Algebras with additional structures and small colenght

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Introduction

Let F be a field of characteristic zero, A an associative F-algebra and $F\langle X\rangle$ the free associative algebra generated by a countable set of variables. We say that A is a PI-algebra if there exists a non zero polynomial $f(x_1,...,x_n) \in F\langle X\rangle$ such that $f(a_1,...,a_n) = 0$, for all $a_1,...,a_n \in A$. In this case, we say that f is an identity of A.

Denote by $Id(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$ the T-ideal of A. In characteristic zero, $Id(A)$ is finitely generated, as a T-ideal, by its multilinear identities. Let $P_n = \text{span}\{x_{\sigma(1)}\cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ be the space of multilinear polynomials in the first n variables. We say that A and B are T-equivalent and we write $A \sim_T B$ if $Id(A) = Id(B)$. Consider

$$P_n(A) = \frac{P_n}{P_n \cap Id(A)},\ n \geq 1.$$

and denote by $c_n(A) := \dim_F P_n(A)$ the nth codimension of A. Notice that S_n acts on P_n via $\sigma\cdot(x_{i_1}\cdots x_{i_n}) = x_{\sigma(i_1)}\cdots x_{\sigma(i_n)}$ and so P_n is a S_n-module. Since $Id(A)$ is invariant by this action of S_n, we have that $P_n(A)$ is also a S_n-module. By complete reducibility, we may consider its character $\chi_n(A) = \bigoplus_{\lambda\vdash n} m_\lambda\chi_\lambda$, called nth cocharacter of A, where χ_λ is the irreducible S_n-character associated to $\lambda \vdash n$ and m_λ is its multiplicity. The nth colenght of A is defined by

$$l_n(A) = \sum_{\lambda\vdash n} m_\lambda.$$

Mishchenko, Regev and Zaicev, 1999: $c_n(A) \leq \alpha n^t \Leftrightarrow l_n(A) \leq k$, for some $k \geq 0$ and $\forall n \geq 1$.

For a fixed constant $k \geq 0$ which algebras A generates varieties such that $l_n(A) = k$?

Giambruno and La Mattina, 2005:

1. $l_n(A) = 0$ if and only if $A \sim_{PI} N$;
2. $l_n(A) = 1$ if and only if $A \sim_{PI} C$;
3. $l_n(A) = 2$ if and only if $A \sim_{PI}$: $D_1 \oplus N$ or $D_2 \oplus N$,

where $D_1 = Fe_{11} + Fe_{12}$, $D_2 = Fe_{22} + Fe_{12}$, N denotes a nilpotent algebra and C a commutative non nilpotent algebra.

Additional structures

Definition. For any group $G = \{g_1,\ldots,g_k\}$ we say that an algebra A is a G-graded algebra if there exist subspaces A_g, $g \in G$, which is called homogeneous component of degree g, such that $A = \bigoplus_{g\in G} A_g$ satisfying $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$.

In 2013, Vieira [6] presented the classification of varieties of $\mathbb{Z}_2$-graded algebras with $\mathbb{Z}_2$-colength bounded by 2.

Definition. An algebra A is called a $*$-algebra if A is endowed with an involution $*$, i.e., a linear map satisfying $(a^*)^* = a$ and $(ab)^* = b^*a^*$, for all $a, b \in A$.

In 2018, La Mattina, Nascimento, and Vieira [4] extended the classification to $*$-varieties whose sequence of $*$-colenght is bounded by 3.

We say that an involution $*$ defined in a G-graded algebra is graded if $A_g^* = A_g$, for all $g \in G$.

Definition. A G-graded algebra endowed with a graded involution $*$ is called a $(G,*)$-algebra.

1. $C_{2,*}$: is the algebra $C_2 = F(e_{11}+e_{22}) + Fe_{12}$ with trivial G-grading and involution $(\alpha(e_{11}+e_{22}) + \beta e_{12})^* = \alpha(e_{11}+e_{22}) - \beta e_{12}$;
2. $C_{3,*}$: is the algebra $C_3 = F(e_{11}+e_{22}+e_{22}) + F(e_{12}+e_{23}) + Fe_{13}$ with trivial G-grading and involution $(e_{12}+e_{23})^* = -(e_{12}+e_{23}), e_{13}^* = e_{13}$;
3. $\mathcal{G}_{2,\tau}$: is the algebra $\mathcal{G}_2 = \langle 1, e_1, e_2 \mid e_ie_j = -e_je_i\rangle$ with trivial G-grading and involution $\tau(e_i) = -e_i$, for $i = 1, 2$;
4. C_2^g : is the algebra C_2 with trivial involution and G-grading $(C_2^g)_1 = F(e_{11}+e_{22})$, $(C_2^g)_g = Fe_{12}$, $(C_2^g)_h = \{0\}$, for all $h \in G\backslash\{1, g\}$;
5. $C_{2,*}^g$: is the algebra C_2 with G-grading and involution defined above.

If $|G|$ is even and $g \in G$ with $|g| = 2$, we consider:

6. C_3^g : is the algebra C_3 with trivial involution and G-grading $(C_3^g)_1 = F(e_{11}+e_{22}+e_{33}) + Fe_{13}$, $(C_3^g)_g = F(e_{12}+e_{23})$, $(C_3^g)_h = \{0\}$, for all $h \in G\backslash\{1, g\}$;
7. $C_{3,*}^g$: is the algebra C_3 with G-grading and involution defined above.

Let $n = n_1 + n_2 + \cdots + n_{2k-1} + n_{2k}$, $\langle n\rangle = (n_1,\ldots,n_{2k})$ and $P_{\langle n\rangle}$ be the vector space of multilinear $(G,*)$-polynomials containing n_{2i-1} symmetric variables of homogeneous degree g_{2i-1} and n_{2i} skew variables of homogeneous degree g_{2i}, $1 \leq i \leq n$. Note that $P_{\langle n\rangle}(A) = \frac{P_{\langle n\rangle}}{P_{\langle n\rangle}\cap Id^{(G,*)}(A)}$ is a $S_{n_1} \times \cdots \times S_{n_{2k}}$-module. Consider $\chi_{\langle n\rangle}(A) = \sum_{\langle\lambda\rangle\vdash\langle n\rangle} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes\cdots\otimes \chi_{\lambda(2k)}$, its cocharacter, where $\lambda(i) \vdash n_i$. The $(G,*)$-colenght of A is denoted by

$$l_n^{(G,*)}(A) = \sum_{n=n_1+\cdots+n_{2k}} \sum_{\langle\lambda\rangle\vdash\langle n\rangle} m_{\langle\lambda\rangle}.$$

Results

Consider $\mathcal{D} = \bigcup_{g\in G\backslash\{1\}} \mathcal{D}^g$, where $\mathcal{D}^g = \{C_2^*, C_2^g, C_{2,*}^g\}$.

Theorem. Let G be a finite abelian group and A be a finite dimensional $(G,*)$-algebra.

1. If $l_n^{(G,*)}(A) = 0$, n large enough, then $A \sim_{T_{(G,*)}} N$.
2. If $l_n^{(G,*)}(A) = 1$, n large enough, then $A \sim_{T_{(G,*)}} C \oplus N$;
3. If $l_n^{(G,*)}(A) = 2$, n large enough, then $A \sim_{T_{(G,*)}}$:
$$C_{2,*} \oplus N,\ C_2^g \oplus N \text{ or } C_{2,*}^g \oplus N, \text{ for some } g \in G\backslash\{1\}.$$
4. If $|G|$ is odd and $l_n^{(G,*)}(A) = 3$, n large enough, then A is $T_{(G,*)}$-equivalent to either:
$$C_3^* \oplus N, \mathcal{G}_{2,\tau} \oplus N \text{ or } D_1 \oplus D_2 \oplus N.$$
5. If $|G|$ is even and $l_n^{(G,*)}(A) = 3$, n large enough, then A is $T_{(G,*)}$-equivalent to either:
$$C_3^* \oplus N, \mathcal{G}_{2,\tau} \oplus N, C_3^h \oplus N, C_{3,*}^h \oplus N \text{ or } D_1 \oplus D_2 \oplus N,$$
for some $D_i \in \mathcal{D}$ with $D_1 \neq D_2$ and $h \in G$ with $|h| = 2$.

This result generalizes the results presented in [5].

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