{(t,\theta) \in U : t < L/2\} \\ (t,\theta) \mapsto& (x = t\cos\theta, y = t\sin\theta), \\ N \mapsto& (0,0), \end{aligned}$$

and

$$\begin{aligned} U_S :=& \{S\} \cup \{(t,\theta) \in U : t > L/2\} \\ (t,\theta) \mapsto& (x = (L-t)\cos\theta, y = (L-t)\sin\theta), \\ S \mapsto& (0,0). \end{aligned}$$

On the one hand, γ_θ is a normalized segment from N to S, and $\theta \mapsto \gamma_\theta$ is a variation of geodesics by geodesics all of them passing through N, so $\partial/\partial t := \gamma'_\theta$ and $\partial/\partial\theta := d\gamma_\theta/d\theta$ are orthogonal vector fields on U, and $|\partial/\partial t| = 1$. (This is a standard result about Jacobi fields along geodesics, and can be found in [doC], Chapter V.) Hence the metric g can be written as

$$g = dt^2 + \rho^2(t)d\theta^2 \tag{3.1}$$

on U, for some smooth function $\rho : (0, L) \to (0, \infty)$. This is a special case of the general form of rotationally symmetric metrics (see [Pe] Chapters 1 and 4 for more examples).

On the other hand, given a smooth function $\rho : (0, L) \to (0, \infty)$, it is natural to ask when a metric on U given by formula (3.1) extends to a metric on the whole sphere $\mathbb{S}^2$.

Lemma 3.1. *Suppose $\rho : (0, L) \to (0, \infty)$ is a smooth function, and let g be the metric on $U = \mathbb{S}^2 \setminus \{N, S\}$ given by equation (3.1). Then g extends to a Riemannian metric on $\mathbb{S}^2$ if and only if ρ extends to a smooth function $\rho : [0, L] \to [0, \infty)$ such that $\rho(0) = \rho(L) = 0$, $\rho'(0) = 1$, $\rho'(L) = -1$, and $\rho^{(2k)}(0) = \rho^{(2k)}(L) = 0$ for all $k \geq 1$.*

Proof. In $U_N \cap U$, the equations

$$x = t\cos\theta, \quad y = t\sin\theta$$

imply

$$x^2 + y^2 = t^2,$$

and

$$dx = \cos\theta dt - t\sin\theta d\theta, \quad dy = \sin\theta dt + t\cos\theta d\theta.$$

In particular, we have

$$dx^2 + dy^2 = dt^2 + t^2 d\theta^2 \tag{3.2}$$

and

$$tdt = xdx + ydy. \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$dt = \frac{xdx + ydy}{t}, \tag{3.4}$$

and

$$d\theta^2 = \frac{1}{t^2}\Big[dx^2 + dy^2 - \frac{1}{t^2}\Big(x^2dx^2 + xydx \otimes dy + xydy \otimes dx + y^2dy^2\Big)\Big]. \tag{3.5}$$

We then apply equations (3.2), (3.4), and (3.5) in the formula (3.1), and obtain

$$dt^2 + \rho^2(t)d\theta^2 = \frac{t^2 - \rho^2(t)}{t^4}\big(x^2dx^2 + xydx\otimes dy + xydy\otimes dx + y^2dy^2\big) + \frac{\rho^2(t)}{t^2}\big(dx^2 + dy^2\big). \tag{3.6}$$

Thus the metric g given by (3.1) on U extends to a metric on $U \cup U_N = \mathbb{S}^2 \setminus \{S\}$ if and only if the functions

$$\frac{t^2 - \rho^2(t)}{t^4} \text{ and } \frac{\rho^2(t)}{t^2}$$

extend smoothly to 0. We have to be careful, however, because $t = \sqrt{x^2 + y^2}$ is not smooth at the origin as a function of x and y. First observe that for

$$\frac{\rho^2(t)}{t^2}$$

to extend smoothly to 0, it is necessary that $\rho(t)$ extends smoothly to 0, with $\rho(0) = 0$. Similarly, for the function

$$\frac{t^2 - \rho^2(t)}{t^4} = \frac{1}{t^2} - \frac{\rho^2}{t^4}$$

to be smooth a 0, it is necessary for $\rho(t)$ to be smooth at 0, and $\rho'(0) = 1$. Indeed, we write $\rho(t) = \sum_{i=1}^{k} a_i t^i + O(t^{k+1})$ for some $k > 1$, and compute

$$\begin{aligned} \frac{t^2 - \rho^2(t)}{t^4} &= \frac{t^2 - \Big(\sum_{i=1}^{k} a_i t^i + O(t^{k+1})\Big)^2}{t^4} \\ &= \frac{(1 - a_1^2)t^2 - 2a_1a_2t^3}{t^4} - \sum_{i=4}^{k+1} b_i t^{i-4} + O(t^{k-2}), \end{aligned} \tag{3.7}$$

where $b_j = \sum_{i=1}^{j} a_i a_{j-i}$. From (3.7), observe that the limit

$$\lim_{t\to 0^+} \frac{t^2 - \rho^2(t)}{t^4}$$

exists if and only if $a_1 = \rho'(0) = 1$ and $a_2 = \rho''(0) = 0$ (it is not possible to have $a_1 = -1$, because $\rho(t) > 0$ for $t \in (0, L)$). Applying this to the function $\rho^2(t)/t^2$, we obtain the Taylor expansion:

$$\frac{\rho^2(t)}{t^2} = \sum_{i=2}^{k+1} b_i t^{i-2} + O(t^k). \tag{3.8}$$

Since t is not smooth at 0, but t^2 is, as functions of x and y, we need that $b_i = 0$ for every i odd in order to (3.8) to be k-times differentiable on x and y at the origin. Using induction on the coefficients, we conclude that the smoothness of $\rho^2(t)/t^2$ is equivalent to $a_{2i} = \rho^{(2i)}(0) = 0$ for all $i \geq 1$.

This shows that the conditions stated in the Lemma are necessary. The sufficiency comes from the fact that both functions $\rho^2(t)/t^2$ and $(t^2 - \rho^2(t))/t^4$ have expansions of the form

$$\frac{\rho^2(t)}{t^2} = \sum_{i=1}^{k+1} b_{2i} t^{2i-2} + O(t^{2k}),$$

and

$$\frac{t^2 - \rho^2(t)}{t^4} = \sum_{i=2}^{k+1} b_{2i} t^{2i-4} + O(t^{2k-2})$$

for $k \geq 2$, and both are smooth in the variables x and y at the origin.

The computations for the case in which g extends to S are analogous. □

From now on, we will assume that g is given by formula (3.1) for some smooth function ρ that satisfies the conditions of Lemma 3.1. Our goal is to study the behavior of the geodesics of g. For this, recall that a geodesic γ, written in coordinates as $\gamma(s) = (t(s), \theta(s))$ on U, is a solution of the system of equations

$$\begin{cases} \dfrac{d^2 t}{ds^2} + \left(\dfrac{dt}{ds}\right)^2 \Gamma^t_{tt} + 2\dfrac{dt}{ds}\dfrac{d\theta}{ds}\Gamma^t_{t\theta} + \left(\dfrac{d\theta}{ds}\right)^2 \Gamma^t_{\theta\theta} = 0, \\ \\ \dfrac{d^2 \theta}{ds^2} + \left(\dfrac{dt}{ds}\right)^2 \Gamma^\theta_{tt} + 2\dfrac{dt}{ds}\dfrac{d\theta}{ds}\Gamma^\theta_{t\theta} + \left(\dfrac{d\theta}{ds}\right)^2 \Gamma^\theta_{\theta\theta} = 0, \end{cases}$$

where Γ^k_{ij}, $i, j, k \in \{t, \theta\}$, are the Christoffel symbols of the Levi-Civita connection of g. These symbols are given by the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{lk} \big(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\big),$$

where $g_{ij} = g(\partial_i, \partial_j)$, and $[g^{ij}] = [g_{ij}]^{-1}$. Hence, after a few computations, we have

$$\Gamma^\theta_{t\theta} = \frac{\rho'(t)}{\rho(t)}, \quad \Gamma^t_{\theta\theta} = -\rho'(t)\rho(t),$$

and all the other symbols are zero. This means that the system of equations satisfied by the geodesics of M on U is:

$$\begin{cases} \dfrac{d^2 t}{ds^2} - \rho'(t)\rho(t)\left(\dfrac{d\theta}{ds}\right)^2 = 0, \\ \\ \dfrac{d^2 \theta}{ds^2} + 2\dfrac{\rho'(t)}{\rho(t)}\dfrac{dt}{ds}\dfrac{d\theta}{ds} = 0. \end{cases} \tag{3.9}$$

Lemma 3.2 (Clairaut's First Integral). *Suppose we have the parametrization $\{U_N, U_S, U\}$ on $\mathbb{S}^2$ as described above, together with cylindrical coordinates (t, θ) on U. Let g be a Riemannian metric of revolution on $\mathbb{S}^2$, written as $g = dt^2 + \rho^2(t)d\theta^2$ on U, where $\rho : (0, L) \to (0, \infty)$ is a smooth function satisfying the hypotheses of Lemma 3.1. Then*

(i) the segments γ_θ (called meridians) extend to simply closed geodesics passing through N and S, and they are the only geodesics that pass through one of these points — i.e. any other geodesic must be entirely contained in U;

(ii) if a geodesic $\gamma(s) = (t(s), \theta(s))$ is not a meridian, then there are numbers $0 < t_1 \leq t_2 < L$ such that $\rho(t_1) = \rho(t_2)$ and $t(s) \in [t_1, t_2]$ for all values of s.

Proof. Statement *(i)* follows from the effectiveness of the $\mathbb{S}^1$ action, together with the local uniqueness of geodesics: these guarantee that every normalized geodesic passing through N or S is a meridian γ_θ. Concatenating γ_θ and $\gamma_{\theta+\pi}$ (with the reverse parametrization), we get a simply closed geodesic through N and S of length $2L$.

For the second statement, consider a normalized geodesic $\gamma(s) = (t(s), \theta(s))$ which is not a meridian. Multiply both sides of the second equation of (2.9) to get

$$0 = \rho^2(t)\frac{d^2\theta}{ds^2} + 2\rho'(t)\rho(t)\frac{dt}{ds}\frac{d\theta}{ds} = \frac{d}{ds}\left(\rho^2(t)\frac{d\theta}{ds}\right),$$

i.e.

$$\rho^2(t)\frac{d\theta}{ds} = c$$

along γ for some constant c. Also, since γ is a normalized geodesic, we have

$$|\gamma'(t)|^2 = \left(\frac{dt}{ds}\right)^2 + \rho^2(t)\left(\frac{d\theta}{ds}\right)^2 = 1.$$

We then multiply both sides of the equation by $\rho^2(t)$, and obtain the inequality

$$c^2 = \rho^4(t)\left(\frac{d\theta}{ds}\right)^2 \leq \rho^2(t);$$

hence $|c| \leq \rho(t)$. Since γ is not a meridian, $d\theta/ds \neq 0$ for all s. (If not, then $\gamma'(s) = \gamma'_{\theta(s)}(t(s))$ for some $s \in \mathbb{R}$, and this would imply $\gamma = \gamma_{\theta(s)}$ by the uniqueness of geodesics.) Thus $d\theta/ds$ is either strictly positive or strictly negative, and $|c| > 0$. Because ρ is continuous and $\rho(0) = \rho(L) = 0$, there must be $0 < t_1 \leq t_2 < L$ such that $\rho(t_1) = \rho(t_2) = |c|$, and $\rho(t) < |c|$ whenever $t < t_1$ or $t > t_2$. Since $\rho(t(s)) \geq |c|$ for all s, we conclude that $t(s) \in [t_1, t_2]$. $\square$

Looking at the system of equations (3.9), we see that, if $\gamma(s) = (t(s), \theta(s))$ is a geodesic that passes through a point $\gamma(s_0) = (t_0, \theta_0)$ with $\rho'(t_0) = 0$ and $t'(s_0) = 0$, then $t(s) = t_0$ for all s, and γ is a closed geodesic of length $2\pi\rho(t_0)$. Indeed, in this case the curve

$$\gamma(s) = \left(t_0, \theta_0 + \frac{s - s_0}{\rho(t_0)}\right)$$

is the (unique) solution of (2.9) with initial condition

$$\gamma(s_0) = (t_0, \theta_0) \text{ and } \gamma'(s_0) = (0, 1/\rho(t_0)),$$

and is parametrized by arc length. Inversely, if $\gamma(s_0) = (t_0, \theta_0)$, but $\rho'(t_0) \neq 0$, then the parallel $t = t_0$ is not a geodesic.

Now, suppose $g \in \mathcal{Z}(\mathbb{S}^2, 2\pi)$. Then $L = \pi$, and $\rho(t_0) = 1$ for any point $t_0 \in [0, \pi]$ satisfying $\rho'(t_0) = 0$ — because the parallel $t = t_0$ is a closed geodesic of length $2\pi\rho(t_0) = 2\pi$. Since the point t_0 at which ρ attains its maximum is such a point, we conclude that ρ maps $(0, \pi)$ onto $(0, 1]$. Furthermore, any Jacobi field J along a geodesic γ such that $J(0) = 0$ is a (2π-)periodic solution of the equation

$$J'' - \frac{\rho''(t(\gamma))}{\rho(t(\gamma))} J = 0;$$

this is simply the fact that the Gaussian curvature of a rotationally symmetric metric on the sphere does not depend on θ, and is given by

$$\kappa(t) = -\frac{\rho''(t)}{\rho(t)}$$

(see [Pe], ch. 4). When γ is the parallel $t = t_0$, for $t_0 \in (0, \pi)$ with $\rho'(t_0) = 0$, the Jacobi equation assumes the form:

$$J'' - \rho''(t_0) J = 0,$$

which admits periodic solutions if and only if $\rho''(t_0) < 0$. Thus any critical point t_0 of ρ must be a local maximum, and so ρ has only one such point.

Lemma 3.3. *Let g be a $\mathcal{Z}_{2\pi}$-metric of revolution on $\mathbb{S}^2$ equipped with the parametrization $\{U, U_N, U_S\}$, and the coordinates (t, θ) on U for which g is written as $g = dt^2 + \rho^2(t) d\theta^2$. Then, by setting $\rho(t) = \sin r$, it is possible to obtain new coordinates (r, θ) on U in such a way that the metric g is written as*

$$g = [f(\cos r)]^2 dr^2 + \sin^2 r d\theta^2, \tag{3.10}$$

for some smooth function $f : (-1, 1) \to (0, \infty)$ that extends smoothly to $[-1, 1]$, and satisfies $f(1) = 1 = f(-1)$.

Remark. Observe that $r = t$ when $g = \text{can}$ (since the function $\rho(t) = \sin t$), and $g = dr^2 + \sin^2 r d\theta^2$, so $f \equiv 1$. The passage from (t, θ) to (r, θ) is justified, in a sense, by the desire to force the geodesic parallel $t = t_0$ (t_0 being the only critical point of ρ) to be placed on the equator $r = \pi/2$.

Proof. Our characterization of the function $\rho : [0, \pi] \to [0, 1]$ is quite similar to the characterization of sin on $[0, \pi]$. Indeed, $\rho(0) = 0 = \rho(\pi)$, $\rho'(0) = 1 = -\rho'(\pi)$, and ρ has only one critical point: its maximum, say t_0, for which the critical value is 1. Hence ρ is strictly increasing between 0 and t_0, and strictly decreasing between t_0 and π. Thus the function

$$r = \begin{cases} \arcsin \rho(t), & \text{for } t \in (0, t_0], \\ \pi - \arcsin \rho(t), & \text{for } t \in [t_0, \pi). \end{cases}$$

is smooth, and (r, θ) are the desired coordinates on U.

What remains to be proved is the existence of a smooth function $f : (-1, 1) \to (0, \infty)$ for which g may be written as (3.10). For this, observe that the equality $\rho(t) = \sin r$ implies

$$\rho'(t)dt = \cos r dr;$$

hence what we seek is a function $\eta(x)$ satisfying $\eta(\cos r) = t$. Indeed, such η allows the definition of f as

$$f(x) = \frac{x}{\rho'(\eta(x))},$$

so that g can be written as (3.10) on U. Define

$$\xi(t) = \begin{cases} \sqrt{1 - \rho^2(t)}, & \text{for } t \in (0, t_0], \\ -\sqrt{1 - \rho^2(t)}, & \text{for } t \in [t_0, \pi), \end{cases}$$

and $\eta := \xi^{-1}$. Since

$$\xi'(t) = \begin{cases} -\frac{\rho(t)\rho'(t)}{\sqrt{1-\rho^2(t)}}, & \text{for } t \in (0, t_0), \\ \\ \frac{\rho(t)\rho'(t)}{\sqrt{1-\rho^2(t)}}, & \text{for } t \in (t_0, \pi), \end{cases}$$

and since $\rho'(t) > 0$ for $t < t_0$, $\rho'(t) < 0$ for $t > t_0$, and $\rho(t) < 1$ for $t \neq t_0$, we see that ξ is a strictly decreasing smooth function with non-vanishing derivative for all $t \neq t_0$. Moreover, ξ is also smooth at t_0, since $\rho'(t_0) = 0$, and $\rho''(t_0) < 0$ — for the first derivative, write

$$\rho(t) = 1 + \frac{1}{2}\rho''(t_0)(t - t_0)^2 + O((t - t_0)^3)$$

to see that

$$\xi'(t)^2 = -\frac{\rho''(t_0)^2(t - t_0)^2 + O((t - t_0)^3)}{\rho''(t_0)(t - t_0)^2 + O((t - t_0)^3)} \xrightarrow[t \to t_0]{} -\rho''(t_0);$$

hence ξ is differentiable at t_0 and $\xi'(t_0) = -\sqrt{-\rho''(t_0)}$. What this shows is that ξ maps $(0, \pi)$ diffeomorphically on $(-1, 1)$ reversing the orientation; thus its inverse $\eta = \xi^{-1}$: $(-1, 1) \to (0, \pi)$ exists, is smooth, and satisfies $\eta(\cos r) = t$. Indeed, ξ was chosen so that $\rho^2(t) + \xi^2(t) = 1$, and since $\sin r = \rho(t)$, we see that $\xi(t) = \cos r$.

Finally, the function f defined as

$$f(x) = \begin{cases} \frac{x}{\rho'(\eta(x))}, & \text{for } x \neq 0, \\ 1, & \text{for } x = 0, \end{cases}$$

is smooth — including at 0, because $\rho'(\eta(x)) = x + O(x^2)$; maps $(-1,1)$ to $(0,\infty)$, because $\eta((-1,0]) = [t_0,\pi)$, where ρ' is nonpositive, and $\eta([0,1)) = (0,t_0]$, where ρ' is nonnegative; and extends smoothly to $[-1,1]$ by $f(-1) = f(1) = 1$, because ξ extends to a reverse orientation diffeomorphism from $[0,\pi]$ to $[-1,1]$, sending t_0 to 0 — this is a consequence of the properties of ρ stated in Lemma 3.1. Actually, all the necessary and sufficient conditions on ρ stated in Lemma 3.1, for a metric given by (3.1) on U to extend to $\mathbb{S}^2$, are replaced by the condition of $f(-1) = f(1) = 1$. The condition of $\rho(0) = \rho(\pi) = 0$ is replaced by the possibility of extending ξ (and so η) to $[0,\pi]$. The condition on the first derivatives $\rho'(0) = 1$ and $\rho'(\pi) = -1$ is replaced by $f(-1) = f(1) = 1$, because $f(-1) = -1/\rho'(\pi)$, $f(1) = 1/\rho'(0)$. The restriction on the even derivatives $\rho^{(2k)}(0) = \rho^{(2k)}(\pi) = 0$ is also replaced by $f(-1) = f(1) = 1$ — this can be seen by applying the chain rule on the identity

$$\rho'(t) = \frac{\sqrt{1-\rho^2(t)}}{f\big(\sqrt{1-\rho^2(t)}\big)}.$$

□

We are now able to state and prove the two main results of this section.

Theorem 3.4 (Darboux). *Let N and S be two distinct points of $\mathbb{S}^2$, let $U := \mathbb{S}^2\backslash\{N,S\}$, and consider cylindrical coordinates $(r,\theta) : U \to (0,\pi)\times[0,2\pi)$. Let g be a Riemannian metric on $\mathbb{S}^2$, and suppose g is written as in* (3.10) *on U for some smooth function $f : [-1,1] \to (0,\infty)$ such that $f(-1) = f(1) = f(0) = 1$. A necessary and sufficient condition for all geodesics of $(\mathbb{S}^2, g)$ to be closed is that, for every $\alpha \in (0,\pi/2)$, one has*

$$\int_{\alpha}^{\pi-\alpha} \frac{f(\cos r)\sin\alpha}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr = \frac{p}{q}\pi, \tag{3.11}$$

for some co-prime integers p and q. In this case, apart from the equator — which is simply closed and of length 2π —, every geodesic γ in U has length $2q\cdot \mathrm{dist}(N,S)$, turns p times, and consists of $2q$ geodesic segments between two consecutive points of contact with the parallels $r = \alpha$ and $r = \pi - \alpha$.

Proof. Similarly to the proof of Lemma 3.2, we study the geodesic equations. For this, observe that the Christoffel's symbols of the for the metric g on U are

$$\Gamma^r_{rr} = -\frac{f'(\cos r)\sin r}{f(\cos r)}, \quad \Gamma^r_{\theta\theta} = -\frac{\sin r\cos r}{[f(\cos r)]^2}, \quad \Gamma^\theta_{r\theta} = \cot r,$$

and all the others are zero. Hence a geodesic $\gamma(s) = (r(s),\theta(s))$ in U is a solution of the system of equations

$$\begin{cases} \dfrac{d^2r}{ds^2} - \dfrac{f'(\cos r)\sin r}{f(\cos r)}\left(\dfrac{dr}{ds}\right)^2 - \dfrac{\sin r\cos r}{f^2(\cos r)}\left(\dfrac{d\theta}{ds}\right)^2 &= 0, \\[2ex] \dfrac{d^2\theta}{ds^2} + 2\cot r\dfrac{d\theta}{ds}\dfrac{dr}{ds} &= 0. \end{cases}$$

Multiplying both sides of the second equation by $\sin^2 r$, we see that

$$\frac{d}{ds}\left(\sin^2 r \frac{d\theta}{ds}\right) = 0.$$

This shows that

$$\sin^2 r \frac{d\theta}{ds} = c. \tag{3.12}$$

for some constant $c \in \mathbb{R}$. (This is, of course, the same equation already known, since $\rho^2(t) = \sin^2 r$ when we changed the variables.) Taking γ to be a normalized geodesic, we also have the equation

$$f(\cos r)^2\left(\frac{dr}{ds}\right)^2 + \sin^2 r\left(\frac{d\theta}{ds}\right)^2 = 1,$$

which, after multiplying both sides by $\sin^2 r$, gives us

$$\left[f(\cos r)\sin r\left(\frac{dr}{ds}\right)\right]^2 + c^2 = \sin^2 r.$$

In particular, we see that $|c| \leq \sin r \leq 1$, so that we can write $|c| = \sin\alpha$ for some (unique) $\alpha \in (0, \pi/2]$ and $r(s) \in [\alpha, \pi - \alpha]$ for all s. Also, we get

$$f(\cos r)\sin r \frac{dr}{ds} = \varepsilon_r\sqrt{\sin^2 r - \sin^2\alpha}, \tag{3.13}$$

for $\varepsilon_r = \pm 1$. Thus, equations (3.12) and (3.13) tell us that, when the geodesic γ is not the parallel $s \mapsto (\pi/2, \theta_0 \pm s)$, it is a solution of the system

$$\begin{cases} \dfrac{d\theta}{ds} = \varepsilon_\theta \dfrac{\sin\alpha}{\sin^2 r} \\[2ex] \dfrac{ds}{dr} = \varepsilon_r \dfrac{f(\cos r)\sin r}{\sqrt{\sin^2 r - \sin^2\alpha}} \end{cases} \implies \frac{d\theta}{dr} = \varepsilon_\theta\varepsilon_r \frac{f(\cos r)\sin\alpha}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}, \tag{3.14}$$

where both $\varepsilon_r, \varepsilon_\theta \in \{-1, 1\}$. The number ε_θ is the sign of $d\theta/ds$; whereas ε_r is 1 if γ is moving from the parallel $r = \alpha$ towards $r = \pi - \alpha$, and is -1 if γ is moving from the parallel $r = \pi - \alpha$ towards $r = \alpha$ — *i.e.* ε_r is the sign of dr/ds.

The angle between two consecutive points of contact with the parallels $r = \alpha$ and $r = \pi - \alpha$ is

$$\hat{\theta}(\alpha) = \int_\alpha^{\pi-\alpha} \frac{d\theta}{dr} dr = \varepsilon_\theta\varepsilon_r \int_\alpha^{\pi-\alpha} \frac{f(\cos r)\sin\alpha}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}} dr.$$

The geodesic is then closed if and only if $\theta(\alpha)$ is a rational multiple of π, say

$$\varepsilon_r\varepsilon_\theta\hat{\theta}(\alpha) = \frac{p(\alpha)}{q(\alpha)}\pi,$$

for $p(\alpha), q(\alpha)$ co-prime integers. Furthermore, it closes at the angle $2p(\alpha)\pi$ and, since γ is parametrized by arc length, has length equal to $2q(\alpha)\varepsilon_r l(\alpha)$, where $\varepsilon_r l(\alpha)$ is the interval of time required for γ to move from the parallel $r = \alpha$ to $r = \pi - \alpha$:

$$\varepsilon_r l(\alpha) = \varepsilon_r \int_\alpha^{\pi-\alpha} \frac{ds}{dr} dr = \int_\alpha^{\pi-\alpha} \frac{f(\cos r)\sin r}{\sqrt{\sin^2 r - \sin^2 \alpha}} dr.$$

Given $\alpha \in (0, \pi/2)$, the normalized geodesic $\gamma_\alpha(s) = (r_\alpha(s), \theta_\alpha(s))$ passing through $(\alpha, 0)$ at time zero with $(dr_\alpha/ds)(0) = 0$ and $(d\theta/ds) > 0$ is the unique geodesic — modulo the $\mathbb{S}^1$ action, and reversing the parametrization — such that $r_\alpha \in [\alpha, \pi - \alpha]$ and touches the extreme parallels $r = \alpha$ and $r = \pi - \alpha$. Thus, the application $\alpha \in (0, \pi/2) \mapsto \varepsilon_r \varepsilon_\theta \hat{\theta}(\alpha) \in \mathbb{R}$ is well defined, and is continuous, because of the smoothness of the exponential map. In particular, if all geodesics are closed, then $\varepsilon_\theta \varepsilon_r \hat{\theta} : (0, \pi/2) \to \mathbb{R}$ is a continuous map with range in $\pi\mathbb{Q}$. Hence $\hat{\theta}$ must be constant, and $p(\alpha) = p$, $q(\alpha) = q$ for all α.

The function $\alpha \in (0, \pi/2) \mapsto \varepsilon_{r_\alpha} l(\alpha)$ is also well defined, continuous, and tends to $\operatorname{dist}(N, S)$ when $\alpha \to 0$ — since γ_α, restricted to $[0, l(\alpha)]$, is approaching a meridian in this case. Using the continuity of $\varepsilon_{r_\alpha} l(\alpha)$, we can construct a (free) homotopy between any two geodesics γ_{α_1} and γ_{α_2} via geodesics of the form γ_α, parametrized in such a way that they all close at time 1. Since every geodesic is a critical point for the energy functional, the homotopy gives a path between two critical points via critical points. Hence the energy must be constant, and also the length; this shows that $\varepsilon_r l(\alpha)$ must be constant equal to $\operatorname{dist}(N, S)$.

The only remaining case is when γ is the equator, but then γ is simply closed and has length

$$\int_0^{2\pi} f(0) d\theta = 2\pi f(0) = 2\pi.$$

□

Characterization of Zoll surfaces of revolution. *A rotationally symmetric metric g on $\mathbb{S}^2$ is a $\mathcal{Z}_{2\pi}$-metric if and only if, on the cylindrical coordinate chart $(U; r, \theta)$, g is written as*

$$g = [1 + h(\cos r)]^2 dr^2 + \sin^2 r d\theta^2, \tag{3.15}$$

where $h : [-1, 1] \to (-1, 1)$ is a smooth odd function mapping 1 to 0.

Proof. First, observe that, since $(\mathbb{S}^2, \mathrm{can})$ is a $\mathcal{Z}_{2\pi}$-surface of revolution, Theorem 3.4 tells us that

$$\int_\alpha^{\pi-\alpha} \frac{\sin \alpha}{\sin r \sqrt{\sin^2 r - \sin^2 \alpha}} dr = \pi$$

for every real number $\alpha \in (0, \pi/2)$. Theorem 3.4 and Lemma 3.3 also tell us that a metric of revolution g is $\mathcal{Z}_{2\pi}$ only if g can be written as in (3.10) for some smooth function $f : [-1, 1] \to (0, \infty)$ satisfying the condition (3.11), and such that $f(-1) = f(0) = f(1) = 1$. Define

$$h(x) := f(x) - 1.$$

Our claim is that

$$\int_{\alpha}^{\pi-\alpha} \frac{h(\cos r)\sin\alpha}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr = 0 \quad \forall\alpha \in (0,\pi/2] \tag{3.16}$$

if and only if h is odd. For this, define $h^e(x) := (h(x) + h(-x))/2$,

$$H(\alpha) := \int_{\alpha}^{\frac{\pi}{2}} \frac{h^e(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr,\ \alpha \in (0,\pi/2],$$

and notice that

$$\int_{\alpha}^{\pi-\alpha} \frac{h(\cos r)\sin\alpha}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr = 2\sin(\alpha)H(\alpha),$$

because

$$\begin{aligned}
\int_{\alpha}^{\pi-\alpha} \frac{h(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr &= \int_{\alpha}^{\frac{\pi}{2}} \frac{h(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr \\
&\quad + \int_{\frac{\pi}{2}}^{\pi-\alpha} \frac{h(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr \\
&= \int_{\alpha}^{\frac{\pi}{2}} \frac{h(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr \\
&\quad + \int_{\alpha}^{\frac{\pi}{2}} \frac{h(-\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr \\
&= \int_{\alpha}^{\frac{\pi}{2}} \frac{2h^e(\cos r)}{\sin r\sqrt{\sin^2 r - \sin^2\alpha}}dr \\
&= 2H(\alpha).
\end{aligned}$$

In particular, h is odd if and only if $h^e \equiv 0$, which implies $H \equiv 0$, and so (3.16) is satisfied.

For the converse, consider the function

$$I(\beta) := \int_{\beta}^{\frac{\pi}{2}} \frac{\sin(\beta)\cos(\beta)H(\alpha)}{\sqrt{\sin^2\alpha - \sin^2\beta}}d\alpha,\ \beta \in (0,\pi/2].$$

Observe that, for each $\beta \in (0,\pi/2]$, the function

$$(r,\alpha) \mapsto \frac{1}{\sqrt{(\sin^2 r - \sin^2\alpha)(\sin^2\alpha - \sin^2\beta)}}$$

is integrable on $\{(r,\alpha) : r \in [\beta,\pi/2], \alpha \in [\beta,\pi/2], \alpha \leq r\}$, and Fubini's theorem tells us that

$$\begin{aligned}
I(\beta) &= \int_{\beta}^{\frac{\pi}{2}} \frac{h^e(\cos r)}{\sin r}\left(\int_{\beta}^{r} \frac{\sin\alpha\cdot\cos\alpha}{\sqrt{(\sin^2 r - \sin^2\alpha)(\sin^2\alpha - \sin^2\beta)}}d\alpha\right)dr \\
&= \int_{\beta}^{\frac{\pi}{2}} \frac{h^e(\cos r)}{\sin r}\left(\int_{0}^{\infty} \frac{dx}{1+x^2}\right)dr,
\end{aligned} \tag{3.17}$$

where we used the change of variables

$$x = \sqrt{\frac{\sin^2 \alpha - \sin^2 \beta}{\sin^2 r - \sin^2 \alpha}}.$$

If $H \equiv 0$, then $I \equiv 0$, which implies, by formula (3.17), $h^e \equiv 0$ — equivalently, h is odd.

Also, since f maps $[-1, 1]$ to $(0, \infty)$, the assumption of $h = f - 1$ being odd implies that h maps $[-1, 1]$ to $(-1, 1)$. Otherwise, there would be some $x \in [-1, 1]$ with $h(x) \geq 1$, and so $f(-x) = 1 - h(x) \leq 0$, which is absurd.

Thus, by Theorem 3.4, we conclude that, if a metric g is given by formula (3.15) for some smooth odd function $h : [-1, 1] \to (-1, 1)$ such that $h(1) = 0$, then its geodesics are all simply closed, and — except possibly for the meridians — of length 2π. Since the length of a meridian is

$$\int_0^\pi (1 + h(\cos r))dr = \pi$$

by the oddness of h, we conclude that g is a Zoll metric. On the other hand, if $g \in \mathcal{Z}(\mathbb{S}^2, 2\pi)$, then the argument above together with Lemma 3.3 and Theorem 3.4 imply that g may be written as in formula (3.15) for some smooth odd function $h : [-1, 1] \to (-1, 1)$ such that $h(1) = 0$. $\square$

Remark. Although we considered only the case of Zoll spheres of revolution in dimension two, it is interesting to mention that the construction done here can be extended to spheres of arbitrary dimension. This was done by Weinstein, and is exposed in [Bes], Chapter 4, Section 4.E. Guillemin also proved that there are many other interesting examples on the sphere. We state his result below.

Theorem 3.5 (Guillemin — see [Gui])**.** *For every odd function $\dot{\rho}$ on $\mathbb{S}^2$, there exists a smooth one-parameter family of C^∞-functions $\rho(t)$ on the sphere such that $\rho(0) = 0$, $\rho'(0) = \dot{\rho}$ and $e^{2\rho(t)} \cdot \mathrm{can}$ is a Zoll metric for small t.*

4 Point-line duality and Green's Theorem

Our main goal for this chapter is to prove the following result:

Theorem 4.1 (Green's Theorem)**.** *If g is a Zoll metric on the real projective plane $\mathbb{RP}^2$, then g is isometric to a constant multiple of the canonical metric.*

When Green first proved this theorem (see [Gre] and also Chapter 5 of [Bes]), the ideas and methods used were entirely in a Riemannian geometric setting. However, the argument we will present here, due to LeBrun and Mason [LM1], goes on a different way: it explores a duality between points and lines that arises from Corollary 2.26. Let g be a Zoll metric on $M \approx \mathbb{RP}^2$, and denote by $\nabla = \nabla^g$ its Levi-Civita connection. We now know that the projective connection $[\nabla]$ is a (tame) Zoll projective structure (see Lemma 2.10), and that such structure on $\mathbb{RP}^2$ implies that any two geodesics of M intersect at exactly one point (see Corollary 2.26). In particular, a point $p \in M$ determines a circle $\ell_p = \nu[\mu^{-1}(p)]$ in the manifold N of unparametrized geodesics of M, and any two distinct ℓ_p and $\ell_q \subset N$ intersect at precisely one point — the point of N representing the unique geodesic that passes through both p and q. In this sense, M becomes the moduli space of the curves $\{\ell_p\}_{p \in M}$ in N.

One could argue, however, that such a general viewpoint does not give us much information. Indeed, how could we possibly hope to distinguish the curves of the family $\{\ell_p : p \in M\}$? The point is that this situation is analogous to the projective duality known in algebraic geometry between the projective plane $\mathbb{KP}^2$ and the dual projective plane $\mathbb{KP}^{2*} = \mathbb{P}(\mathbb{K}^{3*})$ for any field $\mathbb{K}$ (see [Ful]). Even more, what we will argue is that the data we have is not only 'analogous to', but actually 'is' the projective duality, in some sense. To make our argument more transparent, let us work with the following definition.

Definition 4.2. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let M be a (complex, when $\mathbb{K} = \mathbb{C}$) surface. A *point-line dual structure* on M is a collection $\mathscr{C}$ of subsets of M satisfying the following properties:

(i) the elements of $\mathscr{C}$ are non-singular embedded (complex, when $\mathbb{K} = \mathbb{C}$) curves of M, all of them diffeomorphic (biholomorphic, when $\mathbb{K} = \mathbb{C}$) to $\mathbb{KP}^1$, and are called the *lines* of M;

(ii) for every point $p \in M$, and for every $\mathbb{K}$-subspace $\ell \subset T_pM$ of $\mathbb{K}$-dimension one there is a unique line $\mathfrak{C} = \mathfrak{C}_{(p,\ell)} \in \mathscr{C}$ passing through p with $T_p\mathfrak{C} = \ell$;

(iii) any two distinct lines $\mathfrak{C}_1$ and $\mathfrak{C}_2$ intersect at exactly one point; and

(iv) the foliation $\mathcal{F}$ of the lifted lines of M on $\mathbb{P}_{\mathbb{K}}TM = (TM - 0_M)/\mathbb{K}^{\times}$ is locally trivial, in the sense that every lifted line $\hat{\mathfrak{C}} \subset \mathbb{P}TM$ has a neighborhood diffeomorphic (biholomorphic, when $\mathbb{K} = \mathbb{C}$) to $\mathbb{KP}^1 \times \mathbb{K}^2$ in such a way that the lifted lines contained in this neighborhood correspond to the projective lines $\mathbb{KP}^1 \times \{\mathrm{pt}\}$.

Of course the main examples of these structures are $\mathbb{RP}^2$ and $\mathbb{CP}^2$ with their canonical projective lines. What we proved in Section 2.3 is that any (tame) Zoll projective structure on a manifold $M \approx \mathbb{RP}^2$ induces a point-line dual structure on M.

When M is equipped with a point-line dual structure $\mathscr{C}$, then $\mathscr{C}$ is canonically identified with the leaf space $N = \mathbb{P}_{\mathbb{K}}TM/\mathcal{F}$, which is a connected $\mathbb{K}$-surface called the *dual* of M, and the canonical projection $\nu : \mathbb{P}_{\mathbb{K}}TM \to N$ is a $\mathbb{KP}^1$-fiber bundle. For each point $p \in M$, there is a corresponding *dual line* $\ell_p = \nu[\mu^{-1}(p)]$ in N — the set of lines of M passing through p —, and property (iii) tells us that two distinct points $p, q \in M$ determine distinct dual lines ℓ_p and ℓ_q. Using properties (i), (ii) and (iv), we can prove, with almost the same argument as the one given in Lemma 2.14, that any two distinct points $p, q \in M$ are contained in some line $\mathfrak{C}$, which is unique by property (iii). In particular, M must be compact, and any two dual lines ℓ_p and ℓ_q intersect at a unique point of N. When M is a real surface, arguments similar to the proofs of the first lemmas of Section 3.1 tell us that M must be either $\mathbb{S}^2$ or $\mathbb{RP}^2$, and that $N \approx \mathbb{RP}^2$. In fact, M cannot be $\mathbb{S}^2$, since property (iii) cannot be satisfied in this case; hence $M \approx \mathbb{RP}^2$.

When $\mathscr{C}$ comes from a Zoll projective structure on a compact real surface M, we will show that $\mathscr{C}^* := \{\ell_p : p \in M\}$ is a point-line dual structure on N, and that the dual of $(N, \mathscr{C}^*)$ is $(M, \mathscr{C})$.

The most important observation, however, is that this kind of structure is unique when $M \approx \mathbb{CP}^2$, as the following lemma states.

Lemma 4.3. *Let $\mathcal{S}$ be a simply connected compact complex surface, equipped with a fixed class $\alpha \in H_2(\mathcal{S}, \mathbb{Z})$ such that $\alpha \cdot \alpha = 1$. Suppose also that there is a family $\mathscr{C}$ of nonsingular embedded complex curves of genus 0 in $\mathcal{S}$, all of them with homology class α, and such that, for every point $p \in \mathcal{S}$, there is at least one $\mathfrak{C} \in \mathscr{C}$ passing through p. Then $\mathcal{S}$ is biholomorphic to $\mathbb{CP}^2$, in such a way that all of the given curves of the family $\mathscr{C}$ become projective lines. Furthermore, if there is an embedded real surface $S \hookrightarrow \mathcal{S}$, together with an anti-holomorphic involution $\sigma : \mathcal{S} \to \mathcal{S}$ with fixed set S, and such that all the curves of $\mathscr{C}$ are invariant under σ, then the biholomorphism $F : \mathcal{S} \to \mathbb{CP}^2$ can be chosen in a way that σ becomes the standard complex conjugation $[z_0 : z_1 : z_2] \mapsto [\overline{z}_0 : \overline{z}_1 : \overline{z}_2]$, S is identified with $\mathbb{RP}^2 = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : z_0, z_1, z_2 \in \mathbb{R}\}$, and the complex curves $\mathfrak{C} \in \mathscr{C}$ with the complex projective lines $\mathbb{CP}^1 \subset \mathbb{CP}^2$ which are invariant under complex conjugation.*

We will not prove this lemma (see Lemma 3.3 and Theorem 3.4 of [LM1]), but the point is that the result makes us ask if point-line dual structures on $\mathbb{RP}^2$ are also unique. As we will see, the answer is affirmative when the point-line dual structure comes from a tame Zoll projective structure.

The proof we give will be done in three parts. First, we construct a four-dimensional manifold $\mathcal{N}$ containing N. Then we will show that there is a complex structure on $\mathcal{N}$, and that all hypotheses of Lemma 4.3 are satisfied. Finally, we identify $\mathcal{N}$ with $\mathbb{CP}^2$ in such a manner that the lines ℓ_p are the real part of the projective curves $ax + by + cz = 0$, for $a, b, c \in \mathbb{R}$, and identify M with $(\mathbb{RP}^2)^* \subset (\mathbb{CP}^2)^*$. This identification will be proved to be conformal, and a last simple computation will show that, when this duality arises from a metric g, we obtain an isometry between (M, g) and $\mathbb{RP}^2$ equipped with a constant multiple of its canonical metric.

4.1 Embedding the manifold of unparametrized geodesics

For now, fix a point-line dual structure $\mathscr{C}$ on a manifold $M \approx \mathbb{RP}^2$. As we explained above, a point-line dual structure on M induces a diagram as given in (2.5), and we can also construct a map $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ given by $\varphi(z) = \nu_*(\ker \mu_{*,z})$. We will impose an extra restriction on the point-line dual structure $\mathscr{C}$:

(v) the map $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ is a diffeomorphism, such that the submersion $\nu : \mathbb{P}TM \to N$ becomes the canonical projection $\mathbb{P}TN \to N$, and the line bundle $\ker \mu_* \to \mathbb{P}TM$ becomes the tautological real line bundle $L \to \mathbb{P}TN$.

In other words, we assume that the conclusion of Theorem 2.25 holds.

We want to construct $\mathcal{N}$ in such a way that N is embedded in $\mathcal{N}$, and that the dual lines ℓ_p, $p \in M$, become embedded circles inside projective curves $\Sigma_p \hookrightarrow \mathcal{N}$, $\Sigma_p \approx \mathbb{CP}^1$. For this purpose, notice that the dual lines ℓ_p are the images of $\mu^{-1}(p) \approx \mathbb{RP}^1$ under the map ν, so that, by considering the complexification

$$\mathbb{P}T_{\mathbb{C}}M := (\mathbb{C} \otimes \mathbb{P}TM - 0_M)/\mathbb{C}^\times,$$

and denoting by $\hat{\mu} : \mathbb{P}T_{\mathbb{C}}M \to M$ the canonical projection, it would be reasonable to require that there is a map $\Psi : \mathbb{P}T_{\mathbb{C}}M \to \mathcal{N}$ satisfying the following conditions:

(1) Ψ is a submersion, and its restriction to $\mathbb{P}TM \subset \mathbb{P}T_{\mathbb{C}}M$ is the map ν;

(2) the projective lines Σ_p are the sets $\Psi[\hat{\mu}^{-1}(p)]$.

The construction of the manifold $\mathcal{N}$ is then done with the following procedure. First, observe that there is a complex structure $J^{||}$ on the fibers of $\mathbb{P}T_{\mathbb{C}}M$, since those fibers are complex projective lines. In particular, the line bundle $J^{||} \ker \mu_* \to \mathbb{P}TM$ can be identified with the normal bundle of the inclusion $\mathbb{P}TM \hookrightarrow \mathbb{P}T_{\mathbb{C}}M$. Next, fix a tubular neighborhood $\hat{\mathcal{V}} \subset \mathbb{P}T_{\mathbb{C}}M$ of $\mathbb{P}TM$, and fix a diffeomorphism $\hat{\mathcal{V}} \approx J^{||} \ker \mu_*$. Then by identifying $\hat{\mathcal{V}}$ with the tautological line bundle $L \to \mathbb{P}TN$ via the compositions $\hat{\mathcal{V}} \approx J^{||} \ker \mu_* \xrightarrow{-J^{||}} \ker \mu_* \xrightarrow{\tilde{\varphi}} L$, where $\tilde{\varphi}(z, u) = (\varphi(z), \nu_{*,z} u)$ (see the proof of Theorem 2.25), we get a map $\psi : \hat{\mathcal{V}} \to TN$ induced by the blowing-down map $\beta : L \to TN$, $\beta(y, v) = v$. Having done this, denote by $\mathcal{U} = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$, $\mathcal{V} = TN$, and define

$$\mathcal{N} := \mathcal{U} \cup_\psi \mathcal{V}. \tag{4.1}$$

Lemma 4.4. *Suppose $\mathscr{C}$ is a real point-line dual projective structure on a surface $M \approx \mathbb{RP}^2$ that satisfies the extra condition* (v)*. Then there is a real compact simply connected four-dimensional manifold $\mathcal{N}$, together with a submersion $\Psi : \mathbb{P}T_{\mathbb{C}}M \to \mathcal{N}$, in such a manner that N is smoothly embedded in $\mathcal{N}$, the restriction of Ψ to $\mathbb{P}TM$ is the map ν, and each dual line ℓ_p, $p \in M$, is contained in a smooth closed real surface $\Sigma_p := \Psi[\hat{\mu}^{-1}(p)]$ of genus zero.*

Proof. We define $\mathcal{N}$ as in (4.1). The canonical map $\Psi : \mathbb{P}T_{\mathbb{C}}M \to \mathcal{N}$ is taken to be the identity on $\mathcal{U}$, and the 'blowing-down' map $\psi : \hat{\mathcal{V}} \to \mathcal{V}$ on $\hat{\mathcal{V}}$. The topology on $\mathcal{N}$ is the weakest for which the function Ψ is a continuous open map, and N is embedded in $\mathcal{N}$ as the zero section of $TN = \mathcal{V}$. Now observe that the blowing-down map $\beta : L \to TN = \mathcal{V}$ is a diffeomorphism away from the zero section $\mathbb{P}TN$, where it restricts to the canonical projection $\mathbb{P}TN \to N$. Also, since the submersion $\nu : \mathbb{P}TM \to N$ is identified with the canonical projection $\mathbb{P}TN \to N$ under the diffeomorphism $\varphi : \mathbb{P}TM \to \mathbb{P}TN$, the restriction of Ψ to $\mathbb{P}TM$ is the map ν. Hence there is a unique differentiable structure on $\mathcal{N}$ for which the map Ψ is a smooth submersion and induces a diffeomorphism between $\mathcal{U}$ and $\mathcal{N} - N$, and the open set $\mathcal{V} = \Psi(\hat{\mathcal{V}})$ is a tubular neighborhood of $N \hookrightarrow \mathcal{N}$.

For a point $y \in \mathcal{N} - N$, a coordinate chart (U', χ') around $\Psi^{-1}(y)$, with $U \subset \mathcal{U}$, induces a chart $(U, \chi) = (\Psi(U'), \chi' \circ (\Psi|_{\mathcal{U}'})^{-1})$ around y. For the other case, when $y \in N$, we can take a coordinate chart $(W; y^1, y^2)$ of N around y, and consider the induced coordinates (y^1, y^2, c^1, c^2) on $TN|_W$ given by

$$(y^1, y^2, c^1, c^2) \leftrightarrow c^1 \frac{\partial}{\partial y^1}\bigg|_{(y^1,y^2)} + c^2 \frac{\partial}{\partial y^2}\bigg|_{(y^1,y^2)}.$$

Since $V = TN|_W$ is an open set of $\mathcal{V} = TN \subset \mathcal{N}$ (and since $\mathcal{V}$ is open in $\mathcal{N}$), (V, ϑ), where $\vartheta = (y^1, y^2, c^1, c^2)$, is a coordinate chart around y in $\mathcal{N}$. Coordinate transitions between two charts of the form (U, χ) are smooth because $\mathcal{U}$ is, in itself, a smooth manifold. The same is true for coordinate transitions between two charts of type (V, ϑ), by the very differentiable structure of TN. The only nontrivial case is when we consider transitions between charts of type (U, χ) and (V, ϑ). But by our construction, $\vartheta \circ \chi^{-1}$ is the map ψ restricted to an open set of $\mathcal{U} \cap \hat{\mathcal{V}}$. Thus $\vartheta \circ \chi^{-1}$ is a diffeomorphism, for ψ is a diffeomorphism away from $\mathbb{P}TM$. The collection of all charts of type (U, χ) and (V, ϑ) is then the C^∞ atlas of $\mathcal{N}$.

We define $\Sigma_p := \Psi(\hat{\mu}^{-1}(p))$ for each $p \in M$. Because the restriction of $\Psi_{*,z}$ to the tangent space $T_z\hat{\mu}^{-1}(p)$ is injective for each $z \in \hat{\mu}^{-1}(p)$ and each p, the sets Σ_p are real closed surfaces, and Ψ induces a diffeomorphism between $\hat{\mu}^{-1}(p) \approx \mathbb{CP}^1$ and Σ_p.

It only remains to prove that $\mathcal{N}$ is simply connected. For this, observe that $\mathcal{U}$ can be viewed as the set of point-wise complex structures on M. Indeed, an element $[v] \in \mathbb{P}T_{\mathbb{C},p}M - \mathbb{P}T_pM$ induces a decomposition

$$T_{\mathbb{C},p}M = \operatorname{span}_{\mathbb{C}}\{v\} \oplus \operatorname{span}_{\mathbb{C}}\{\overline{v}\}$$

for which we associate the complex structure $I_{[v]} : T_{\mathbb{C},p}M \to T_{\mathbb{C},p}M$ acting as

$$I_{[v]} = \begin{bmatrix} -i & 0 \\ 0 & +i \end{bmatrix}.$$

On the other hand, the $(-i)$-eigenspace of a complex structure $I : T_{\mathbb{C},p}M \to T_{\mathbb{C},p}M$ can be viewed as an element of $\mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$. (Our convention for the choice of $(+i)$- and $(-i)$-eigenspaces follows the discussion done in the next section.) But the set of point-wise complex structures on M can be identified with the set of pairs $([g_p], \mathcal{O}_p)$, where $[g_p]$ is a conformal class of a inner product g_p on $T_pM \approx \mathbb{R}^2$, and $\mathcal{O}_p$ is a choice of orientation for T_pM. Since the space of inner products is a convex cone, $\mathcal{U}$ deform retracts to the orientation covering $\widetilde{M} \approx \mathbb{S}^2$ of M.

We can thus finish our proof. By construction, $\mathcal{N} = \mathcal{U} \cup \mathcal{V}$. On the one hand, $\mathcal{U}$ deform retracts to $\mathbb{S}^2$, and $\mathcal{V} \approx TN$ deform retracts to N. On the other hand, the inclusion $\imath : \mathcal{U} \cap \mathcal{V} \hookrightarrow \mathcal{V}$ is homotopic to the bundle projection $\pi : TN - 0_N \to N$. Since $\mathcal{U}$ is simply connected and $\mathcal{U} \cap \mathcal{V}$ is connected, the Seifert-van Kampen Theorem tells us that

$$\pi_1(\mathcal{N}) = \frac{\pi_1(\mathcal{V})}{\imath_\#(\pi_1(\mathcal{U} \cap \mathcal{V}))} = \frac{\pi_1(N)}{\pi_\#(\pi_1(TN - 0_N))}.$$

At the same time, the fibers of $\pi : TN - 0_N \to N$ are path connected, so $\pi_\# : \pi_1(TN - 0_N) \to \pi_1(N)$ is onto. Thus $\mathcal{N}$ is simply connected. $\square$

4.2 The complex structure

Now that we have constructed the four-dimensional manifold $\mathcal{N}$, we want to prove that it is the complex projective plane. For that we will construct a complex structure J on $\mathcal{N}$, and verify that all conditions stated in Lemma 4.3 are satisfied. This is the point of the argument where the Zoll projective structure $[\nabla]$ plays a prominent role: it determines a special decomposition of $T\mathbb{P}T_{\mathbb{C}}M$ as the direct sum of two vector sub-bundles.

Fix $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$, and consider a manifold M of dimension n, together with an affine $\mathbb{K}$-vector bundle $(E, \nabla) \to M$, and denote by $\mu : \mathbb{P}E \to M$, $\mathbb{P}E = E/\mathbb{K}^\times$, the induced projective bundle. When $\mathbb{K} = \mathbb{C}$, assume that ∇ is complex linear, *i.e.* $\nabla(is) = i\nabla s$ for every section $s \in \Gamma(E)$. Then the connection ∇ induces a decomposition $T\mathbb{P}E = \mathbf{H} \oplus \mathbf{V}$, where $\mathbf{H}$ and $\mathbf{V}$ are called the *horizontal* and the *vertical* bundles respectively, in such a manner that $\mu_* : \mathbf{H} \to \mu^*TM$ is an isomorphism. The vertical bundle is simply taken to be $\mathbf{V} = \ker \mu_*$, while the horizontal bundle is defined as follows: For a given point p, and a class $[e]$ in the fiber $\mathbb{P}E_p$, $\mathbf{H}_{[e]}$ is generated by vectors of the form $\sigma'(0)$, where $\sigma = [s] : (-\varepsilon, \varepsilon) \to \mathbb{P}E$ is a section of $\mathbb{P}E$ along a smooth curve $c : (-\varepsilon, \varepsilon) \to M$, starting at $p = c(0)$, that can be represented by a parallel section s of E along c, satisfying the initial condition $s(0) = e \in [e]$.

Turning back to our case of interest, we have a compact surface M^2, equipped with a Zoll projective structure $[\nabla]$ of conjugacy number equal to one, and we fix a torsion-free connection $\nabla \in [\nabla]$. The vector bundle here considered is $T_{\mathbb{C}}M$, which we turn into an affine bundle by extending ∇ to be complex linear. Then ∇ decomposes $T\mathbb{P}T_{\mathbb{C}}M$ as a direct sum $\mathbf{H} \oplus \mathbf{V}$, that can be complexified as

$$T_{\mathbb{C}}\mathbb{P}T_{\mathbb{C}}M = \mathbf{H}_{\mathbb{C}} \oplus \mathbf{V}_{\mathbb{C}}.$$

On the one hand, $\mathbf{V} = \ker \hat{\mu}_*$ is tangent to the fibers of $\mathbb{P}T_{\mathbb{C}}M \to M$, and since all of them are copies of $\mathbb{CP}^1$, there is a fiber-wise complex structure $J^{||} : \mathbf{V} \to \mathbf{V}$, $(J^{||})^2 = -1$. In particular, we can consider the $(-i)$-eigenspace of $J^{||}$:

$$\mathbf{L}_1 := \mathbf{V}^{0,1}_{J^{||}} \subset \mathbf{V}_{\mathbb{C}}. \tag{4.2}$$

On the other hand, $\hat{\mu}_* : \mathbf{H}_{\mathbb{C}} \to T_{\mathbb{C}}M$ is an isomorphism, so there is a 'tautological' line sub-bundle $\mathbf{L}_2 \subset \mathbf{H}_{\mathbb{C}}$ given by

$$(\mathbf{L}_2)_{[v]} := (\hat{\mu}_{*,[v]})^{-1}\big(\mathrm{span}_{\mathbb{C}}\{v\}\big). \tag{4.3}$$

We then define

$$\mathbf{D} := \mathbf{L}_1 \oplus \mathbf{L}_2. \tag{4.4}$$

The idea behind this construction is that determining a complex structure J on $\mathcal{N}$ is the same as defining an involutive complex distribution $D \subset T_{\mathbb{C}}\mathcal{N}$, having $\dim_{\mathbb{C}} D_y = 2$, and $D_y \cap \overline{D_y} = \{0\}$ for all $y \in \mathcal{N}$. Indeed, for a given J, we define $D = T^{0,1}_J\mathcal{N}$; while a sub-bundle D with those properties induces an almost complex structure J having D as its $(-i)$-eigenspace, and $\overline{D}$ as its i-eigenspace. In the second case, the integrability of J follows from D being involutive, because for any two $X, Y \in \Gamma(D)$, the Nijenhuis tensor is

$$\begin{aligned}
\tau(X,Y) &= [X,Y] - [JX,JY] + J[X,JY] + J[JX,Y] \\
&= [X,Y] - (-i)^2[X,Y] + (-i)J[X,Y] + (-i)J[X,Y] \\
&= 2[X,Y] - 2[X,Y] = 0.
\end{aligned}$$

Of course we could interchange the roles of D and its conjugate $\overline{D}$, but this would only act as a change of orientation.

Lemma 4.5. *Let $(M, [\nabla])$ be a surface diffeomorphic to $\mathbb{RP}^2$, equipped with a Zoll projective structure. Then the distribution $\mathbf{D}$ defined in* (4.4) *is an involutive two-dimensional complex sub-bundle of $T_{\mathbb{C}}\mathbb{P}T_{\mathbb{C}}M$, and*

$$\dim_{\mathbb{C}}(\mathbf{D}_z \cap \overline{\mathbf{D}}_z) = \begin{cases} 0, & \textit{for } z \notin \mathbb{P}TM \\ 1, & \textit{for } z \in \mathbb{P}TM. \end{cases}$$

Furthermore, $\mathbf{D}$ does not depend on the choice of the torsion-free connection ∇ representing the class $[\nabla]$.

Proof. $\mathbf{D}$ is a complex two-dimensional vector sub-bundle of $T_{\mathbb{C}}\mathbb{P}T_{\mathbb{C}}M$ by definition. The other assertions are proved by local computations. Let $z = (p,[v]) \in \mathbb{P}T_{\mathbb{C}}M$, consider a smooth path $c : (-\varepsilon, \varepsilon) \to M$ starting at p, and fix a section $\sigma = [X] : (-\varepsilon,\varepsilon) \to \mathbb{P}TM$ along c, passing through z at time zero, and represented by a parallel complex vector field $X(t)$ along c. Choose a coordinate chart $(U; x^1, x^2)$ of M around p, in such a way that

$X(0) = \partial_1|_p + \zeta\partial_2|_p$, and denote by $(x^1, x^2, \zeta) : \hat{\mu}^{-1}(U) \to \mathbb{R}^2 \times \mathbb{C}$ the induced coordinates on $\hat{\mu}^{-1}(U) \subset \mathbb{P}T_{\mathbb{C}}M$, given by

$$(x^1, x^2, \zeta) \leftrightarrow \left[\left(\frac{\partial}{\partial x^1} + \zeta\frac{\partial}{\partial x^2}\right)\Big|_{(x^1,x^2)}\right].$$

We can then write, on U and on $\hat{\mu}^{-1}(U)$, $c(t) = (c^1(t), c^2(t))$, $X(t) = a^1(t)(\partial_1 \circ c) + a^2(t)(\partial_2 \circ c)$, and $\sigma(t) = (c^1(t), c^2(t), a^2(t)/a^1(t))$, where $a^1(t), a^2(t) \in \mathbb{C}$, and $a^1(t) \neq 0$ for all t. In particular,

$$\begin{aligned} \sigma'(0) &= \dot{c}^1(0)\frac{\partial}{\partial x^1} + \dot{c}^2(0)\frac{\partial}{\partial x^2} + \left[\frac{d}{dt}\Big|_{t=0}\big(a^2/a^1\big)\right]\frac{\partial}{\partial \zeta} + \left[\frac{d}{dt}\Big|_{t=0}\overline{\big(a^2/a^1\big)}\right]\frac{\partial}{\partial \overline{\zeta}} \\ &= \dot{c}^1(0)\frac{\partial}{\partial x^1} + \dot{c}^2(0)\frac{\partial}{\partial x^2} + \big(\dot{a}^2(0) - \zeta\dot{a}^1(0)\big)\frac{\partial}{\partial \zeta} + \overline{\big(\dot{a}^2(0) - \zeta\dot{a}^1(0)\big)}\frac{\partial}{\partial \overline{\zeta}}. \end{aligned} \tag{4.5}$$

Since X is parallel by assumption, we have

$$\begin{aligned} 0 = \nabla_{\frac{d}{dt}} X &= \nabla_{\frac{d}{dt}}\left[a^j\left(\frac{\partial}{\partial x^j} \circ c\right)\right] \\ &= \dot{a}^l\left(\frac{\partial}{\partial x^l} \circ c\right) + a^j\nabla_{\frac{d}{dt}}\left(\frac{\partial}{\partial x^j} \circ c\right) \\ &= \big[\dot{a}^l + a^j\dot{c}^k(\Gamma^l_{jk} \circ c)\big]\left(\frac{\partial}{\partial x^l} \circ c\right), \end{aligned}$$

where Γ^l_{jk} are the Christoffel symbols of ∇ with respect to the coordinates (x^1, x^2); hence

$$\dot{a}^l(0) = -a^j(0)\dot{c}^k(0)\Gamma^l_{jk}(p), \quad l = 1, 2. \tag{4.6}$$

After substituting equations (4.6) on (4.5), we obtain

$$\begin{aligned} \sigma'(0) = \sum_{j=1}^{2}\dot{c}^j(0)\Bigg\{\frac{\partial}{\partial x^j}\Big|_z &+ \Big[\zeta\Big(\Gamma^1_{j1}(p) + \zeta\Gamma^1_{j2}(p)\Big) - \Gamma^2_{j1}(p) - \zeta\Gamma^2_{j2}(p)\Big]\frac{\partial}{\partial \zeta}\Big|_z \\ &+ \Big[\overline{\zeta}\Big(\Gamma^1_{j1}(p) + \overline{\zeta}\Gamma^1_{j2}(p)\Big) - \Gamma^2_{j1}(p) - \overline{\zeta}\Gamma^2_{j2}(p)\Big]\frac{\partial}{\partial \overline{\zeta}}\Big|_z\Bigg\}. \end{aligned}$$

By the definition of horizontal bundle, what all these computations show us is that $\mathbf{H}_{\mathbb{C},z}$, $z = (x^1, x^2, \zeta)$, is the complex vector space spanned (over $\mathbb{C}$) by

$$\frac{\partial}{\partial x^j}\Big|_z + P_j(x^1, x^2, \zeta)\frac{\partial}{\partial \zeta}\Big|_z + P_j(x^1, x^2, \overline{\zeta})\frac{\partial}{\partial \overline{\zeta}}\Big|_z, \quad j = 1, 2,$$

where

$$P_j(x^1, x^2, \zeta) = -\Gamma^2_{j1} + \zeta\big(\Gamma^1_{j1} - \Gamma^2_{j2}\big) + \zeta^2\Gamma^1_{j2}.$$

Observe that the calculations done here also prove that $\hat{\mu}_* : \mathbf{H}_{\mathbb{C}} \to \hat{\mu}^* T_{\mathbb{C}} M$ is an isomorphism.

Now we see that, in the chart $(\hat{\mu}^{-1}(U); x^1, x^2, \zeta)$, the complex line bundle $\mathbf{L}_1$ is generated by $\partial/\partial\overline{\zeta}$, while the fiber of $\mathbf{L}_2$ on a point $z = (x^1, x^2, \zeta)$ is the complex vector space spanned by

$$\begin{aligned}(\hat{\mu}_{*,z})^{-1}\left(\frac{\partial}{\partial x^1} + \zeta\frac{\partial}{\partial x^2}\right) &= \frac{\partial}{\partial x^1}\bigg|_z + \zeta\frac{\partial}{\partial x^2}\bigg|_z + \big(P_1(x^1,x^2,\zeta) + \zeta P_2(x^1,x^2,\zeta)\big)\frac{\partial}{\partial\zeta}\bigg|_z \\ &\qquad + \big(P_1(x^1,x^2,\overline{\zeta}) + \zeta P_2(x^1,x^2,\overline{\zeta})\big)\frac{\partial}{\partial\overline{\zeta}}\bigg|_z \\ &= \frac{\partial}{\partial x^1}\bigg|_z + \zeta\frac{\partial}{\partial x^2}\bigg|_z + Q(x,\zeta,\zeta)\frac{\partial}{\partial\zeta}\bigg|_z + Q(x,\zeta,\overline{\zeta})\frac{\partial}{\partial\overline{\zeta}}\bigg|_z \\ &=: \Xi_{0,z},\end{aligned} \tag{4.7}$$

here we use the notation

$$Q(x,u,v) = -\Gamma^2_{11} + v\Gamma^1_{11} - (u+v)\Gamma^2_{12} + v(u+v)\Gamma^1_{12} - uv\Gamma^2_{22} + uv^2\Gamma^1_{22}.$$

Thus locally we have

$$\mathbf{D} = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial\overline{\zeta}}, \Xi\right\},$$

where

$$\begin{aligned}\Xi &= \Xi_0 - Q(x,\zeta,\overline{\zeta})\frac{\partial}{\partial\overline{\zeta}} \\ &= \frac{\partial}{\partial x^1} + \zeta\frac{\partial}{\partial x^2} + \tilde{Q}(x,\zeta)\frac{\partial}{\partial\zeta},\end{aligned} \tag{4.8}$$

for $\tilde{Q}(x,\zeta) := Q(x,\zeta,\zeta)$. Since $\partial\zeta/\partial\overline{\zeta} = 0$ and $\partial\tilde{Q}/\partial\overline{\zeta} = 0$, we see that

$$\left[\frac{\partial}{\partial\overline{\zeta}}, \Xi\right] = 0,$$

which implies that $\mathbf{D}$ is, in fact, involutive.

Observe that a change of torsion-free connection ∇ in $[\nabla]$ is characterized, on local coordinates, as a replacement of Γ^l_{jk} by $\Gamma^l_{jk} + \delta^l_j\omega_k + \omega_j\delta^l_k$. Any such substitution leaves

$$\tilde{Q}(x,\zeta) = -\Gamma^2_{11} + \big(\Gamma^1_{11} - 2\Gamma^2_{12}\big)\zeta + \big(2\Gamma^1_{12} - \Gamma^2_{22}\big)\zeta^2 + \Gamma^1_{22}\zeta^3$$

unchanged, so that Ξ does not depend on the choice of $\nabla \in [\nabla]$. Also, the sub-bundle $\mathbf{L}_1$ was defined independently of the projective structure. Thus $\mathbf{D}$ is projectively invariant, since it is locally spanned by $\partial/\partial\overline{\zeta}$ and Ξ, which do not depend on the choice of $\nabla \in [\nabla]$.

Finally, it comes from the definition that $\big(\mathbf{L}_1 \cap \overline{\mathbf{L}}_1\big)_z = \{0\}$ for all $z \in \mathbb{P}T_{\mathbb{C}}M$ — in our coordinate representation, we see that $\overline{\mathbf{L}}_1 = \operatorname{span}_{\mathbb{C}}\{\partial/\partial\zeta\} = \mathbf{V}^{1,0}_{J||}$. On the other hand, notice that formula (4.7) tells us that the imaginary part of Ξ_0 does not vanish when z is not real — *i.e.* $z \notin \mathbb{P}TM$ — while Ξ_0 is real for $z \in \mathbb{P}TM$. This is because, writing $\zeta = \xi + i\eta$, we know that $\mathbb{P}TM$ is locally the set of points for which $\eta = 0$, so that Ξ_0 on $\mathbb{P}TM$ is

$$\Xi_0 = \frac{\partial}{\partial x^1} + \xi \frac{\partial}{\partial x^2} + \tilde{Q}(x, \xi) \left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}} \right) = \frac{\partial}{\partial x^1} + \xi \frac{\partial}{\partial x^2} + \tilde{Q}(x, \xi) \frac{\partial}{\partial \xi}$$

As a conclusion, we have

$$\dim(\mathbf{D}_z \cap \overline{\mathbf{D}}_z) = \dim(\mathbf{L}_{2,z} \cap \overline{\mathbf{L}}_{2,z}) = \begin{cases} 0, & \text{for } z \notin \mathbb{P}TM \\ 1, & \text{for } z \in \mathbb{P}TM. \end{cases}$$

□

Our goal is to prove that $\Psi_* \mathbf{D}$ induces a unique complex structure on $\mathcal{N}$. The next lemma is an important step in this direction.

Lemma 4.6. *Let $(M, [\nabla])$ be a surface diffeomorphic to $\mathbb{RP}^2$, equipped with a Zoll projective structure. Then*

$$\Psi_{*,z} \mathbf{D}_z \cap \overline{\Psi_{*,z} \mathbf{D}_z} = 0$$

for every $z \in \mathbb{P}T_{\mathbb{C}}M$. Moreover, if $z_0, z_1 \in \mathbb{P}TM$ are two distinct points such that $\Psi(z_0) = \Psi(z_1) = y$, then

$$\Psi_{*,z_0} \mathbf{V}_{z_0} \oplus \Psi_{*,z_1} \mathbf{V}_{z_1} = T_y \mathcal{N}.$$

Proof. The construction of the manifold $\mathcal{N}$ was done by gluing $\mathcal{U} = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$ and $\mathcal{V} = TN$ via a 'blowing-down' map from a tubular neighborhood of $\mathbb{P}TM$ to $\mathcal{V}$. On $\mathcal{U}$, Lemma 4.5 tells us that $\mathbf{D}$ induces a complex structure, as discussed previously. At the same time, the restriction of $\mathbf{L}_2$ to $\mathbb{P}TM$ is the line-bundle generated by the directions tangent to the geodesics of M, and Ψ restricts to $\mathbb{P}TM$ as the submersion ν, which collapses all lifted geodesics into points. Consequently, $\Psi_* \mathbf{L}_{2,z} = 0$ whenever $z \in \mathbb{P}TM$, so that

$$\Psi_{*,z} \mathbf{D}_z \cap \overline{\Psi_{*,z} \mathbf{D}_z} = (\Psi_{*,z} \mathbf{L}_{1,z} + \Psi_{*,z} \mathbf{L}_{2,z}) \cap \overline{(\Psi_{*,z} \mathbf{L}_{1,z} + \Psi_{*,z} \mathbf{L}_{2,z})} = \Psi_{*,z} \mathbf{L}_{1,z} \cap \overline{\Psi_{*,z} \mathbf{L}_{1,z}}. \quad (4.9)$$

Our goal is to prove that

$$\Psi_{*,z} \mathbf{L}_{1,z} \cap \overline{\Psi_{*,z} \mathbf{L}_{1,z}} = 0.$$

To better understand this, fix a direction $[v] \in \mathbb{P}TM$, and let $\mathfrak{C}$ be the unique geodesic passing through $\mu([v])$ with $T_{\mu([v])} \mathfrak{C} = [v]$. Choose a representative connection $\nabla \in [\nabla]$, and consider an affine parametrization $\gamma : (-\varepsilon, \varepsilon) \to M$ starting at $\mu([v])$. Because $t \mapsto \gamma'(t) \in TM$ is a parallel vector field along γ, the section $\sigma : t \mapsto [\gamma'(t)] \in \mathbb{P}TM$ induces a nonzero element $\sigma'(0) \in \mathbf{H}_{\mathbb{C},[v]}$. Actually we have more: $\sigma'(t) \in \mathbf{H}_{\mathbb{C},\sigma(t)}$ for all t. But $\sigma'(0)$ is contained in $\mathbf{L}_2$, for the equality $\hat{\mu} \circ \sigma = \gamma$ implies $\sigma'(t) = (\hat{\mu}_*|_{\mathbf{H}_{\mathbb{C}}})^{-1}(\gamma'(t)) \in (\mathbf{L}_2)_{[\gamma'(t)]}$.

Since $\mathbf{L}_2$ is a line-bundle, and by the arbitrary choice of $[v] \in \mathbb{P}TM$, our conclusion is what was said above: that $\mathbf{L}_2$ on $\mathbb{P}TM$ is formed by the lines tangent to the geodesics of M.

It remains to examine what happens to $\mathbf{L}_1$ under the map Ψ_*, when considered on the restriction of $T_{\mathbb{C}}\mathbb{P}T_{\mathbb{C}}M$ to $\mathbb{P}TM$. We proceed by constructing local coordinates (r, θ), (y^1, y^2) and (y^1, y^2, θ) on M, N and $\mathbb{P}TM$, respectively, in such a way that the projections $\mu : \mathbb{P}TM \to M$ and $\nu : \mathbb{P}TM \to N$ are locally expressed as

$$\mu(y^1, y^2, \theta) = (r(y^1, y^2, \theta), \theta) \text{ and } \nu(y^1, y^2, \theta) = (y^1, y^2).$$

Let $z_0 \in \mathbb{P}TM$ be an arbitrary point, and denote by $p_0 = \mu(z_0) \in M$ and by $\hat{\mathfrak{C}} = \hat{\mathfrak{C}}_{z_0}$ the unique lifted geodesic passing through z_0. Take a neighborhood $\hat{U}$ of $\hat{\mathfrak{C}}$ diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$, in a way that the geodesics contained in $\hat{U}$ correspond to the circles $\{\mathrm{pt}\} \times \mathbb{S}^1$, and $\hat{\mathfrak{C}}$ becomes $\{0\} \times \mathbb{S}^1$. Denote the 'trivializing' diffeomorphism $\hat{U} \approx \mathbb{R}^2 \times \mathbb{S}^1$ by (y^1, y^2, t). (Here we think of t as varying in the interval $(-\pi, \pi]$, and as being zero on z_0.) By taking $\hat{U}$ sufficiently small, we may assume that $U = \mu(\hat{U})$ is contained in a tubular neighborhood of $\mathfrak{C} \subset M$. In particular, there is a smooth retraction $f : U \to \mathfrak{C}$ which induces a submersion $\theta : U \to \mathbb{S}^1$, by identifying $\mathfrak{C} \approx \hat{\mathfrak{C}} \approx \mathbb{S}^1$ via the coordinate t. Then the function $\hat{\theta} := \mu^*\theta : \hat{U} \to \mathbb{S}^1$ is also a submersion, and both t and $\hat{\theta}$ coincide on $\hat{\mathfrak{C}}$ from the construction of θ. Therefore, by restricting $\hat{U}$ once more, if necessary, we can assume that $d\hat{\theta}/dt$ never vanishes, so that $(y^1, y^2, \hat{\theta})$ is a 'cylindrical' coordinate system on $\hat{U}$.

The problem is that U is a Möbius band, which does not allow us to work with global coordinates. Fortunately, this can be easily overcome if we fix some number $\varepsilon \in (0, \pi]$ and consider $U_\varepsilon := \{p \in U : |\theta(p)| < \varepsilon\}$ and $\hat{U}_\varepsilon := \{z \in \hat{U} : |\hat{\theta}(z)| < \varepsilon\}$. Now U_ε and $\hat{U}_\varepsilon$ are neighborhoods of p_0 and z_0, respectively, and U_ε is the image of $\hat{U}_\varepsilon$ under the map μ. Moreover, $\hat{U}_\varepsilon$ is diffeomorphic to $\mathbb{R}^2 \times (-\varepsilon, \varepsilon)$ via the coordinate system $(y^1, y^2, \hat{\theta})$, and there is a parametrization $(r, \theta) : U_\varepsilon \approx \mathbb{R} \times (-\varepsilon, \varepsilon)$, because the line $\mathfrak{C} \cap U_\varepsilon \approx (-\varepsilon, \varepsilon)$ has trivial normal bundle. From now on we use θ to denote both coordinates θ on U_ε and $\hat{\theta}$ on $\hat{U}_\varepsilon$.

For these coordinates, the canonical projection $\mu : \mathbb{P}TM \to M$ takes the form $(y^1, y^2, \theta) \mapsto (r(y^1, y^2, \theta), \theta)$, so that

$$\ker \mu_* = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial r}{\partial y^2} \frac{\partial}{\partial y^1} - \frac{\partial r}{\partial y^1} \frac{\partial}{\partial y^2} \right\}$$

on $\hat{U}_\varepsilon$. Observe that $\partial r / \partial y^1$ and $\partial r / \partial y^2$ cannot vanish at the same time, for μ is a submersion. Next, write

$$X := \frac{\partial r}{\partial y^2} \frac{\partial}{\partial y^1} - \frac{\partial r}{\partial y^1} \frac{\partial}{\partial y^2}.$$

Then $J^{||} \ker \mu_*$ becomes the bundle spanned by $J^{||} X$ (over the reals), and we parametrize $J^{||} \ker \mu_*$ as

$$(y^1, y^2, \theta, \lambda) \leftrightarrow \lambda J^{||} X_{(y^1, y^2, \theta)}.$$

We now turn to the local description of the quotient map $\nu : \mathbb{P}TM \to N$. Since ν collapses the lifted geodesics, we can give $V = \nu(\hat{U}) = \nu(\hat{U}_\varepsilon)$ coordinates (y^1, y^2) in such a manner that ν becomes the projection $(y^1, y^2, \theta) \mapsto (y^1, y^2)$. As a consequence, we have

$$\nu_{*,z} X_z = \frac{\partial r}{\partial y^2} \frac{\partial}{\partial y^1}\bigg|_{\nu(z)} - \frac{\partial r}{\partial y^1} \frac{\partial}{\partial y^2}\bigg|_{\nu(z)}$$

for all $z \in \hat{U}_\varepsilon$. Similarly to what was done in the proof of Lemma 4.5, the system (y^1, y^2) induces local parametrizations of the form (y^1, y^2, c^1, c^2), (y^1, y^2, ξ), and (y^1, y^2, ξ, η) on TN, $\mathbb{P}TN$, and on the 'tautological' bundle $L \to \mathbb{P}TN$, respectively. The identifications are:

$$\begin{aligned} (y^1, y^2, c^1, c^2) &\leftrightarrow c^1 \frac{\partial}{\partial y^1}\bigg|_{(y^1,y^2)} + c^2 \frac{\partial}{\partial y^2}\bigg|_{(y^1,y^2)}, \\ (y^1, y^2, \xi) &\leftrightarrow \left[\left(\frac{\partial}{\partial y^1} + \xi \frac{\partial}{\partial y^2} \right)\bigg|_{(y^1,y^2)} \right], \\ (y^1, y^2, \xi, \eta) &\leftrightarrow \eta \left(\frac{\partial}{\partial y^1} + \xi \frac{\partial}{\partial y^2} \right)\bigg|_{(y^1,y^2)}. \end{aligned}$$

This allows us to represent the projection $L \to \mathbb{P}TN$ as $(y^1, y^2, \xi, \eta) \mapsto (y^1, y^2, \xi/\eta)$, and the blowing-down map $\beta : L \to TN$ as $(y^1, y^2, \xi, \eta) \mapsto (y^1, y^2, \eta, \xi\eta)$.

On the other hand, interchanging y^1 and y^2, and taking ε smaller, if necessary, we may assume that $\partial r/\partial y^2$ does not vanish. Hence $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ is locally written as

$$\varphi(y^1, y^2, \theta) = \left(y^1, y^2, -\frac{\partial r/\partial y^1}{\partial r/\partial y^2} \right), \tag{4.10}$$

while the composition $\beta \circ \tilde{\varphi} \circ (-J^{||}) : J^{||} \ker \mu_* \to TN$ becomes

$$(y^1, y^2, \theta, \lambda) \mapsto \left(y^1, y^2, \lambda \frac{\partial r}{\partial y^2}\bigg|_{(y^1,y^2,\theta)}, -\lambda \frac{\partial r}{\partial y^1}\bigg|_{(y^1,y^2,\theta)} \right). \tag{4.11}$$

Moreover, we also know that

$$\frac{\partial}{\partial \theta}\bigg|_{(y^1,y^2,\theta)} \left(\frac{\partial r/\partial y^1}{\partial r/\partial y^2} \right)$$

never vanishes, for the map φ is a diffeomorphism.

The point of all these computations is that, in our construction of $\mathcal{N}$, we fixed a tubular neighborhood $\hat{\mathcal{V}}$ of $\mathbb{P}TM$, together with a diffeomorphism $\hat{\mathcal{V}} \approx J^{||} \ker \mu_*$. In other words, we view the coordinates $(y^1, y^2, \theta, \lambda)$ as local parametrizations of $\hat{\mathcal{V}}$, and the composition $\beta \circ \tilde{\varphi} \circ (-J^{||}) : J^{||} \ker \mu_* \to TN$ as the map Ψ.

For these coordinate systems, the canonical projection $\hat{\mu} : \mathbb{P}T_{\mathbb{C}}M \to M$ is locally expressed as

$$\hat{\mu}(y^1, y^2, \theta, \lambda) = (r(y^1, y^2, \theta), \theta).$$

Hence the vertical bundle $\mathbf{V} = \ker \hat{\mu}_*$ is generated by

$$X = \frac{\partial r}{\partial y^2} \frac{\partial}{\partial y^1} - \frac{\partial r}{\partial y^1} \frac{\partial}{\partial y^2} \text{ and } \frac{\partial}{\partial \lambda}.$$

But observe that $\mathbb{P}TM$ is locally the hypersurface $\lambda = 0$, and that

$$\left.\frac{\partial}{\partial\lambda}\right|_{(y^1,y^2,\theta,0)} = \left.\frac{d}{ds}\right|_{s=0}(y^1,y^2,\theta,s) = \left.\frac{d}{ds}\right|_{s=0} sJ^{||}X_{(y^1,y^2,\theta,0)} = J^{||}X_{(y^1,y^2,\theta,0)}. \tag{4.12}$$

Thus the line bundle $\mathbf{L}_1$, restricted to $\mathbb{P}TM$, is spanned (over $\mathbb{C}$) by

$$\frac{1}{2}(X + iJ^{||}X) = \frac{1}{2}\left(X + i\frac{\partial}{\partial\lambda}\right)$$

on a neighborhood of z_0 in $\mathbb{P}T_{\mathbb{C}}M$.

Our conclusion is: from formulas (4.11) and (4.12), we have

$$\Psi_{*,z_0}(X_{z_0} + iJ^{||}X_{z_0}) = \left.\frac{\partial r}{\partial y^2}\right|_{(0,0,0)}\left.\left(\frac{\partial}{\partial y^1} + i\frac{\partial}{\partial c^1}\right)\right|_{\Psi(z_0)} - \left.\frac{\partial r}{\partial y^1}\right|_{(0,0,0)}\left.\left(\frac{\partial}{\partial y^2} + i\frac{\partial}{\partial c^2}\right)\right|_{\Psi(z_0)}$$

so that $\Psi_{*,z_0}(\mathbf{L}_{1,z_0}) \cap \overline{\Psi_{*,z_0}(\mathbf{L}_{1,z_0})} = 0$. We then apply formula (4.9) to obtain the desired result:

$$\Psi_*\mathbf{D} \cap \overline{\Psi_*\mathbf{D}} = 0,$$

since $z_0 \in \mathbb{P}TM$ was arbitrary.

Furthermore, if z_1 is another point in $\hat{U}_\varepsilon$ contained in the lifted geodesic $\hat{\mathfrak{C}}$, then $z_1 = (0,0,\theta_1,0)$, with $\theta_1 \neq 0$, in our coordinate system on a neighborhood of z_0 in $\mathbb{P}T_{\mathbb{C}}M$. Since $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ is a diffeomorphism, formula (4.10) tells us that

$$\left.\frac{\partial r/\partial y^1}{\partial r/\partial y^2}\right|_{(0,0,\theta_1)} \neq \left.\frac{\partial r/\partial y^1}{\partial r/\partial y^2}\right|_{(0,0,0)}.$$

Hence we have

$$\begin{aligned}\Psi_{*,z_0}\left(\left.\frac{\partial}{\partial\lambda}\right|_{z_0}\right) &= \left.\frac{\partial r}{\partial y^2}\right|_{(0,0,0)}\left.\frac{\partial}{\partial c^1}\right|_y - \left.\frac{\partial r}{\partial y^1}\right|_{(0,0,0)}\left.\frac{\partial}{\partial c^2}\right|_y \\ &\neq \left.\frac{\partial r}{\partial y^2}\right|_{(0,0,\theta_1)}\left.\frac{\partial}{\partial c^1}\right|_y - \left.\frac{\partial r}{\partial y^1}\right|_{(0,0,\theta_1)}\left.\frac{\partial}{\partial c^2}\right|_y \\ &= \Psi_{*,z_1}\left(\left.\frac{\partial}{\partial\lambda}\right|_{z_1}\right),\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\Psi_{*,z_0}(X_{z_0}) &= \left.\frac{\partial r}{\partial y^2}\right|_{(0,0,0)}\left.\frac{\partial}{\partial y^1}\right|_y - \left.\frac{\partial r}{\partial y^1}\right|_{(0,0,0)}\left.\frac{\partial}{\partial y^2}\right|_y \\ &\neq \left.\frac{\partial r}{\partial y^2}\right|_{(0,0,\theta_1)}\left.\frac{\partial}{\partial y^1}\right|_y - \left.\frac{\partial r}{\partial y^1}\right|_{(0,0,\theta_1)}\left.\frac{\partial}{\partial y^2}\right|_y \\ &= \Psi_{*,z_1}(X_{z_1}),\end{aligned} \tag{4.14}$$

where $y = \Psi(z_0) = \Psi(z_1)$. Since $\mathbf{V} = \ker \hat{\mu}_*$ is locally spanned by X and $\partial/\partial\lambda$, formulas (4.13) and (4.14) also imply

$$\Psi_{*,z_0}\mathbf{V}_{z_0} \oplus \Psi_{*,z_1}\mathbf{V}_{z_1} = T_y\mathcal{N}$$

whenever $z_0, z_1 \in \mathbb{P}TM$ are two distinct points contained in the same lifted geodesic, and $y = \Psi(z_0) = \Psi(z_1)$, for $\Psi_*(X_{z_0})$, $\Psi_*(X_{z_1})$, $\Psi_*(\partial/\partial\lambda|_{z_0})$ and $\Psi_*(\partial/\partial\lambda|_{z_1})$ are all linearly independent. $\square$

The next proposition is the key technical result in [LM1] (compare [LM1], Proposition 3.1).

Proposition 4.7. *Let $(M, [\nabla])$ be a surface diffeomorphic to $\mathbb{RP}^2$, equipped with a Zoll projective structure. Then there is a unique complex structure J on $\mathcal{N}$ such that*

$$\Psi_*\mathbf{D} \subset T^{0,1}(\mathcal{N}, J). \tag{4.15}$$

Proof. Since the proof is long, we divide it in three parts as follows:

Claim 1. *There is a unique rough almost complex structure J on $\mathcal{N}$ that satisfies equation* (4.15).

Recall that this means that there is a not necessarily continuous section J of $\mathrm{End}(T\mathcal{N})$, such that $J^2 = -\mathrm{Id}$.

Proof of Claim 1. On $\mathcal{N} - N = \Psi(\mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM) \approx \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$, there is only one choice of J: since

$$T_{\mathbb{C},y}\mathcal{N} = \Psi_{*,z}\mathbf{D}_z \oplus \overline{\Psi_{*,z}\mathbf{D}_z} \tag{4.16}$$

whenever $y \in \mathcal{N} - N$ and $z = \Psi^{-1}(y)$, J must be the only almost complex structure whose $(-i)$-eingenspace and i-eigenspace are $\Psi_{*,z}\mathbf{D}_z$ and $\overline{\Psi_{*,z}\mathbf{D}_z}$, respectively. In terms of the decomposition (4.16),

$$J = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

on $\mathcal{N} - N$.

It remains to define J on N. For this, observe that

$$\mathbf{V} = \ker \hat{\mu}_* = \ker \mu_* \oplus J^{||} \ker \mu_*,$$

so that, according to this decomposition, $J^{||}$ acts on $\mathbf{V}$ as

$$J^{||} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

At the same time, Ψ restricts to $\mathbb{P}TM$ as the map ν — hence $\ker \mu_*$ is mapped to the tangent bundle of N under Ψ_*, while the image of $J^{||} \ker \mu_*$ lies in the normal bundle of N, viewed as an embedded submanifold of $TN = \mathcal{V}$. This shows that the natural choice of J on N is the following: For a point $y \in N$, any choice of coordinate chart $(V; y^1, y^2)$ around y induces a parametrization (y^1, y^2, c^1, c^2) on $TN|_V$, given by

$$(y^1, y^2, c^1, c^2) \leftrightarrow c^1 \frac{\partial}{\partial y^1}\bigg|_{(y^1,y^2)} + c^2 \frac{\partial}{\partial y^2}\bigg|_{(y^1,y^2)}.$$

In particular, we can decompose $T\mathcal{V}|_V$ as

$$TN|_V \oplus T^{\perp}N|_V, \tag{4.17}$$

where $TN|_V$ is spanned by $\partial/\partial y^1$ and $\partial/\partial y^1$, while $T^{\perp}N|_V$ is spanned by $\partial/\partial c^1$ and $\partial/\partial c^2$ fiber-wise. Since any other choice of coordinates $(\hat{y}^1, \hat{y}^2)$ around y produces a local trivialization $(\hat{y}^1, \hat{y}^2, \hat{c}^1, \hat{c}^2)$ on TN in such a way that

$$\frac{\partial}{\partial \hat{y}^i} = \frac{\partial y^j}{\partial \hat{y}^i}\frac{\partial}{\partial y^j} \quad \text{and} \quad \frac{\partial}{\partial \hat{c}^i} = \frac{\partial y^j}{\partial \hat{y}^i}\frac{\partial}{\partial c^j},$$

we see that decomposition (4.17) is invariant under change of local coordinates (y^1, y^2) on N. As a consequence, we can write

$$T\mathcal{V}|_N = TN \oplus T^{\perp}N,$$

and define

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

on N, according to this decomposition.

Although it is not at all obvious that J is even continuous on $\mathcal{N}$, we claim that J is the unique rough almost complex structure for which the identity $\Psi_* \circ J^{||} = J \circ \Psi_*$ holds. In particular, this will imply that $\Psi_*\mathbf{D} \subset T^{0,1}(\mathcal{N}, J)$. Indeed, when we consider coordinates $(y^1, y^2, \theta, \lambda)$ on $\mathbb{P}T_{\mathbb{C}}M$ and (y^1, y^2, c^1, c^2) on $\mathcal{N}$ as before, the vertical bundle $\mathbf{V}$ is seen to be locally generated by

$$X = \frac{\partial r}{\partial y^2}\frac{\partial}{\partial y^1} - \frac{\partial r}{\partial y^1}\frac{\partial}{\partial y^2} \text{ and } \frac{\partial}{\partial \lambda},$$

while the map $\Psi : \mathbb{P}T_{\mathbb{C}}M \to \mathcal{N}$ is written as in (4.11):

$$\Psi(y^1, y^2, \theta, \lambda) = \left(y^1, y^2, \lambda\frac{\partial r}{\partial y^2}, -\lambda\frac{\partial r}{\partial y^1} \right).$$

We also have

$$\frac{\partial}{\partial \lambda}\bigg|_{(y^1,y^2,\theta,0)} = J^{||}X_{(y^1,y^2,\theta,0)}$$

from (4.12), and for any two points $z_0 = (y^1, y^2, \theta_0, 0)$ and $z_1 = (y^1, y^2, \theta_1, 0)$ of $\mathbb{P}TM$, the tangent space $T_y\mathcal{N}$ of $y = \Psi(z_0) = \Psi(z_1)$ is spanned by $\Psi_{*,z_0}(X_{z_0})$, $\Psi_{*,z_1}(X_{z_1})$,

$\Psi_{*,z_0}(\partial/\partial\lambda|_{z_0})$ and $\Psi_{*,z_1}(\partial/\partial\lambda|_{z_1})$ — see (4.13) and (4.14). Hence, equality $\Psi_* \circ J^{||} = J \circ \Psi_*$ is satisfied on N if and only if

$$J\Psi_{*,z_j}(X_{z_j}) = \Psi_{*,z_j}(\partial/\partial\lambda|_{z_j}), \text{ and } J\Psi_{*,z_j}(\partial/\partial\lambda|_{z_j}) = -\Psi_{*,z_j}(X_{z_j}).$$

But this is equivalent to

$$J\left(\frac{\partial}{\partial y^j}\right) = \frac{\partial}{\partial c^j} \text{ and } J\left(\frac{\partial}{\partial c^j}\right) = -\frac{\partial}{\partial y^j},$$

because

$$\Psi_{*,z_j}\left(\frac{\partial}{\partial\lambda}\bigg|_{z_j}\right) = \frac{\partial r}{\partial y^2}\bigg|_{(y^1,y^2,\theta_j)} \frac{\partial}{\partial c^1}\bigg|_y - \frac{\partial r}{\partial y^1}\bigg|_{(y^1,y^2,\theta_j)} \frac{\partial}{\partial c^2}\bigg|_y$$

and

$$\Psi_{*,z_j}(X_{z_j}) = \frac{\partial r}{\partial y^2}\bigg|_{(y^1,y^2,\theta_j)} \frac{\partial}{\partial y^1}\bigg|_y - \frac{\partial r}{\partial y^1}\bigg|_{(y^1,y^2,\theta_j)} \frac{\partial}{\partial y^2}\bigg|_y.$$

On $\mathcal{N} - N$, the identity $\Psi_* \circ J^{||} = J \circ \Psi_*$ is trivially true. Moreover, all these computations show that such almost complex structure must be the unique satisfying (4.15). □

Claim 2. *If J is smooth, then J is integrable.*

Proof of Claim 2. Assume for a moment that J is of class C^∞. Then the properties of the distribution $\mathbf{D}$ stated in Lemma 4.5, together with the fact that Ψ is a diffeomorphism between $\mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$ and $\mathcal{N} - N$ imply that J is a complex structure away from N. Since N is a closed two-dimensional submanifold of $\mathcal{N}$, the smoothness of J tells us that its Nijenhuis tensor vanishes everywhere, because it is continuous on $\mathcal{N}$ and vanishes in the dense open set $\mathcal{N} - N$. □

Claim 3. *J is smooth.*

Proof of Claim 3. Observe that J is smooth if and only if the bundle $T^{0,1}(\mathcal{N}, J)$ is of class C^∞. With this in mind, we will prove that $T^{0,1}(\mathcal{N}, J)$ is of class C^k for every integer $k > 0$. The approach will be to construct C^k local frames for $T^{0,1}(\mathcal{N}, J)$ for any $k > 0$.

Let k be a positive integer greater than 1, fix a coordinate chart $(V; y^1, y^2)$ on N, and denote by $z^j = \nu^* y^j : \nu^{-1}(V) \to \mathbb{R}$. Consider also a coordinate system (x^1, x^2, ζ) on $\mathbb{P}TM$ as the one used in Lemma 4.5, *i.e.* (x^1, x^2) is a local parametrization of M and

$$(x^1, x^2, \zeta) \leftrightarrow \left[\left(\frac{\partial}{\partial x^1} + \zeta\frac{\partial}{\partial x^2}\right)\bigg|_{(x^1,x^2)}\right].$$

Construct smooth complex-valued functions $\mathfrak{z}^1$, $\mathfrak{z}^2$ on an open set of $\mathbb{P}T_{\mathbb{C}}M$ by requiring that $\mathfrak{z}^j(x^1, x^2, \zeta) = z^j(x^1, x^2, \xi) + O(\eta)$ and $\partial\mathfrak{z}^j/\partial\overline{\zeta} = O(\eta^k)$, where $\zeta = \xi + i\eta$. (Observe that, in the coordinates (x^1, x^2, ζ), $\mathbb{P}TM$ is parametrized by $\eta = 0$. In other words, we require that $\mathfrak{z}^j = z^j$ and that $\partial\mathfrak{z}^j/\partial\overline{\zeta} = 0$ to the k^{th} order in η on $\mathbb{P}TM$.) This implies that $\mathfrak{z}^j$ is locally written as

$$\mathfrak{z}^j(x^1, x^2, \zeta) = \sum_{r=0}^{k} \frac{i^r}{r!}\eta^r \frac{\partial^r z^j}{\partial \xi^r}\bigg|_{(x^1,x^2,\xi)} + O(\eta^{k+1}), \tag{4.18}$$

in term of the coordinates (x^1, x^2, ζ), where $\zeta = \xi + i\eta$. This is seen by observing that

$$\begin{aligned}
\frac{\partial\mathfrak{z}^j}{\partial\overline{\zeta}} &= \frac{1}{2}\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)\left(\sum_{r=0}^{k}\frac{i^r}{r!}\eta^r\frac{\partial^r z^j}{\partial\xi^r} + O(\eta^{k+1})\right) \\
&= \frac{1}{2}\sum_{r=0}^{k}\frac{i^r}{r!}\eta^r\frac{\partial^{r+1}z^j}{\partial\xi^{r+1}} + \frac{i}{2}\sum_{r=0}^{k-1}\frac{i^{r+1}}{r!}\eta^r\frac{\partial^{r+1}z^j}{\partial\xi^{r+1}} + O(\eta^k) \\
&= \frac{i^k}{2k!}\eta^k\frac{\partial^{k+1}z^j}{\partial\xi^{k+1}} + O(\eta^k) \\
&= O(\eta^k),
\end{aligned}$$

and since the cancellation is done term by term, uniqueness of the Taylor expansion is guaranteed. In particular, the functions $\mathfrak{z}^j$ are independent of the choice of given parametrization (x^1, x^2, ζ), so that they can be glued together on the intersection of two charts, giving us complex-valued maps on a neighborhood of $\nu^{-1}(V)$, viewed as a subset of $\mathbb{P}T_{\mathbb{C}}M$.

On the one hand, the complex functions $\mathfrak{z}^j$ are constant along the lifted geodesics of $\mathbb{P}TM$, so that they can be viewed as maps on some neighborhood of V in $\mathcal{N}$. On the other hand, since J is smooth when restricted to $T\mathcal{N}|_N$, we can construct smooth complex-valued coordinates $(\tilde{\mathfrak{z}}^1, \tilde{\mathfrak{z}}^2)$ around V by requiring that both $\tilde{\mathfrak{z}}^j = y^j$ and $d\tilde{\mathfrak{z}}^j(Y) = 0$ on V for any $Y \in T^{0,1}(\mathcal{N}, J)|_N$. By the uniqueness of the Taylor expansion (4.18), and since $\partial\Psi^*\tilde{\mathfrak{z}}^j/\partial\overline{\zeta} = 0$ on $\nu^{-1}(V)$, we see that

$$\Psi^*\tilde{\mathfrak{z}}^j = \mathfrak{z}^j + O(\eta^2).$$

Thus $(\mathfrak{z}^1, \mathfrak{z}^2)$ may be viewed both as functions on some open set of $\mathbb{P}T_{\mathbb{C}}M$, or as a smooth complex-valued coordinate system on $\mathcal{N}$. In particular, the map Ψ is then represented as $(x^1, x^2, \zeta) = (x^1, x^2, \xi, \eta) \mapsto (\mathfrak{z}^1, \mathfrak{z}^2)$.

The point of all this is that we want to express $T^{0,1}(\mathcal{N}, J)$ in terms of the coordinates $(\mathfrak{z}^1, \mathfrak{z}^2)$. With this in mind, first notice that

$$T^{0,1}(\mathcal{N}, J)|_{\mathrm{Im}(\mathfrak{z}^j)=0} = \mathrm{span}_{\mathbb{C}}\{\partial/\partial\bar{\mathfrak{z}}^1, \partial/\partial\bar{\mathfrak{z}}^2\}.$$

Indeed, the points where $\mathrm{Im}(\mathfrak{z}^j)$ is zero for $j = 1, 2$ are the points in N, and there we have

$$\begin{aligned}
\Psi_{*,(x^1,x^2,\xi,0)}\left(\frac{\partial}{\partial\overline{\zeta}}\right) &= \left[\Psi_{*,(x^1,x^2,\xi,0)}\left(\frac{\partial}{\partial\overline{\zeta}}\right)\mathfrak{z}^j\right]\frac{\partial}{\partial\mathfrak{z}^j} + \left[\Psi_{*,(x^1,x^2,\xi,0)}\left(\frac{\partial}{\partial\overline{\zeta}}\right)\bar{\mathfrak{z}}^j\right]\frac{\partial}{\partial\bar{\mathfrak{z}}^j} \\
&= \frac{\partial\mathfrak{z}^j}{\partial\overline{\zeta}}\bigg|_{(x^1,x^2,\xi,0)}\frac{\partial}{\partial\mathfrak{z}^j} + \frac{\partial\bar{\mathfrak{z}}^j}{\partial\overline{\zeta}}\bigg|_{(x^1,x^2,\xi,0)}\frac{\partial}{\partial\bar{\mathfrak{z}}^j} \\
&= \frac{\partial\bar{\mathfrak{z}}^j}{\partial\overline{\zeta}}\bigg|_{(x^1,x^2,\xi,0)}\frac{\partial}{\partial\bar{\mathfrak{z}}^j}
\end{aligned}$$

by construction. Since $\Psi_*(\Xi_0)$ vanishes along $\mathbb{P}TM$ (see (4.7) for the definition of Ξ_0), and since $T^{0,1}(\mathcal{N}, J)$ is generated by the image of $\mathbf{D}$ under Ψ_* (this is a consequence of Lemma 4.6), we get the inclusion $T^{0,1}(\mathcal{N}, J) \subset \mathrm{span}_{\mathbb{C}}\{\partial/\partial\bar{\mathfrak{z}}^1, \partial/\partial\bar{\mathfrak{z}}^2\}$ on N. The equality then follows because both sides have complex dimension two point-wise. In particular, we see that $\Xi\mathfrak{z}^j \equiv 0$ when $\eta = 0$ (see (4.8) for the definition of Ξ).

Elsewhere, we have

$$T^{0,1}(\mathcal{N}, J)|_{\mathcal{N}-N} = \mathrm{span}_{\mathbb{C}}\{\Psi_*(\partial/\partial\bar{\zeta}), \Psi_*(\Xi)\}.$$

Hence, by writing

$$\Psi_*(\Xi) = (\Xi\mathfrak{z}^j)\frac{\partial}{\partial\mathfrak{z}^j} + (\Xi\bar{\mathfrak{z}}^j)\frac{\partial}{\partial\bar{\mathfrak{z}}^j} \text{ and } \Psi_*\left(\frac{\partial}{\partial\bar{\zeta}}\right) = \frac{\partial\mathfrak{z}^j}{\partial\bar{\zeta}}\frac{\partial}{\partial\mathfrak{z}^j} + \frac{\partial\bar{\mathfrak{z}}^j}{\partial\bar{\zeta}}\frac{\partial}{\partial\bar{\mathfrak{z}}^j},$$

we see that

$$T^{0,1}(\mathcal{N}, J)|_{\cup_j\{\mathrm{Im}(\mathfrak{z}^j)\neq 0\}} = \mathrm{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial\bar{\mathfrak{z}}^j} - a_j^l\frac{\partial}{\partial\mathfrak{z}^l}\right\}_{j=1,2},$$

where a_j^l are the solutions of

$$\begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}\begin{bmatrix} \Xi\bar{\mathfrak{z}}^1 & \frac{\partial\bar{\mathfrak{z}}^1}{\partial\bar{\zeta}} \\ \Xi\bar{\mathfrak{z}}^2 & \frac{\partial\bar{\mathfrak{z}}^2}{\partial\bar{\zeta}} \end{bmatrix} = -\begin{bmatrix} \Xi\mathfrak{z}^1 & \frac{\partial\mathfrak{z}^1}{\partial\bar{\zeta}} \\ \Xi\mathfrak{z}^2 & \frac{\partial\mathfrak{z}^2}{\partial\bar{\zeta}} \end{bmatrix}. \tag{4.19}$$

(This is a simple Linear Algebra result — see Lemma 4.8 below.)

Our goal then is to analyse what happens to the coefficients a_j^l when $\mathrm{Im}(\mathfrak{z}^j) \to 0$. In order to do so, notice that, because $[\Xi, \partial/\partial\bar{\zeta}] = 0$, the function $\Xi\mathfrak{z}^j$ satisfies

$$\frac{\partial^l}{\partial\bar{\zeta}^l}\Xi\mathfrak{z}^j = \Xi\frac{\partial^l}{\partial\bar{\zeta}^l}\mathfrak{z}^j = \Xi O(\eta^{k-l+1}) = O(\eta^{k-l}).$$

At the same time, $\Xi\mathfrak{z}^j = 0$ along $\eta = 0$, so that

$$\frac{\partial^l}{\partial\eta^l}(\Xi\mathfrak{z}^j)\bigg|_{\eta=0} \equiv 0$$

for $l = 0, ..., k-1$. Hence

$$\Xi\mathfrak{z}^j = O(\eta^k).$$

The other terms of equation (4.19) are estimated as follows:

$$\frac{\partial}{\partial\bar{\zeta}}\bar{\mathfrak{z}}^j = \frac{\partial}{\partial\bar{\zeta}}\left(-i\eta\frac{\partial z^j}{\partial\xi}\bigg|_{\eta=0} + O(\eta^2)\right) = \frac{\partial z^j}{\partial\xi} + O(\eta),$$

and

$$\begin{aligned}
\Xi(\bar{\mathfrak{z}}^j) &= \Xi\bigg(-i\eta\frac{\partial z^j}{\partial \xi}\bigg|_{\eta=0} + O(\eta^2)\bigg) \\
&= -i\eta\left[\Xi, \frac{\partial}{\partial\xi}\right]\bigg|_{\eta=0}\mathfrak{z}^j + O(\eta) \\
&= i\eta\bigg(\frac{\partial z^j}{\partial x^2} + \tilde{Q}'(\xi)\frac{\partial z^j}{\partial \xi}\bigg) + O(\eta).
\end{aligned}$$

Combining everything, we obtain

$$\begin{vmatrix} \Xi\bar{\mathfrak{z}}^1 & \frac{\partial\bar{\mathfrak{z}}^1}{\partial\bar{\zeta}} \\ \Xi\bar{\mathfrak{z}}^2 & \frac{\partial\bar{\mathfrak{z}}^2}{\partial\bar{\zeta}} \end{vmatrix} = i\eta\begin{vmatrix} \frac{\partial z^1}{\partial x^2} + \tilde{Q}'(\xi)\frac{\partial z^1}{\partial\xi} & \frac{\partial z^1}{\partial\xi} \\ \frac{\partial z^2}{\partial x^2} + \tilde{Q}'(\xi)\frac{\partial z^1}{\partial\xi} & \frac{\partial z^2}{\partial\xi} \end{vmatrix} + O(\eta^2) = i\eta\frac{\partial(z^1,z^2)}{\partial(x^2,\xi)} + O(\eta^2)$$

Also, $\partial(z^1,z^2)/\partial(x^2,\xi) \neq 0$ everywhere, because Ξ is linearly independent from both $\partial/\partial x^2$ and $\partial/\partial\xi$. Thus

$$\begin{aligned}
\begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} &= -\begin{bmatrix} \Xi\mathfrak{z}^1 & \frac{\partial\mathfrak{z}^1}{\partial\bar{\zeta}} \\ \Xi\mathfrak{z}^2 & \frac{\partial\mathfrak{z}^2}{\partial\bar{\zeta}} \end{bmatrix}\begin{bmatrix} \Xi\bar{\mathfrak{z}}^1 & \frac{\partial\bar{\mathfrak{z}}^1}{\partial\bar{\zeta}} \\ \Xi\bar{\mathfrak{z}}^2 & \frac{\partial\bar{\mathfrak{z}}^2}{\partial\bar{\zeta}} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} O(\eta^k) & O(\eta^k) \\ O(\eta^k) & O(\eta^k) \end{bmatrix}\frac{1}{i\eta}\bigg(\frac{\partial(x^2,\xi)}{\partial(z^1,z^2)} + O(\eta)\bigg)\begin{bmatrix} \frac{\partial\bar{\mathfrak{z}}^2}{\partial\bar{\zeta}} & -\frac{\partial\bar{\mathfrak{z}}^1}{\partial\bar{\zeta}} \\ -\Xi\bar{\mathfrak{z}}^2 & \Xi\bar{\mathfrak{z}}^1 \end{bmatrix} \\
&= \bigg(\frac{\partial(x^2,\xi)}{\partial(z^1,z^2)} + O(\eta)\bigg)\begin{bmatrix} O(\eta^{k-1}) & O(\eta^{k-1}) \\ O(\eta^{k-1}) & O(\eta^{k-1}) \end{bmatrix}\begin{bmatrix} O(1) & O(1) \\ O(\eta) & O(\eta) \end{bmatrix} \\
&= O(\eta^{k-1})
\end{aligned}$$

What all those computations tell us is that, when we take (x^1, x^2, ξ, η) in any compact set, there is a constant C satisfying

$$|a_j^l| \leq C|\eta|^{k-1}.$$

Passing to $\mathcal{N}$, what we see is that

$$|a_j^l| \leq C'|\mathrm{Im}(\mathfrak{z})|^{k-1},$$

where $\mathfrak{z} = (\mathfrak{z}^1, \mathfrak{z}^2)$, at least on the image under Ψ of the compact set where the coordinates (x^1, x^2, ξ, η) are defined. Furthermore, Ψ is proper, so that we can cover the pre-image of any compact subset of $\mathcal{N}$ by a finite number of compact coordinate charts on $\mathbb{P}T_{\mathbb{C}}M$, and obtain the estimate

$$|a_j^l| \leq C''|\mathrm{Im}(\mathfrak{z})|^{k-1},$$

whenever $(\mathfrak{z}^1, \mathfrak{z}^2)$ lie in a fixed compact set.

We have checked that the coefficients a^l_j are smooth away from N and can be extended to N as zero, in such a way that we obtain C^{k-2} functions. In other words, J is, at least, of class C^{k-2}. Since k was chosen arbitrarily, we conclude that J is thus smooth as we claimed. □

This ends the proof of Proposition 4.7. □

Lemma 4.8. *Let $\{v_1, v_2\}$ and $\{w_1, w_2\}$ be two pairs of linearly independent vectors of $\mathbb{C}^4$, in such a way that $V \cap \overline{V} = W \cap \overline{W} = \{0\}$, for $V = \operatorname{span}\{v_1, v_2\}$ and $W = \operatorname{span}\{w_1, w_2\}$. In particular $\mathbb{C}^4 = V \oplus \overline{V} = W \oplus \overline{W}$, and we can write*

$$w_j = a^k_j v_k + b^k_j \overline{w}_k.$$

We write

$$A = \begin{bmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} b^1_1 & b^1_2 \\ b^2_1 & b^2_2 \end{bmatrix},$$

and assume that A is invertible. Then $W = \operatorname{span}\{v_j - c^k_j \overline{v}_k\}_{j=1,2}$, where

$$C = \begin{bmatrix} c^1_1 & c^1_2 \\ c^2_1 & c^2_2 \end{bmatrix}.$$

is a solution of the matrix equation

$$CA = -B. \tag{4.20}$$

Proof. The linear transformation T defined by $Tw_j = 0$ and $T\overline{w}_j = \overline{w_j}$ is determined, in terms of the basis $\{v_1, v_2, \overline{v}_1, \overline{v}_2\}$, by the system of equations

$$T(a^k_j v_k + b^k_j \overline{v}_j) = 0 \text{ and } T(\overline{b}^k_j v_k + \overline{a}^k_j \overline{v}_j) = \overline{b}^k_j v_k + \overline{a}^k_j \overline{v}_j,$$

which is written in matrix notation as

$$T \begin{bmatrix} A & \overline{B} \\ B & \overline{A} \end{bmatrix} = \begin{bmatrix} 0 & \overline{B} \\ 0 & \overline{A} \end{bmatrix},$$

or equivalently

$$\begin{aligned} T &= \begin{bmatrix} 0 & \overline{B} \\ 0 & \overline{A} \end{bmatrix} \begin{bmatrix} A & \overline{B} \\ B & \overline{A} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & \overline{B} \\ 0 & \overline{A} \end{bmatrix} \begin{bmatrix} A^{-1} + A^{-1}\overline{B}(\overline{A} - BA^{-1}\overline{B})^{-1}BA^{-1} & -A^{-1}\overline{B}(\overline{A} - BA^{-1}\overline{B})^{-1} \\ -(\overline{A} - BA^{-1}\overline{B})^{-1}BA^{-1} & (\overline{A} - BA^{-1}\overline{B})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} -\overline{B}(\overline{A} - BA^{-1}\overline{B})^{-1}BA^{-1} & \overline{B}(\overline{A} - BA^{-1}\overline{B})^{-1} \\ -\overline{A}(\overline{A} - BA^{-1}\overline{B})^{-1}BA^{-1} & \overline{A}(\overline{A} - BA^{-1}\overline{B})^{-1} \end{bmatrix}. \end{aligned}$$

We want to find a matrix

$$C = \begin{bmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{bmatrix}.$$

in a way that $W = \operatorname{span}\{v_j - c_j^k \overline{v}_k\}_{j=1,2}$. That is, we want to determine the coeficients c_j^k in such a way that

$$T(v_j - c_j^k \overline{v}_k) = 0, \quad j = 1, 2.$$

In matrix notation, these equations are

$$0 = T \begin{bmatrix} \mathrm{Id} \\ -C \end{bmatrix} = \begin{bmatrix} -\overline{B}(\overline{A} - BA^{-1}\overline{B})^{-1}(BA^{-1} + C) \\ -\overline{A}(\overline{A} - BA^{-1}\overline{B})^{-1}(BA^{-1} + C) \end{bmatrix}.$$

Thus $C = -BA^{-1}$ is a solution, and the invertibility of A guarantees its existence. Rewriting this equation, we conclude that C must satisfy the desired equation

$$CA = -B.$$

□

Remark. In the specific case dealt in Claim 3 above, $v_j = \partial/\partial\overline{\mathfrak{z}}^j$, $w_1 = \Xi$ and $w_2 = \partial/\partial\overline{\zeta}$, and (4.20) becomes equation (4.19).

Proposition 4.9. *Let $(M, [\nabla])$ be a surface diffeomorphic to $\mathbb{RP}^2$, equipped with a Zoll projective structure, and denote by N its space of unparametrized geodesics. Then there is a compact simply connected complex surface $\mathcal{N}$, together with an embedding $N \hookrightarrow \mathcal{N}$, such that*

(1) there is an anti-holomorphic involution $\sigma : \mathcal{N} \to \mathcal{N}$ with fixed-point set N;

(2) for all $p \in M$, there is a complex curve $\Sigma_p \subset \mathcal{N}$, $\Sigma_p \approx \mathbb{CP}^1$, in a manner that

$$\ell_p = \Sigma_p \cap N;$$

(3) the surfaces Σ_p, $p \in M$, represent the same element of $\pi_2(\mathcal{N})$; and

(4) if p_1 and p_2 are two distinct points of M, then Σ_{p_1} and Σ_{p_2} are transverse and intersect exactly at one point.

Proof. We already constructed $\mathcal{N}$, proved that it is a compact simply connected complex surface, and defined the complex curves $\Sigma_x := \Psi[\hat{\mu}^{-1}(x)]$. Now notice that each fiber $\hat{\mu}^{-1}(x)$ is a $\mathbb{CP}^1$ with the complex structure $J^{||}$, and that Ψ induces a holomorphic map between $\hat{\mu}^{-1}(x)$ and Σ_x, by the construction of the complex structure J on $\mathcal{N}$. Since Ψ is a diffeomorphism away from $\mathbb{P}TM$, its holomorphic restriction to $\hat{\mu}^{-1}(x)$ must be of degree one, so that $\Sigma_x \approx \mathbb{CP}^1$. Moreover, all Σ_x are freely homotopic to each other in $\mathcal{N}$, since this is true for the fibers of $\mathbb{P}T_{\mathbb{C}}M$.

Statement *(2)* is trivially true by the construction of $\mathcal{N}$, and *(4)* comes from the fact that two different fibers of $\mathbb{P}T_{\mathbb{C}}M$ are disjoint, so that $\Sigma_{x_1} \cap \Sigma_{x_2} = \ell_{x_1} \cap \ell_{x_2} \subset N$, for Ψ is a diffeomorphism on $\mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$. But we already proved that two distinct geodesics of M intersect transversely at only one point (see Corollary 2.26).

To finish the proof, observe that there is a canonical fiber-wise anti-holomorphic involution $\hat{\sigma} : \mathbb{P}T_{\mathbb{C}}M \to \mathbb{P}T_{\mathbb{C}}M$ — this is simply the function $\hat{\sigma}([v]) = [\overline{v}]$. Such $\hat{\sigma}$ has fixed-point set $\mathbb{P}TM$, and thus induces a map $\sigma : \mathcal{N} \to \mathcal{N}$ with fixed-point set N. At the same time, the complex structure J on $\mathcal{N}$ is such that $\Psi_* \circ J^{||} = J \circ \Psi_*$. Hence σ is, in fact, an anti-holomorphic involution. □

Corollary 4.10. *The complex manifold $\mathcal{N}$ is biholomorphic to $\mathbb{CP}^2$ in such a way that the antiholomorphic involution $\sigma : \mathcal{N} \to \mathcal{N}$ becomes the standard complex conjugation $[z_0 : z_1 : z_2] \mapsto [\overline{z}_0 : \overline{z}_1 : \overline{z}_2]$, N is identified with $\mathbb{RP}^2 = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : z_0, z_1, z_2 \in \mathbb{R}\}$, and the complex curves Σ_p become projective lines $\mathbb{CP}^1 \subset \mathbb{CP}^2$ invariant under complex conjugation.*

Proof. Proposition 4.9 tells us that $\mathcal{N}$ and the family $\{\Sigma_p\}_{p \in M}$ satisfy the hypotheses of Lemma 4.3. □

4.3 The final argument

Now that we constructed a complex surface $\mathcal{N}$ containing N that is biholomorphically equivalent to $\mathbb{CP}^2$, we are able to prove the 'rigidity' of the point-line dual structure on $\mathbb{RP}^2$, at least when it comes from a Zoll projective structure.

Theorem 4.11 (LeBrun and Mason, [LM1] Theorem 3.4)**.** *Let $[\nabla]$ a Zoll projective structure on a surface M^2 diffeomorphic to $\mathbb{RP}^2$. Then there is a diffeomorphism $\Phi : M \xrightarrow{\approx} \mathbb{RP}^2$ such that $[\nabla] = [\Phi^* \nabla^{\mathrm{can}}]$, where ∇^{can} is the Levi-Civita connection of the canonical Riemannian metric* can *on $\mathbb{RP}^2$*

Proof. By Corollary 4.10, there is a biholomorphism $F : \mathcal{N} \to \mathbb{CP}^2$ in such a way that $F \circ \sigma \circ F^{-1}$ is the standard complex conjugation on $\mathbb{CP}^2$, $F(N) = \mathbb{RP}^2 \subset \mathbb{CP}^2$, and $F(\Sigma_p)$ is a projective line invariant under complex conjugation for all $p \in M$. Consider now the dual projective plane $\mathbb{CP}^{2*} = \mathbb{P}(\mathbb{C}^{3*})$ of $\mathbb{CP}^2$, and define the map

$$\begin{aligned} \Phi_0 : M &\to \mathbb{CP}^{2*} \\ p &\mapsto F(\Sigma_p)^{\perp}, \end{aligned}$$

where $\perp$ denotes the usual correspondence between lines in $\mathbb{CP}^2$ and points in $\mathbb{CP}^{2*}$, *i.e.* a line $\Sigma = \{[x : y : z] \in \mathbb{CP}^2 : ax + by + cz = 0\}$ is identified with the element $\Sigma^{\perp} \in \mathbb{CP}^{2*}$ represented by the linear functional $l : \mathbb{C}^3 \to \mathbb{C}$ given by $l(x, y, z) = ax + by + cz$.

The smoothness of Φ_0 is proved in the following way. For any given local section s of $\hat{\mu} : \mathcal{U} \to M$, we know that $F[\Psi(s(p))] \in F[\Psi(\hat{\mu}^{-1}(p))] = F(\Sigma_p)$. Since $\Psi(s(p)) \in \mathcal{N} - N$ and $F(\Sigma_p) = \overline{F(\Sigma_p)}$, we have $F[\Psi(s(p))] \notin F(N) = \mathbb{RP}^2 \subset \mathbb{CP}^2$. Therefore $F(\Sigma_p)$ is the unique projective line passing through both $F[\Psi(s(p))]$ and $\overline{F[\Psi(s(p))]}$. Hence Φ_0 is locally written as

$$\Phi_0(p) = F[\Psi(s(p))] \times \overline{F[\Psi(s(p))]},$$

where $\times : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^{3*}$ is the vector cross-product — in other words, if $F[\Psi(s(p))]$ is represented by a vector $v \in \mathbb{C}^3 - \{0\}$, then $F[\Psi(s(p))] \times \overline{F[\Psi(s(p))]}$ is represented by $v \times \overline{v}$. This proves that Φ_0 is smooth, for it is locally the composition of smooth maps.

The same argument also shows that Φ_0 is an immersion, because F is a biholomorphism, Ψ is a diffeomorphism on $\mathcal{U}$, and s — being a local section — is an immersion.

With all this said, observe that $\Phi_0(p)$ is actually an element of $\mathbb{RP}^{2*} \subset \mathbb{CP}^{2*}$, for it is invariant under complex conjugation. Thus Φ_0 induces a smooth function $\Phi : M \to \mathbb{RP}^{2*}$, which a proper local diffeomorphism since M is compact, Φ_0 is an immersion, and both M and $\mathbb{RP}^{2*}$ have the same dimension. Hence Φ is a covering map, and because $\pi_1(M) \cong \pi_1(\mathbb{RP}^{2*}) \cong \mathbb{Z}_2$, it is in fact a diffeomorphism.

Finally, the function $\Phi : M \xrightarrow{\approx} \mathbb{RP}^{2*}$ can be represented as

$$p \mapsto F(\ell_p)^{\perp},$$

and it sends a geodesic $\mathfrak{C}_y$, represented by a point $y \in N$, to $F(y)^{\perp}$, which is the set of all lines passing through the point $F(y) \in \mathbb{RP}^2$. In other words, we successfully identified N with $\mathbb{RP}^2$ and M with its dual $\mathbb{RP}^{2*}$ in such a manner that the geodesics of M become the geodesics of the canonical metric on $\mathbb{RP}^{2*} \approx \mathbb{RP}^2$. Thus $[\nabla] = [\Phi^*\nabla^{\mathrm{can}}]$, and the proof is finished. $\square$

Remark. In other words, what Theorem 4.11 proves is the uniqueness of point-line dual structures on $\mathbb{RP}^2$, at least when it comes from a Zoll projective structure. Another consequence is that the collection $\mathscr{C}^* := \{\ell_p : p \in M\}$ gives a point-line dual structure on N, that is dual to $(M, \mathscr{C})$, where $\mathscr{C}$ is the collection of geodesics induced by the Zoll projective structure.

Proof of Theorem 4.1 (Green's Theorem, see [LM1] Theorem 3.5*).* Let g be a Zoll metric on $M \approx \mathbb{RP}^2$. After a possible multiplication by a constant, we may assume that $g \in \mathscr{Z}_\pi$, *i.e.* the length of all its geodesics is π.

Extend g to $T_{\mathbb{C}}M$ to be complex bi-linear, and define

$$\mathcal{C} = \{[v] \in \mathbb{P}T_{\mathbb{C}}M : g(v,v) = 0\}.$$

Notice that, when we consider local coordinates (x^1, x^2) around a point $p \in M$ with induced coordinates

$$(x^1, x^2, \zeta^1, \zeta^2) \leftrightarrow \zeta^1 \frac{\partial}{\partial x^1} + \zeta^2 \frac{\partial}{\partial x^2} \quad (\zeta^j = \xi^j + i\eta^j)$$

on $T_{\mathbb{C}}M$, the set of vectors $v \in T_{\mathbb{C}}M$ for which $g(v,v) = 0$ is seen as the zero locus of a quadratic homogeneous polynomial in ζ^1 and ζ^2 that varies smoothly as a function of x^1 and x^2. Hence $\mathcal{C}$ is a smooth curve of $\mathbb{P}T_{\mathbb{C}}M$ that intersects $\mathbb{P}T_{\mathbb{C},p}M$ in two points, counted with multiplicity. At the same time, g is a positive-definite inner product on T_pM that was extended to $T_{\mathbb{C},p}M$ to be complex linear, and thus $\mathcal{C}$ is invariant under complex conjugation $[v] \mapsto [\overline{v}]$. In particular, if $\mathcal{C} \cap T_{\mathbb{C},p}M$ consists of a unique point with multiplicity two, then it must lie in $\mathbb{P}TM$. But this is impossible, since $g(v,v) > 0$ for every nonzero $v \in T_pM$. By the arbitrary choice of $p \in M$ we conclude that $\mathcal{C}$ intersects each fiber of $\mathbb{P}T_{\mathbb{C}}M$ in exactly two distinct points, each intersection counted with multiplicity one.

There is an intuitive way to think of $\mathcal{C}$, which also justifies its definition. When we think of $\mathcal{U} = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$ as the set of point-wise almost complex structures on M, $\mathcal{C}$ can be identified with the set of those structures that are orthogonal transformations with respect to g. Indeed, as discussed in the proof of Lemma 4.4, an element $[v] \in \mathbb{P}T_{\mathbb{C},p}M - \mathbb{P}T_pM$ induces a decomposition

$$T_{\mathbb{C},p}M = \operatorname{span}_{\mathbb{C}}\{v\} \oplus \operatorname{span}_{\mathbb{C}}\{\overline{v}\}$$

for which we associate the complex structure $I_{[v]} : T_{\mathbb{C},p}M \to T_{\mathbb{C},p}M$ acting (in accordance with our convention) as

$$I_{[v]} = \begin{bmatrix} -i & 0 \\ 0 & +i \end{bmatrix}.$$

If $[v] \in \mathcal{C}$, we can compute, for any $a, b, c, d \in \mathbb{C}$,

$$\begin{aligned} g(I_{[v]}(av + b\overline{v}), I_{[v]}(cv + d\overline{v})) &= g(-iav + ib\overline{v}, -icv + id\overline{v}) \\ &= -acg(v,v) + (ad+bc)g(v,\overline{v}) - bdg(\overline{v},\overline{v}) \\ &= (ad+bc)g(v,\overline{v}) \\ &= acg(v,v) + (ad+bc)g(v,\overline{v}) + bdg(\overline{v},\overline{v}) \\ &= g(av + b\overline{v}, cv + d\overline{v}), \end{aligned}$$

thus proving that $I_{[v]}$ is an orthogonal transformation of (T_pM, g_p). Of course when we consider $[\overline{v}]$ instead of $[v]$, we get the same decomposition of $T_{\mathbb{C},p}M$, but the complex structure associated is the only other that also is compatible with g_p: the one corresponding to the opposite choice of orientation on T_pM, *i.e.* $I_{[\overline{v}]} = -I_{[v]}$. This observation gives another proof that $\mathcal{C}$ intersects the fibers of $T_{\mathbb{C}}M$ in exactly two distinct points, since there are exactly two distinct ways to orient any tangent space T_pM.

This interpretation also has an important implication: that $\mathcal{C}$ intersects the fibers of $\mathbb{P}T_{\mathbb{C}}M$ transversely. In fact, we claim that $\mathcal{C}$ is horizontal with respect to the Levi-Civita connection ∇ of g. To see this, take a smooth curve $c : (-\varepsilon, \varepsilon) \to M$, together with two parallel vector fields X_t and Y_t along c. Consider also a nonvanishing complex vector field V_t along c, in such a way that $[V_t] \in \mathcal{C}$ for all $t \in (-\varepsilon, \varepsilon)$, and let $I_t = I_{[V_t]}$ be the corresponding complex structure along c. As in the previous section, we extend ∇ to $T_{\mathbb{C}}M$ to be complex linear, in the sense that $\nabla \circ i = i \circ \nabla$. Since I_t is an orthogonal transformation and X, Y are parallel,

$$g(I_tX_t, I_tY_t) = g(X_t, Y_t) \equiv g(X_0, Y_0),$$

so that, because X and Y were arbitrary, we obtain:

$$0 = \nabla_{\frac{d}{dt}}(I_tX_t) = \left(\nabla_{\frac{d}{dt}} I_t\right) X_t$$

for every parallel vector field X along c, *i.e.*

$$\nabla_{\frac{d}{dt}} I_t \equiv 0.$$

On the other hand, we may assume that X_t is nonvanishing (for example, we can suppose $|X_t| \equiv 1$, since parallel transport preserves the metric), and write $X = aV + \overline{aV}$ for some nonvanishing smooth function $a : (-\varepsilon, \varepsilon) \to \mathbb{R}$. Because

$$0 = \nabla_{\frac{d}{dt}} X = \frac{da}{dt} V + \overline{\frac{da}{dt} V} + a \nabla_{\frac{d}{dt}} V + \overline{a \nabla_{\frac{d}{dt}} V}$$

and

$$0 = \nabla_{\frac{d}{dt}} (IX) = i \overline{\frac{da}{dt} V} - i \frac{da}{dt} V + i \overline{a \nabla_{\frac{d}{dt}} V} - ia \nabla_{\frac{d}{dt}} V,$$

we see that

$$0 = \frac{1}{2i} \Big(i \nabla_{\frac{d}{dt}} X - \nabla_{\frac{d}{dt}} (IX) \Big) = \frac{da}{dt} V + a \nabla_{\frac{d}{dt}} V = \nabla_{\frac{d}{dt}} (aV).$$

Hence the derivative of the curve $t \mapsto [V_t] = [a(t)V_t] \in \mathcal{C}$ is an element of the horizontal bundle $\mathbf{H} \subset T\mathbb{P}T_{\mathbb{C}}M$. But any tangent vector of $\mathcal{C}$ can be written as the derivative of a curve of the form $t \mapsto [a(t)V_t]$, thus proving that $T\mathcal{C} \subset \mathbf{H}$. We actually have $T\mathcal{C} = \mathbf{H}|_{\mathcal{C}}$, for both sides have dimension two fiber-wise.

As a consequence, the projection $\hat{\mu} : \mathbb{P}T_{\mathbb{C}}M \to M$ restricts to a local diffeomorphism $\pi : \mathcal{C} \to M$. At the same time, as argued above, a point in $\mathcal{C}$ is identified with a point-wise orthogonal complex structure, which is equivalent to a point-wise choice of orientation on M. In other words, $\mathcal{C}$ is viewed in this way as the orientation double cover of M, which is diffeomorphic to the sphere, and $\pi : \mathcal{C} \to M$ is actually the orientation covering map.

Furthermore, observe that $\mathcal{U} = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$ has a complex structure induced by the distribution $\mathbf{D}$ (this is a consequence of Lemma 4.5), and that $\mathcal{C} \subset \mathcal{U}$ is an embedded complex curve. Indeed, as our previous computations show,

$$T_{\mathbb{C}}\mathcal{C} = \mathbf{H}_{\mathbb{C}}|_{\mathcal{C}} = (\mathbf{L}_2 \oplus \overline{\mathbf{L}_2})|_{\mathcal{C}}$$

so that the induced complex structure on $\mathcal{U}$ restricts to a complex structure I on $\mathcal{C}$ for which

$$T^{0,1}\mathcal{C} = \mathbf{L}_2|_{\mathcal{C}} = \operatorname{span}_{\mathbb{C}}\{\Xi_0\}$$

and

$$T^{1,0}\mathcal{C} = \overline{\mathbf{L}_2}|_{\mathcal{C}} = \operatorname{span}_{\mathbb{C}}\{\overline{\Xi_0}\}.$$

The fact that $\mathcal{C}$ is actually a Riemann surface has two important consequences. The first is that the covering $\pi : \mathcal{C} \to M$ is a conformal map from $\mathcal{C}$ to the Riemannian manifold (M, g). This is true because

$$\pi_* \big[T^{0,1}_{[v]} \mathcal{C} \big] = \operatorname{span}_{\mathbb{C}}\{v\} \subset T_{\mathbb{C}}M$$

from the definition of $\mathbf{L}_2$, and $g(v,v) = 0$ — hence the pull-back metric $\pi^* g$ is compatible with I, as one can easily see from the following calculation:

$$\begin{aligned}
\pi^* g(I(a\Xi_0 + \overline{a\Xi_0}), I(b\Xi_0 + \overline{b\Xi_0})) &= \pi^* g(-ia\Xi_0 + i\overline{a\Xi_0}, -ib\Xi_0 + i\overline{b\Xi_0}) \\
&= -abg(\pi_*\Xi_0, \pi_*\Xi_0) - \overline{ab}g(\overline{\pi_*\Xi_0}, \overline{\pi_*\Xi_0}) \\
&\quad + (a\bar{b} + \bar{a}b)g(\pi_*\Xi_0, \overline{\pi_*\Xi_0}) \\
&= (a\bar{b} + \bar{a}b)g(\pi_*\Xi_0, \overline{\pi_*\Xi_0}) \\
&= \pi^* g(a\Xi_0 + \overline{a\Xi_0}, b\Xi_0 + \overline{b\Xi_0}).
\end{aligned}$$

The second is that $\mathcal{Q} := \Psi(\mathcal{C})$ is also a genus zero Riemann surface, for Ψ is a biholomorphism from $\mathcal{U}$ to $\mathcal{N} - N$.

We now turn to the study of the complex curve $\mathcal{Q}$. By Lemma 4.3, we know that there is a biholomorphism $F : \mathcal{N} \to \mathbb{CP}^2$ that identifies N with $\mathbb{RP}^2$ and the curves Σ_p with the complex projective curves invariant under complex conjugation. Consequently, $F(\mathcal{Q})$ is an embedded Riemann surface of genus zero in $\mathbb{CP}^2$. Moreover, it must be either a projective line or a conic by Chow's theorem (see [Dem], ch. 2) and the degree-genus formula (see [Don], ch. 7). At the same time, since $\mathcal{C}$ intersects each fiber of $\mathbb{P}T_{\mathbb{C}}M$ transversely exactly at two points away from $\mathbb{P}TM$, $\mathcal{Q}$ also intersects the curves Σ_p, $p \in M$, transversely precisely at two points away from N. Hence $F(\mathcal{Q})$ meets certain projective lines transversely in two points away from $\mathbb{RP}^2$. Bézout's Theorem (see [Ful]) then tells us that $F(\mathcal{Q})$ is a conic, that is the zero locus of a homogeneous quadratic polynomial

$$0 = q(z) = \sum_{j,k=0}^{2} q_{jk} z_j z_k.$$

On the other hand, the fact that $\mathcal{C}$ is invariant under fiber-wise complex conjugation on $\mathbb{P}T_{\mathbb{C}}M$ implies that $\mathcal{Q}$ is invariant under the involution σ, and hence that $F(\mathcal{Q})$ is invariant under complex conjugation on $\mathbb{CP}^2$. Consequently, $F(\mathcal{Q})$ is also the zero locus of

$$\overline{q(\overline{z})} = \sum_{j,k=0}^{2} \overline{q_{jk}} z_j z_k.$$

If we then consider $q(z) + \overline{q(\overline{z})}$ and $q(z) - \overline{q(\overline{z})}$, we see that both

$$\sum_{j,k=0}^{2} \mathrm{Re}(q_{jk}) z_j z_k \quad \text{and} \quad \sum_{j,k=0}^{2} \mathrm{Im}(q_{jk}) z_j z_k$$

vanish at $F(\mathcal{Q})$, and that at least one of then is non-trivial, since $q \neq 0$. In other words, what we discovered is that $F(\mathcal{Q})$ is, in fact, the zero locus of a real homogeneous quadratic polynomial, which is completely described by a symmetric 3×3 real matrix A. Such a matrix is similar, over $GL(3, \mathbb{R})$, to a diagonal matrix whose entries are either 1, 0 or -1.

Because $F(\mathcal{Q})$ does not intersect $\mathbb{RP}^2$, A must be definite. Thus, after a suitable real change of coordinates, we may assume that the biholomorphism $F : \mathcal{N} \to \mathbb{CP}^2$ identifies $\mathcal{Q}$ with the standard conic $\mathcal{Q}_0 \subset \mathbb{CP}^2$, given by

$$z_0^2 + z_1^2 + z_2^2 = 0,$$

without losing any of the other properties of F.

All these procedures where done for an arbitrary $\mathcal{Z}_\pi$-manifold $(M, g) = (\mathbb{RP}^2, g)$, so that we can repeat them to the particular case of $(\mathbb{RP}^2, \mathrm{can})$. The diffeomorphism $\Phi : M \to \mathbb{RP}^2$ constructed in Theorem 4.11 is then described by

$$\Phi(p) = \tilde{p} \iff F(\Sigma_p) = \tilde{F}(\tilde{\Sigma}_{\tilde{p}}), \tag{4.21}$$

where untilded letters are those related to the construction on (M, g), and tilded ones are from $(\mathbb{RP}^2, \mathrm{can})$. Indeed, what was actually constructed in Theorem 4.11 was an identification $M \to \mathbb{RP}^{2*}$ induced by the map $p \mapsto F(\Sigma_p)^\perp$, while the same construction on $(\mathbb{RP}^2, \mathrm{can})$ gives a diffeomorphism $\mathbb{RP}^2 \to \mathbb{RP}^{2*}$ from the application $\tilde{p} \mapsto \tilde{F}(\tilde{\Sigma}_{\tilde{p}})^\perp$. Since we view $\Phi : M \xrightarrow{\approx} \mathbb{RP}^2$ as the composition of $M \to \mathbb{RP}^{2*}$ with the inverse $\mathbb{RP}^{2*} \to \mathbb{RP}^2$, Φ is then characterized by (4.21). In particular, both curves $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are mapped biholomorphically to $\mathcal{Q}_0$, so that

$$F\big(\Psi\big(\pi^{-1}(p)\big)\big) = F(\Sigma_p) \cap \mathcal{Q}_0 \quad \text{and} \quad \tilde{F}\big(\tilde{\Psi}\big(\tilde{\pi}^{-1}(\tilde{p})\big)\big) = \tilde{F}(\tilde{\Sigma}_{\tilde{p}}) \cap \mathcal{Q}_0.$$

Hence the holomorphic map

$$\hat{\Phi} := \Big((\tilde{F} \circ \tilde{\Psi})|_{\tilde{\mathcal{C}}}\Big)^{-1} \circ (F \circ \Psi) : \mathcal{C} \to \tilde{\mathcal{C}}$$

makes the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\hat{\Phi}"] \arrow[d, "\pi"] & \tilde{\mathcal{C}} \arrow[d, "\tilde{\pi}"] \\ M \arrow[r, "\Phi"] & \mathbb{RP}^2 \end{tikzcd} \tag{4.22}$$

commute.

We can now finish the proof. For this, notice that, since both π and $\tilde{\pi}$ are conformal, and because $\hat{\Phi}$ is a biholomorphism of Riemann surfaces, the diffeomorphism Φ is also conformal. In particular, $h := \Phi^*\mathrm{can} = e^{2u}g$ for some function $u \in C^\infty(M)$. Denote by ∇^g and ∇^h the respective Levi-Civita connections of g and h. By the formula relating conformal metrics, we have

$$\nabla^g_X Y - \nabla^h_X Y = du(X)Y + du(Y)X - g(X, Y)\mathrm{grad}_g(u)$$

for every $X, Y \in \mathfrak{X}(M)$. Since ∇^g and ∇^h are projectively equivalent by construction,

$$\nabla^g_X Y - \nabla^h_X Y = \omega(X)Y + \omega(Y)X$$

for some 1-form ω. Hence

$$\omega(X_p)Y_p + \omega(Y_p)X_p = du(X_p)Y_p + du(Y_p)X_p - g(X_p, Y_p)\mathrm{grad}_g(u) \tag{4.23}$$

for every pair of tangent vectors $X_p, Y_p \in T_pM$, for any point $p \in M$. Taking X_p, Y_p to be orthonormal and such that $\omega(Y_p) = 0$, we then see from (4.23) that

$$\omega(X_p)Y_p = du(X_p)Y_p + du(Y_p)X_p.$$

Because X_p and Y_p are linearly independent, we see that $\omega(X_p) = du(X_p)$ and $du(Y_p) = 0 = \omega(Y_p)$. This implies that $du = \omega$, since they are equal on a basis for each tangent space. Thus (4.23) can be further simplified to

$$g(X_p, Y_p)\mathrm{grad}_g(u)_p = 0, \quad \forall X_p, Y_p \in T_pM, \ \forall p \in M.$$

If we take $X_p = Y_p \neq 0$, we see that $\mathrm{grad}_g(u) \equiv 0$. Consequently, u is constant. Since all geodesics of (M, g) and $(\mathbb{RP}^2, \mathrm{can})$ have the same length π, e^{2u} must be identically equal to 1. This proves that $\Phi : (M, g) \to (\mathbb{RP}^2, \mathrm{can})$ is an isometry, as desired. $\square$

5 Conclusions and further directions

Our objective was to give a brief introduction to the theory of Zoll manifolds. The scope was limited to the two-dimensional case, and even so we did not study in depth the Zoll metrics on the sphere. This chapter tries to give a panorama of what comes next.

5.1 Topological and geometric facts about Zoll manifolds

More can be said about the topology of Zoll manifolds than what we proved in section 2.1. Perhaps the best result in this direction is the Bott-Samelson Theorem, which can be stated as follows:

Theorem 5.1 (Bott-Samelson — [Bot], [Sam], see also [Bes], ch. 7)**.** *The integral cohomology ring of a Zoll manifold is the same as that of a CROSS.*

As for the geometric properties of such manifolds, we mention two results. The first is about their volume. There is an interesting theorem by A. Weinstein relating the volume of a n-dimensional Zoll manifold to the volume of the n-dimensional sphere with its canonical metric.

Theorem 5.2 (Weinstein — [Wei], see also [Bes], ch. 2)**.** *If (M, g) is a n-dimensional $\mathcal{Z}_l$-manifold, then the ratio*

$$\frac{\mathrm{Vol}(M,g)}{\mathrm{Vol}(\mathbb{S}^n,\mathrm{can})}\left(\frac{2\pi}{l}\right)^n$$

is an integer $i(M, g)$, called the Weinstein integer *of the Zoll manifold (M, g). In particular, if (M, g) is a $\mathcal{Z}_{2\pi}$-manifold, then the volume of (M, g) is an integral multiple of the volume of $(\mathbb{S}^n, \mathrm{can})$.*

The Weinstein integer depends continuously on the metric g, so it stays constant under continuous deformations. However, we do not know in general if the space $\mathcal{Z}(M)$ is connected — this is an open question even for the 2-sphere (see Section 5.4 below). Therefore, a priori there could be two different Zoll metrics on a given manifold with different Weinstein integers. Fortunately, this is not the case for the spheres.

Theorem 5.3 (Weinstein and Yang — [Wei], [Yan], see also [Bes], ch. 2)**.** *The Weinstein integer of a Zoll sphere $(\mathbb{S}^n, g)$ is 1. In particular, $\mathrm{Vol}(\mathbb{S}^n, g) = \mathrm{Vol}(\mathbb{S}^n, \mathrm{can})$ whenever $g \in \mathcal{Z}(\mathbb{S}^n, 2\pi)$.*

Another interesting property of Zoll manifolds is that their Laplace spectra are asymptotically well behaved. We do not state the result here, but refer the reader to [Bes], Chapter 8, specially to the Duistermaat-Guillemin Theorem in Section 8.B.

5.2 The Blaschke conjecture

A notion closely related to that of a Zoll surface is that of a *Blaschke surface*. Put simply, a Blaschke surface is a Riemannian surface (M^2, g) for which the following condition holds: For every pair of points $p, q \in M$ such that $q \in \mathrm{Cut}(p)$, there are exactly two shortest normalized geodesics (called segments) $\gamma_1, \gamma_2 : [0, l] \to M$ from p to q and $\gamma_1'(0) = -\gamma_2'(0)$. With a bit of work, it is not hard to show that the Green Theorem proved in Chapter 4 is equivalent to the result below (see [LM1] or [Bes], ch. 5):

Theorem 5.4 (Green)**.** *If (M, g) is a Blaschke surface, then it is isometric either to $(\mathbb{S}^2, k\mathrm{can})$ or to $(\mathbb{RP}^2, k\mathrm{can})$ for some constant $k > 0$.*

The definition of a Blaschke surface generalizes, in higher dimensions, to that of a *Blaschke manifold*. These are Riemannian manifolds (M^n, g) with the extra condition that, for any pair of points $p, q \in M$ with $q \in \mathrm{Cut}(p)$, the set

$$\{\gamma'(0) : \gamma \text{ is a segment from } p \text{ to } q\} \subset T_pM$$

is a whole great sphere of the unit tangent space $U_pM = \{u \in T_pM : g(u, u) = 1\}$. Similarly to the Zoll case, all CROSS'es are Blaschke manifolds. In dimension two, Green's theorem tells us that the CROSS'es are the only examples of Blaschke manifolds. In higher dimensions, however, the question whether every Blaschke manifold is isometric to a CROSS is, to the author's knowledge, still widely open.

Blaschke conjecture. *Every Blaschke manifold is isometric to a CROSS.*

5.3 Other types of Zoll structures

Zoll projective structures are not the only possible generalization of Zoll metrics. As discussed at the beginning of Section 2.2, geodesics can be interpreted in two different way: as solutions of a variational problem, or as curves with zero acceleration. The notion of Zoll projective structures emerged from this second viewpoint, but from the first is derived another possible perspective. While in dimension one geodesics are critical points for the length functional, in higher dimensions minimal submanifolds are critical points for the area functional. Hence, if we substitute the word "geodesic" with the expression "minimal submanifold", what we get is an entirely new definition:

Definition 5.5. Let Σ^k be a closed k-dimensional manifold. A Σ-*Zoll manifold* is a Riemannian manifold (M, g) of dimension $n > k$, together with a family $\mathcal{Z}$ of embedded minimal submanifolds Z in (M, g) that satisfies:

1) every $Z \in \mathcal{Z}$ is diffeomorphic to Σ;

2) for every $p \in M$ and every k-dimensional vector subspace $\pi \subset T_pM$, there is a unique $Z = Z_{p,\pi}$ that passes through p with $T_pZ = \pi$; and

3) all $Z \in \mathcal{Z}$ have the same area in (M, g).

The family $\mathcal{Z}$ is called a (Σ-)*Zoll family* and the metric g, a Σ-*Zoll metric*. We write $g \in \mathcal{Z}(M, \Sigma)$ or $g \in \mathcal{Z}_A(M, \Sigma)$ if we want to emphasize that the area of the submanifolds $Z \in \mathcal{Z}$ is A.

When $(M^n, g, \mathcal{Z})$ is a Σ^k-Zoll manifold, each surface $Z \subset M$ lifts canonically to

$$Gr_k(TM) = \{(p, \pi) : p \in M \text{ and } \pi \text{ is a } k\text{-vector subspace of } T_pM\}$$

as $\hat{Z} = \{(p, T_pZ) \in Gr_k(TM) : p \in Z\}$, and the collection $\mathcal{Z}$ induces a foliation $\mathcal{F}$ of $Gr_k(TM)$. We can then define $N = Gr_k(TM)/\mathcal{F}$ the leaf space, called the *space of* Σ-*submanifolds of* M. This space has a canonical bijection to $\mathcal{Z}$, but the point is that we now get a picture analogous to that of the double fibration (2.5) in Chapter 2. We may impose the extra condition:

4) the Σ-Zoll manifold $(M, g, \mathcal{Z})$ is said to be *tame* when its space of Σ-submanifolds N has the structure of a smooth manifold for which the canonical projection $\nu : Gr_k(TM) \to N$ is a fiber bundle with fiber Σ. In this case, we also call the metric g tame.

As for the classical notion of Zoll manifolds, there are many interesting questions in this generalized setting. For example, the discussion of Zoll manifolds presented in this text started by asking if there were other metrics on the 2-sphere all of whose geodesics were simply closed and of the same length. This question can be posed for Σ-Zoll manifolds in many different ways. Here is one of them:

Question. *Are there other $\mathbb{S}^k$-Zoll metrics on the sphere $\mathbb{S}^n$ aside the (multiples of the) canonical metric?*

Very recently, L. Ambrozio, F. Marques and A. Neves [AMN] gave an affirmative answer for the codimension 1 case. In fact, they generalized Guillemin's theorem in the following way:

Theorem 5.6 (Ambrozio-Marques-Neves — see [AMN], Theorem A)**.** *Let $\dot{\rho}$ be a smooth odd function on the sphere $\mathbb{S}^n$, $n \geq 3$. Then there exists a smooth one-parameter family of smooth functions $\rho(t)$ on S^n, $-\delta < t < \delta$, with $\rho(0) = 0$ and $\rho'(0) = \dot{\rho}$ such that*

i) the metric $g(t) = e^{2\rho(t)}\mathrm{can} \in \mathcal{Z}(\mathbb{S}^n, \mathbb{S}^{n-1})$ for every t;

ii) the triple $(\mathbb{S}^n, g(t), \mathcal{Z}_t)$, where $\mathcal{Z}_t$ is the Zoll family of embedded $\mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$ of $g(t)$, is tame for all t; and

iii) the space of $\mathbb{S}^{n-1}$-submanifolds N_t of $e^{2\rho(t)}\mathrm{can}$ is diffeomorphic to $\mathbb{RP}^n$ for each t.

In higher dimensions, the projective duality between $\mathbb{KP}^n$ and its dual $\mathbb{KP}^{n*} = \mathbb{P}((\mathbb{K}^{n+1})^*)$ is not about points and lines, but rather about points and hyperplanes. It then would be natural to ask if there is some kind of rigidity as for the case of dimension two. However, if there is, it is not so strong as to have a uniqueness resut such as that of Green's theorem. Indeed, in dimension greater than two, this uniqueness already breaks down as states the next theorem.

Theorem 5.7 (Ambrozio-Marques-Neves — see [AMN], page 65 for all $n \geq 3$)**.** *For all $n \geq 3$, there exist non-homogeneous Riemannian metrics on $\mathbb{RP}^n$ with minimal projective hyperplanes.*

5.4 Comments on LeBrun and Mason's approach

The ideas from Twistor Theory used by LeBrun and Mason changed the way we view the theory of Zoll surfaces. In their work [LM1], they also considered the case of Zoll projective structures on the sphere, and they found a relation between these structures and holomorphic discs. Unfortunately, nothing of this was discussed in this brief monograph, and the reader is referred to the papers [LM1] and [LM2]. We only mention a conjecture they pose (see [LM1] for the terminology):

LeBrun and Mason's conjecture. *The moduli space of Zoll metrics on $\mathbb{S}^2$ is connected. Moreover, once we mark our Zoll structures by choosing an orthonormal frame at some base-point, the moduli space of marked Zoll structures is in natural 1-1 correspondence with the set of totally real Lagrangian embeddings of $\mathbb{RP}^2 \hookrightarrow (\mathbb{CP}^2 - \mathcal{C}, \omega)$ which are homotopic to the standard embedding.*

As interesting as this Twistor-theoretic approach may look, it has some limitations. For instance, it is not likely to give much information about Zoll manifolds of higher dimensions, even when we consider Zoll manifolds in the generalized sense of Definition 5.5. This is because the whole argument given in the proof of Green's theorem used two properties that only coexist in dimension two:

(i) the duality between points and lines on the projective plane; and

(ii) each embedded submanifold considered (in this case, the geodesics) has codimension n in a $2n$-dimensional manifold ($n = 1$ for Zoll surfaces).

As said before, the projective duality in higher dimensions relates points and hyperplanes. Hence, if we want to find a generalization of LeBrun and Mason's construction, we should perhaps consider manifolds with Zoll families of minimal hypersurfaces. However, we used the fact that, in dimension two, fiber-wise complex structures in a surface M^2 are in one-to-one correspondence with the space $\mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$. When we deal with manifolds of (real) dimension $2n$, this should be replaced by $Gr_{\mathbb{C},n}(T_{\mathbb{C}}M) - Gr_n(TM)$, where $Gr_{\mathbb{C},n}(T_{\mathbb{C},p}M)$ is the set of all complex vector subspaces of $T_{\mathbb{C},p}M$ of complex dimension n, and we view $Gr_n(TM) \subset Gr_{\mathbb{C},n}(T_{\mathbb{C}}M)$ by identifying $\pi \in Gr_n(TM)$ with $\pi \otimes \mathbb{C} \subset Gr_{\mathbb{C},n}(T_{\mathbb{C}}M)$.

In other words, there are two different possibilities if we want to work with some generalization of LeBrun and Mason's ideas: the first is to consider manifolds with Zoll families of hypersurfaces; the second is to study Σ^n-Zoll manifolds of dimension $2n$. Unfortunately, not being able to have both properties at the same time may be a serious limitation that does not seem easy to overcome. Even so, in view of Theorem 5.7 and Theorem 4.11, it sounds reasonable to ask the following:

Question. *Is it possible to find, for every tame $\mathbb{RP}^{n-1}$-Zoll metric g on $\mathbb{RP}^n$, a diffeomorphism $F : \mathbb{RP}^n \to \mathbb{RP}^n$ in such a way that $F(Z)$ is a hyperplane for every minimal submanifold Z in the Zoll family $\mathcal{Z}$ of g?*

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