The Structure of 3-Pyramidal Groups

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1 Introduction

A Steiner triple system STS(v) is a pair (V, B) where V is a set of v points and B is a set of triples (blocks) of V with the property that any two distinct points are contained in exactly one block. A Kirkman triple system KTS(v) is an STS(v) which admits a resolution of its block-set, i.e. the triples can be partitioned into parallel classes, each of which is a partition of the point-set. It is well known that there exists an STS(v) and KTS(v) iff $v \equiv 1, 3 (\mathrm{mod}\ 6)$ (Kirkman, 1847) and $v \equiv 3 (\mathrm{mod}\ 6)$ (Ray-Chaudhuri, Wilson, 1971) respectively. An automorphism $\mathrm{Aut}(D)$ of an STS (resp. KTS) D is a permutation of its points leaving the block-set (resp. resolution) invariant, which forms a group with composition. An STS (resp. KTS) D is called 3-pyramidal if there exists a subgroup G of $\mathrm{Aut}(D)$ fixing 3 points and acting regularly on the other points. If this happens, we say that the STS (resp. KTS) is 3-pyramidal under G.

2 Some Results

Let D be a Steiner Triple System, M. Buratti, G. Rinaldi and T. Traetta in [1] proved that if D is 3-pyramidal under G then G has precisely 3 involutions. Moreover, they proved the following result:

Lemma 2.1. *A 3-pyramidal Steiner Triple System* $\mathrm{STS}(v)$ *exists if and only if* $v \equiv 7, 9, 15 (\mathrm{mod}\ 24)$ *or* $v \equiv 3, 19 (\mathrm{mod}\ 48)$.

In [2], S. Bonvicini, M. Buratti, M. Garonzi, G. Rinaldi and T. Traetta proved the following result:

Lemma 2.2. *A necessary condition for the existence of a 3-pyramidal* $\mathrm{KTS}(v)$ *is that* $v = 24n + 9$ *or* $v = 24n + 15$ *or* $v = 48n + 3$ *for some* n *which, in the last case, must be of the form* $4^e m$ *with* m *odd. This condition is also sufficient in each of the following cases:*

- $v = 24n + 9$ *and* $4n + 1$ *is a sum of two squares.*
- $v = 24n + 15$ *and either* $2n + 1 \equiv 0 (\mathrm{mod}\ 3)$ *or the square-free part of* $2n + 1$ *does not have any prime* $p \equiv 11 (\mathrm{mod}\ 12)$.
- $v = 48n + 3$.

Moreover, they proved that if D is a 3-pyramidal Kirkman Triple System, then the corresponding group G has exactly 3 involutions, which are all conjugate to each other. So we give the following definition.

Definition 2.3. *We say that a finite group* G *is 3-pyramidal if it has exactly 3 involutions, which are all conjugate to each other.*

3 Main Theorem

Theorem 3.1 (X. Gao, M. Garonzi, see [3])**.** *Let* G *be a finite group and* $O(G)$ *the largest normal subgroup of* G *of odd order. Let* K *be the subgroup generated by the involutions of* G*. Then* G *is 3-pyramidal if and only if one of the following holds.*

- G *is isomorphic to* $S_3 \times H$ *where* H *is a group of odd order.*
- $O(G) \leqslant C_G(K)$ *and* $G/O(G)$ *is isomorphic to* $N \rtimes A$ *where* N *is the Suzuki 2-group of order 64 and* A *is a subgroup of* $\mathrm{Aut}(N)$ *of order 3 or 15.*
- $O(G) \leqslant C_G(K)$ *and* $G/O(G)$ *is isomorphic to* $(C_{2^n} \times C_{2^n}) \rtimes A$ *where* A *is the cyclic group of order 3 generated by the automorphism* $(a, b) \mapsto (b, (ab)^{-1})$.

In the first item $K \cong S_3$ *while in the last two items* $K \cong C_2 \times C_2$.

4 Strategy of the proof

The first main step is to prove that G is solvable.

Sketch of proof: Let $C = C_G(K)$. We assume that G is a nonsolvable 3-pyramidal group of minimal order, and we divide the proof into several steps.

Step 1 : We prove that $\Phi(G)$ is a 2-group containing K and there exists a normal subgroup N of G with $\Phi(G) < N \leqslant C$ such that G/N is a cyclic 3-group and $N/\Phi(G)$ is isomorphic to S^m for some nonabelian simple group S.

$$\{1\} \underset{2^a}{\text{———}} \Phi(G) \underset{S^m}{\text{———}} N \underset{3^b}{\text{———}} C \underset{3}{\text{———}} G$$

Step 2 : Let M be a maximal subgroup of G with $M \neq C$. We prove that $M_G = \Phi(G)$ and consequently $G/\Phi(G)$ is a primitive group.

Step 3 : We prove that $G/\Phi(G)$ is an alsomt simple group with $S^m \leqslant G/\Phi(G) \leqslant S \wr P$ where P is a cyclic 3-subgroup of $Sym(m)$.

Step 4 : We construct a contradiction. Consider $\Delta = \{(s, s, \cdots, s) | s \in S\} \leqslant S^m$, $H := N_{G/\Phi(G)}(\Delta)$. In fact, we can get that the preimage of H in G is a proper 3-pyramidal subgroup of G. Obviously, H is nonsolvable since $S \cong \Delta \leqslant H$, this contradicts the minimality of G.

This concludes the proof of the solvability of 3-pyramidal group.

We have proved that 3-pyramidal groups are solvable, so the following result is very useful.

Lemma 4.1 (Thompson [4])**.** *Suppose that* G *is a solvable group of even order and that the Sylow 2-subgroup of* G *contains more than one involution. Suppose that all the involutions in* G *are conjugate. Then the 2-length of* G *is 1 and the Sylow 2-subgroups of* G *are either homocyclic or Suzuki 2-groups.*

Next, let $W := G/O(G)$. In the light of Lemma 4.1 we have the Sylow 2-subgroup P of W is normal in W and P is a homocyclic 2-group or Suzuki 2-group.

Finally, we use $W/C_W(P) \lesssim Aut(P)$ to finish our proof.

The main theorem implies that, if G is a 3-pyramidal group, then the quotient group $G/O(G)$ is one of the following: SmallGroup(192,1025), SmallGroup(960,5748), $(C_{2^n} \times C_{2^n}) \rtimes C_3$, C_2.

References

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