Sharp Convergence for Noisy Stochastic SIS Model: An Approach by Curie-Weiss Model

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Introduction

The SIS (Susceptible-Infected-Susceptible) model describes the dissemination of a single communicable disease in a susceptible population of size n. We investigate the convergence towards equilibrium of the Noisy SIS Model, which is a perturbation of SIS Model. In the Noisy SIS Model, individuals can change states randomly due to external noise. We find the profile of convergence, which is given as the total variation distance between the marginals of a suitable Gaussian process and its stationary state. Furthermore, this analysis demonstrates that the model exhibits cut-off, under suitable noise intensity conditions. Our approach relies on Stein's method, and Yau's relative entropy method, which may be of independent interest for quantitatively studying other families of Markov chains.

Background

The Dynamics

Let $n \in \mathbb{N}$ be a scaling parameter. The number n represents the number of individuals on a population of fixed size. Let $\Lambda_n := \{1, \dots, n\}$, $Q := \{0, 1\}$, and $\Omega_n := Q^{\Lambda_n} = \{0,1\}^{\Lambda_n}$. The elements $x \in \Lambda_n$ are called *individuals* or *sites*. On the other hand, we denote the elements of Ω_n by $\eta = \{\eta_x | x \in \Lambda_n\}$ and we call them *configurations*. Given a configuration $\eta \in \Omega_n$, we say that the individual $x \in \Lambda_n$ is infected (susceptible) when $\eta_x = 1$ ($\eta_x = 0$). Let $\eta^x \in \Omega_n$ be the configuration obtained from η by changing the status of the individual at site x. More precisely,

$$(\eta^x)_y := \begin{cases} 1 - \eta_x & \text{if } y = x \\ \eta_y & \text{if } y \neq x. \end{cases}$$

One relevant feature of a configuration $\eta \in \Omega_n$ is given by

$$\begin{aligned} M^n : \Omega_n &\longrightarrow [0,1] \\ \eta &\longmapsto M^n(\eta) := \frac{1}{n} \sum_{x \in \Lambda_n} \eta_x. \end{aligned}$$

In other words, $M^n(\eta)$ represents the *global density* of infected individuals, in the configuration η. Let us fixed $\eta_0 \in \Omega_n$. Let $a > 0$ be noise-intensity parameters and $\lambda > 0$. For each $x \in \Lambda_n$, let $c_x : \Omega_n \to \mathbb{R}$ be given by

$$c_x(\eta) := (1 - \eta_x)\left[a + \lambda M(\eta)\right] + \eta_x$$

for every $\eta \in \Omega_n$. We consider the linear operator, which acts on functions $f : \Omega_n \to \mathbb{R}$ as

$$L_n f(\eta) := \sum_{x \in \Lambda_n} c_x(\eta)\left(f(\eta^x) - f(\eta)\right) \qquad \forall \eta \in \Omega_n,$$

The Markov process induced by L_n, $(\eta^n(t) | t \geq 0)$ is also known as the *Noisy Stochastic SIS Model*.

Kantorovich Distance and Total Variation Distance

Given $d \in \mathbb{N}$, we let $\mathscr{P}_1(\mathbb{R}^d)$ stand for the set of probability measures with finite first moment, defined on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, where $\mathscr{B}$ is the Borel σ-algebra. We say that $f : \mathbb{R}^d \to \mathbb{R}$ is *Lipschitz* if

$$\|f\|_{\text{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|} < \infty.$$

Given two measures $\mu, \nu \in \mathscr{P}_1(\mathbb{R}^d)$, the *Kantorovich distance* between μ and ν is defined as

$$d_K(\mu, \nu) := \sup_{\|f\|_{\text{Lip}} \leq 1} \{\int_{\mathbb{R}^d} f d\mu - \int_{\mathbb{R}^d} f d\nu\}.$$

Given two probability measures μ and ν on Ω_n, the *the total variation distance* between μ and ν is defined as

$$d_{\text{TV}}(\mu, \nu) := \frac{1}{2} \sum_{\sigma \in \Omega_n} |\mu(\sigma) - \nu(\sigma)|.$$

Results

Lemma 1 (LLN for global density): For every initial condition $\eta_0 \in \Omega_n$, every $n \in \mathbb{N}$ and every $t \geq 0$, there exists a constant $C = C(a, \lambda)$ such that

$$\mathbb{E}_{\eta_0}\left[(M_t - \rho_t)^2\right] \leq \frac{Ct}{n}, \tag{1}$$

where ρ_t solves the ODE:

$$\begin{cases} \partial_t \rho_t = F(\rho_t), \\ \rho_0 = M(\eta_0), \end{cases}$$

and $L_n M_t := F(M_t)$.

Lemma 2 (CLT for global density fluctuation): Define $\xi_t := \sqrt{n}(M_t - \rho_t)$. There exists a centered Gaussian random variable Z_t, and for any $\epsilon > 0$, there exists $n_0 = n_0(a, b, \epsilon)$ and $t_0 = t_0(a, b, \epsilon)$, such that

$$d_K(\mathscr{L}(\xi_t; \text{P}_{\eta_0}), \mathscr{L}(Z_t)) \leq \epsilon \tag{2}$$

for any $t \geq t_0$.

Curie-Weiss measure we consider the reference Curie-Weiss measure μ^n_{β,h_t} as follows

$$\mu^n_{\beta,h_t}(\sigma) = \frac{1}{\mathcal{Z}^n_{\beta,h_t}} \exp\left(-\mathcal{H}^n_{\beta,h_t}(\sigma)\right),$$

where h_t satisfies the mean-field equation: $\tanh(\beta\rho_t + h_t) = \rho_t$. We consider the high temperature regime ($\beta < 1$), and the Glauber dynamics associated to μ^n_{β,h_t}. We define $\gamma_t := \sqrt{n}(M_t - \rho_t)$.

Lemma 3: There exists a constant $C = C(\beta, a, \lambda)$ and $t_0 = t_0(\beta, a, \lambda)$ and a centered, non-degenerate Gaussian random variable $(Z_t; t \geq 0)$ such that

$$d_K(\mathscr{L}(\gamma_t; \mu^n_{\beta,h_t}), \mathscr{L}(Z_t)) \leq \frac{C}{n^{1/6}}.$$

Lemma 4 (Estimate in total variation): Consider the Markov chain $(\sigma(t); t \geq 0)$ evolving with the dynamics of the Noisy SIS model with initial condition η_0. For any $\epsilon > 0$, and $\frac{1}{2\sqrt{\Delta}} \log n \leq T \leq \frac{1}{c} \log n$, there exists n_0, such that

$$d_{\text{TV}}(\mathscr{L}(\sigma(T+t); \mathbb{P}_{\eta_0}), \mathscr{L}(\bar{\sigma}(T+t); \mu_{\beta,h_{T+t}})) \leq \epsilon, \tag{3}$$

for any $n \geq n_0$ and $t = O(\frac{1}{n^\alpha})$, $\alpha > 0$.

Theorem (Cutoff profile): Consider the profile function $\text{H}(t) := d_{\text{TV}}(N_t, N_\infty)$, then for $\frac{1}{2\sqrt{\Delta}} \log n \leq t \leq \frac{1}{c} \log n$ and $|\rho_* - \frac{1}{2}| < \delta_0$, for any $\epsilon > 0$, there exists n_0, when $n \geq n_0$, such that

$$|d_{\text{TV}}(\mathscr{L}(M_t; \mathbb{P}_{\eta_0}), \mathscr{L}(M_\infty; \mathbb{P}_{\eta_0})) - d_{\text{TV}}(N_t, N_\infty)| \leq \epsilon,$$

where $N_t \sim \text{N}(\rho_t, \frac{v_t^2}{n})$ and $N_* \sim \text{N}(\rho_*, \frac{v_*^2}{n})$, and for some $\delta_0(a, \lambda) > 0$. The profile function $\text{H}(t)$ has the explicit formula:

$$\frac{1}{2}\mathbb{E}_N\left[\left|\exp\left\{W_t \cdot N - \frac{1}{2}W_t^2\right\} - 1\right|\right],$$

where $N \sim \text{N}(0,1)$ and $W_t := \sqrt{n} v_*^{-1} \cdot (\rho_t - \rho_*)$. In particular, the profile function $\text{H}(t)$ has exponential decay, and the mixing time is $t_{\text{mix}} = \frac{1}{2\sqrt{\Delta}} \log n + O(1)$. From this formula, we can conclude that the Markov chain exhibits a cutoff phenomenon.

References

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