The Injectivity of the Galois Map for Inverse Semigroup Actions on Rings

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Abstract

In this work it will be provided a proof of the injectivity of the Galois map for actions of inverse semigroups on commutative rings using E-unitary inverse semigroup theory. It will also be shown that the Galois map restricted to a class of inverse subsemigroups is bijective and relate it to the Galois theory for partial group actions via the relation between actions of inverse semigroups and partial actions of groups given by Ruy Exel in [4].

Introduction

S. U. Chase, D. K. Harrison and A. Rosenberg developed in 1965 a Galois theory for finite groups acting on commutative ring extensions, in which they exhibited a generalization of the Fundamental Theorem of Galois theory [2]. Many generalizations of this theorem have been developed since. One example is in the case of partial actions of groups on rings [3].

On the groupoids acting on rings case, we can cite [1] and [6]. Both of these works required the action to be orthogonal.

Our goal in this work is to achieve a Galois theory for inverse semigroups acting on rings in where the action need not to be orthogonal.

Goals

We will use the results of Lawson [5] as tools to create a Galois theory for inverse semigroup actions. The main argument follows from a result that states that from every E-unitary inverse semigroup action a partial group action can arise. This relation allows us to translate some results of the Galois theory for partial group actions [3] to the E-unitary inverse semigroup case. From that we will use these new results to obtain the Galois map and its injectivity.

First Definitions

Let S be an inverse semigroup and denote by $E(S)$ its set of idempotents and by $\preceq$ its natural partial order.

We define the relation σ in S by $s\sigma t \Leftrightarrow \exists u \preceq s, t$, for $s, t \in S$. We say that σ is the *minimum group congruence* in S.

We say that S is E*-unitary* if $e \in E(S)$ and $e \preceq s$ imply $s \in E(S)$, for all $s \in S$.

Assume that S is an E-unitary inverse subsemigroup of the inverse monoid $\mathcal{I}(A)$ of the isomorphisms between ideals of the ring A. We can consider the element $\alpha_f = \bigcup_{g\in\sigma(f)} g \in \mathcal{I}(A)$. Consider $G' = \{\alpha_f : f \in S\}$. By [5, Lemma 7.2.1], for all pair $\alpha, \beta \in G'$, there is a unique $\gamma \in G'$ such that $\alpha \circ \beta \leq \gamma$.

Define the binary operation $\odot$ on G' by $\alpha \odot \beta = \gamma$. In this case, by [5, Lemma 7.2.2], we have that $(G', \odot) \simeq S/\sigma$ as groups.

Let S be an inverse semigroup and A a ring. A unital action of S on A is a homomorphism $\beta : S \to \mathcal{I}_u(A)$ that covers A, where $\mathcal{I}_u(A)$ is the inverse monoid of isomorphisms between unital ideals of A. Given $s \in S$, we denote by $\beta_s := \beta(s)$, $A_s := \mathrm{im}(\beta_s)$, $A_{s^{-1}} = \mathrm{dom}(\beta_s)$ and 1_s the unity of A_s.

Results

We will consider S a finite E-unitary inverse semigroup acting on A via unital and injective action β. We define

$$A^\beta = \{a \in A : \beta_s(a1_{s^{-1}}) = a1_s, \text{ for all } s \in S\},$$

the *subring of invariants* of A by β.

Let

$$A_{\sigma(s)} = \sum_{t\in\sigma(s)} A_t$$

and

$$\alpha_{\sigma(s)} = \bigcup_{t\in\sigma(s)} \beta_t.$$

Then $\alpha = (A_g, \alpha_g)_{g\in G}$ is a unital partial action of the group $G = S/\sigma$ on A.

The next results follow from a translation of the partial action α to our context.

We say that A is a β-Galois extension of A^β if there is $\{x_i, y_i\}_{i=1}^n \subseteq A$ such that

$$\sum_{i=1}^n x_i\beta_s(y_i 1_{s^{-1}}) = \sum_{e\in E(S)} 1_e \delta_{e,s}.$$

Let B be a A^β-subalgebra of A. We define the full inverse subsemigroup of S that fixes B by

$$S_B = \{s \in S : \beta_s(b1_{s^{-1}}) = b1_s, \text{ for all } b \in B\}.$$

We say that a subring B of A is β-strong if for all $s, t \in S$ with $ss^{-1} = tt^{-1}$, $s^{-1}t \notin S_B$ and for any $0 \neq e \in E(A_s)$, there is an element $b \in B$ such that $\beta_s(b1_{s^{-1}})e \neq \beta_t(b1_{s^{-1}})e$.

Now denote by $A^e = A \otimes_{A^\beta} A^{op}$ the enveloping algebra of A. Notice that A is a left A^e-module via $(x \otimes y) \cdot a = xay$, for all $a, x, y \in A$. We say that A is a *separable* A^β-algebra if it is a projective left A^e-module [2].

Theorem: Assume that A is β-Galois over A^β. Let T be a full inverse subsemigroup of S. Then $\beta|_T$ is a unital action of T on A and A is $\beta|_T$-Galois over $B := A^{\beta|_T}$, B is A^β-separable and β-strong.

Conversely, let B be a separable, β-strong A^β-subalgebra of A. Then $A^{\beta|_{S_B}} = B$.

Conclusion

The Galois map that associates every separable, β-strong A^β-subalgebra B of A to a full inverse subsemigroup S_B of S is injective. If we restrict this map to the cases in where S_B satisfies the following: $s \in S_B$ implies $\sigma(s) \subseteq S_B$; then this restriction is bijective with inverse $S_B \mapsto A^{\beta|_{S_B}}$.

References

[1] D. Bagio and A. Paques. Partial groupoid actions: globalization, Morita theory, and Galois theory. *Commun. Algebra*, 40(10):3658–3678, 2012.

[2] S. U. Chase, D. K. Harrison, and A. Rosenberg. *Galois theory and cohomology of commutative rings*, volume 52. Am. Math. Soc., 1965.

[3] M. Dokuchaev, M. Ferrero, and A. Paques. Partial actions and galois theory. *J. Pure Appl. Algebra*, 208(1):77–87, 2007.

[4] R. Exel. Partial actions of groups and actions of inverse semigroups. *Proc. Am. Math. Soc.*, 126(12):3481–3494, 1998.

[5] M. V. Lawson. *Inverse semigroups, the theory of partial symmetries*. World Scientific, 1998.

[6] A. Paques and T. Tamusiunas. The Galois correspondence theorem for groupoid actions. *J. Algebra*, 509:105–123, 2018.

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