Lebesgue solvability of elliptic homogeneous linear equations with measure data

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Abstract

In this work, we present new results on solvability of the equation $A^*(D)f = \mu$ for $f \in L^p$ and positive measure data μ associated to an elliptic homogeneous linear differential operator $A(D)$ of order m. Our method is based on $(m,p)-$energy control of μ giving a natural characterization for solutions when $1 \leq p < \infty$. We also obtain sufficient conditions in the limiting case $p = \infty$ using new L^1 estimates on measures for elliptic and canceling operators.

Main results

Inspired by the work in [5], we study the Lebesgue solvability for the equation

$$A^*(D)f = \mu, \tag{1}$$

where $A^*(D)$ is the (formal) adjoint operator associated to the homogeneous linear differential operator $A(D)$.

Our first result concerns the Lebesgue solvability for the equation (1) when $1 \leq p < \infty$.

Theorem A. *Let $A(D)$ be a homogeneous linear differential operator of order $1 \leq m < N$ on $\mathbb{R}^N$, $N \geq 2$, from E to F and $\mu \in \mathcal{M}_+(\mathbb{R}^N, E^*)$.*

(i) If $1 \leq p \leq N/(N-m)$ and $f \in L^p(\mathbb{R}^N, F^*)$ is a solution for (1) then $\mu \equiv 0$.

(ii) If $N/(N-m) < p < \infty$ and $f \in L^p(\mathbb{R}^N, F^*)$ is a solution for (1) then μ has finite $(m,p)-$energy. Conversely, if $|\mu|$ has finite $(m,p)-$energy and $A(D)$ is elliptic, then there exists a function $f \in L^p(\mathbb{R}^N, F^*)$ solving (1).

We recall that ellipticity means the symbol $A(\xi) : E \to F$ given by

$$A(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

is injective for $\xi \in \mathbb{R}^N \setminus \{0\}$. We say that μ has finite $(m,p)-$energy if

$$\|I_m\mu\|_{L^p} := \left(\int_{\mathbb{R}^N} |I_m\mu(x)|^p \, dx \right)^{1/p} < \infty.$$

The proof of Theorem A follows from Gauss-Green Theorem, the $L^p - L^p$ boundedness of the Riesz transform operators, the ellipticity of $A(D)$ and a duality argument.

Our second and main result deals with the case $p = \infty$.

Theorem B. *Let $A(D)$ be a homogeneous linear differential operator of order $1 \leq m < N$ on $\mathbb{R}^N$ from E to F and $\mu \in \mathcal{M}(\mathbb{R}^N, E^*)$. If $A(D)$ is elliptic and canceling, and μ satisfies*

$$\|\mu\|_{0,N-m} := \sup_{r>0} \frac{|\mu|(B_r)}{r^{N-m}} < \infty, \tag{2}$$

and the potential control

$$[[\mu]]_{N-m} := \sup_{y\in\mathbb{R}^N} \int_0^{|y|/2} \frac{|\mu|(B(y,r))}{r^{N-m+1}} \, dr < \infty, \tag{3}$$

then there exists $f \in L^\infty(\mathbb{R}^N, F^*)$ solving (1).

The canceling property means

$$\bigcap_{\xi\in\mathbb{R}^N\setminus\{0\}} A(\xi)[E] = \{0\}.$$

The main ingredient in the proof Theorem B is to investigate sufficient conditions on μ in order to obtain

$$\left| \int_{\mathbb{R}^N} u(x) \, d\mu(x) \right| \lesssim \|A(D)u\|_{L^1}, \quad \forall\, u \in C_c^\infty(\mathbb{R}^N, E), \tag{4}$$

which is in fact equivalent to a particular case of the following generalization of Stein-Weiss type inequalities previously studied in [2].

Lemma 1. Assume $N \geq 2$, $0 < \ell < N$ and $K(x,y) \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R}^N, \mathcal{L}(F;V))$ satisfying

$$|K(x,y)| \leq C\, |x-y|^{\ell-N}, \quad x \neq y$$

and

$$|K(x,y) - K(x,0)| \leq C \frac{|y|}{|x|^{N-\ell+1}}, \quad 2|y| \leq |x|.$$

Suppose $1 \leq q < \infty$ and let $\nu \in \mathcal{M}_+(\mathbb{R}^N)$ satisfying

$$\|\nu\|_{0,(N-\ell)q} < \infty, \tag{5}$$

and the potential condition

$$[[\nu]]_{(N-\ell)q} < \infty. \tag{6}$$

If $L(D)$ is cocanceling then

$$\left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} K(x,y) g(y) \, dy \right|^q d\nu(x) \right)^{1/q} \lesssim \int_{\mathbb{R}^N} |g(x)| \, dx, \tag{7}$$

for all $g \in L^1(\mathbb{R}^N; F)$ satisfying $L(D)g = 0$ in the sense of distributions.

The operator $L(D)$ is said to be cocanceling if

$$\bigcap_{\xi\in\mathbb{R}^N\setminus\{0\}} ker\, L(\xi) = \{0\}.$$

Application

We show the validity of the limiting case of a trace inequality presented in [3].

Theorem 1. *Let $A(D)$ be a homogeneous linear differential operator of order m on $\mathbb{R}^N$, $N \geq 2$, from E to F. Then for all $\nu \in \mathcal{M}_+(\mathbb{R}^N)$ satisfying (5) and (6) there exists $C > 0$ such that*

$$\int_{\mathbb{R}^N} |D^{m-1}u(x)| \, d\nu \leq C \|A(D)u\|_{L^1},$$

for every $u \in C_c^\infty(\mathbb{R}^N, E)$.

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