Optimal plane curves of degree $q-1$ over a finite field

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Abstract

Let $q \geq 5$ be a prime power. In [1], we prove that if a plane curve $\mathcal{X}$ of degree $q-1$ defined over $\mathbb{F}_q$ without $\mathbb{F}_q$-linear components attains the Sziklai upper bound $(d-1)q+1=(q-1)^2$ for the number of its $\mathbb{F}_q$-rational points, then $\mathcal{X}$ is projectively equivalent over $\mathbb{F}_q$ to the curve $\mathcal{C}_{(\alpha,\beta,\gamma)} : \alpha X^{q-1} + \beta Y^{q-1} + \gamma Z^{q-1} = 0$ for some $\alpha, \beta, \gamma \in \mathbb{F}_q^*$ such that $\alpha+\beta+\gamma=0$. This completes the classification of curves that are extremal with respect to the Sziklai bound. Also, since the Sziklai bound is equal to the Stöhr-Voloch's bound for plane curves of degree $q-1$, our main result classifies the $\mathbb{F}_q$-Frobenius extremal classical nonsingular plane curves of degree $q-1$.

Introduction

Let $\mathcal{X}$ be a (projective, geometrically irreducible, algebraic) curve defined over a finite field $\mathbb{F}_q$ where $q=p^h$ for some prime p and some positive integer h. It is a classical problem to count the number $\mathrm{N}_q(\mathcal{X})$ of $\mathbb{F}_q$-rational points of $\mathcal{X}$. However, since this problem is rather hard to solve, it is often desirable to find good upper bounds for $\mathrm{N}_q(\mathcal{X})$ depending on some invariants of the curve $\mathcal{X}$. Once we have a bound, it is a natural question to see whether such a bound is sharp or not, and then, it is also natural to try and classify the optimal curves, that is, the curves attaining said bound.

Let $C_d(\mathbb{F}_q)$ be the set of plane curves of degree $d \geq 2$ defined over $\mathbb{F}_q$ without $\mathbb{F}_q$-linear components. In [6], if $\mathcal{X} \in C_d(\mathbb{F}_q)$, Sziklai conjectured the following result: $\mathrm{N}_q(\mathcal{X}) \leq (d-1)q+1$. The unique exception to Sziklai's conjecture was found in [4, Section 3] and it is given by the curve over $\mathbb{F}_4$ with homogeneous equation

$$X^4+Y^4+Z^4+X^2Y^2+Y^2Z^2+Z^2X^2+G(X,Y,Z)=0,$$

where $G(X,Y,Z) = X^2YZ + XY^2Z + XYZ^2$. The Sziklai bound was later proved by Homma and Kim in a sequence of three papers [2, 4, 3]. We are interested in the curves attaining the Sziklai bound.

Optimal plane curves

In [4, section 5], Homma and Kim observe that the possible degrees d of a nonsingular curve with $(d-1)q+1$ rational points are $q+2, q+1, q, q-1, \sqrt{q}+1$ and 2. Also, for each degree d in the list, there is a nonsingular curve of degree d attaining the bound. For $d \neq q-1$, the complete classification of such optimal curves is known. For $d=q-1$, as it was mentioned by Sziklai in [6], the curve

$$\mathcal{C}_{(\alpha,\beta,\gamma)} : \alpha X^{q-1} + \beta Y^{q-1} + \gamma Z^{q-1} = 0$$

with $\alpha,\beta,\gamma \in \mathbb{F}_q^*$ and $\alpha+\beta+\gamma=0$ has $(q-1)^2$ rational points. This curve is nonsingular and the set of its $\mathbb{F}_q$-rational points is

$$\mathcal{C}_{(\alpha,\beta,\gamma)}(\mathbb{F}_q) = \mathbb{P}^2(\mathbb{F}_q)\backslash\{X=0\}\cup\{Y=0\}\cup\{Z=0\}.$$

Recently, in 2021, Homma has stated the following question:

Question 1: Are there curves of degree $q-1$ that attain the Sziklai's upper bound such that are not projectively equivalent over $\mathbb{F}_q$ to a curve of type $\mathcal{C}_{(\alpha,\beta,\gamma)}$?

In the same paper, he gives a positive solution to this problem for $q=4$, since in this case, the Hermitian cubic attains Sziklai's bound but is not projectively equivalent to any $\mathcal{C}_{(\alpha,\beta,\gamma)}$.

In [1], we give a negative answer to Question 1 for $q \geq 5$, thus completing the classification of optimal Sziklai curves.

Main Result

In [1], we proved the following result:

Theorem: Let $\mathcal{X} \in C_{q-1}(\mathbb{F}_q)$. If $\mathrm{N}_q(\mathcal{X}) = (q-1)^2$ and $q \geq 5$, then there exist $\alpha,\beta,\gamma \in \mathbb{F}_q^*$ with $\alpha+\beta+\gamma=0$ such that $\mathcal{X}$ is projectively equivalent over $\mathbb{F}_q$ to the curve $\mathcal{C}_{(\alpha,\beta,\gamma)}$ over $\mathbb{F}_q$.

Final Remarks

In [5], Stöhr and Voloch gave a geometric method to obtain upper bounds for the number of rational points of a curve of $\mathbb{P}^n$. Let $\mathcal{X} \subseteq \mathbb{P}^2$ be an irreducible nonsingular algebraic curve of degree d defined over $\mathbb{F}_q$. If $\mathcal{X}$ is q-Frobenius classical, by Stöhr and Voloch Theory, we have

$$\mathrm{N}_q(\mathcal{X}) \leq \frac{d(d+q-1)}{2}. \tag{1}$$

Note that if $d=q-1$ then $d(d+q-1)/2=(q-1)^2$. This means that the Stöhr-Voloch upper bound for a nonsingular q-Frobenius classical plane curve of degree $q-1$ is equal to Sziklai's upper bound. Also, by the proof of our main result, it is inferred that the curves attaining the Sziklai bound are q-Frobenius classical and have no $\mathbb{F}_q$-rational point of inflection. In other words, our main result classifies the $\mathbb{F}_q$-Frobenius classical nonsingular curves of degree $q-1$ attaining the Stöhr-Voloch upper bound (1) up to projective equivalence.

References

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