Homological invariants of ternary graphs

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Abstract

In 2022, Jinha Kim proved a conjecture by Engstrom that states the independence complex of graphs with no induced cycle of length divisible by 3 is either contractible or homotopy equivalent to a sphere, which means the minimal free resolution of the edge ideal of these graphs is characteristic-free. We apply this result to give a combinatorial description of projective dimension and depth of the edge ideal of these graphs. As a consequence, we give a complete description of the multigraded betti numbers of edge ideals of ternary graphs in terms of its combinatorial structure and classify ternary graphs whose independence complex is contractible.

1 Introduction

One of the central problems of combinatorial commutative algebra is the study of free resolutions of (squarefree) monomial ideals. Using algebraic techniques such as polarization and initial ideals, and combinatorial techniques such as barycentric subdivision, one can reduce the problem of computing many algebraic invariants of homogeneous ideals in a polynomial ring over a field, such as Hilbert functions, to computing the minimal free resolution of ideals generated by monomials of the form x_ix_j. These ideals are called **edge ideals**, we can think of their generators as edges of a graph G:

$$I(G) \subset k[x_1, \dots, x_n],\ I(G) = (x_ix_j | x_ix_j \in E(G))$$

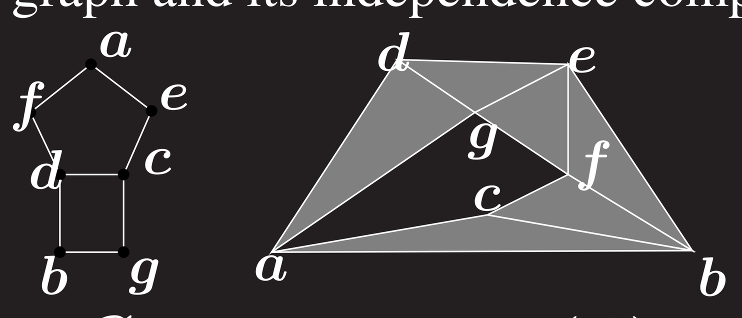
Preliminaries

Let G be a graph. A subset S of the vertices of G is said to be **independent** if there are no edges between elements of S. Since subsets of of independent sets are also independent, the independent sets of G have a simplicial complex structure.

Definition 1. Let G be a graph. The **independence complex of** G, denoted by $\mathrm{Ind}(G)$ is the simplicial complex on vertex set $V(G)$ and the faces are the independent sets of G. Let S be a subset of the vertices of G, the induced subgraph $G[S]$ of G is the graph on vertex set S and edges $E[S] = \{uv | uv \in E(G)$ and $u, v \in S\}$.

The set $N[S] := \bigcup_{v \in S} (N(v) \cup v)$ is called the closed neighborhood of S. We write $G - S$ for the induced subgraph $G[V(G) \setminus S]$.

Example 2. A graph and its independence complex:



G $\mathrm{Ind}(G)$

Given a graph G on vertex set $\{x_1, \dots, x_n\}$ and a field k, one can define the **edge ideal** of G as follows:

$$I(G) \subset k[x_1, \dots, x_n],\ I(G) = (x_ix_j | x_ix_j \in E(G))$$

The connection between independence complexes and free resolutions of edge ideals follows from the theorem below:

Theorem 3. Let G be a graph and $x_\tau = \prod_{x_i \in \tau} x_i$ where $\tau \subset V(G)$. Then the multigraded betti numbers $b_{i,m}(R/I(G))$ of the quotient of the edge ideal of G are given by

$$b_{i,x_\tau}(R/I(G)) = \dim \tilde{H}_{|\tau|-i-1}(\mathrm{Ind}(G[\tau]); k)$$

where $R = k[x_1, \dots, x_n]$ and $\tilde{H}$ denotes reduced simplicial homology.

Definition 4. A graph G is called **ternary** if it does not contain any cycle of length divisible by 3.

In 2022, J. Kim proved the following theorem:

Theorem 5 (J. Kim (2022))**.** A graph G is ternary if and only if the independence complex of every induced subgraph of G is either contractible or homotopy equivalent to a sphere.

Although the theorem completely describes the independence complexes of ternary graphs, it does not say when the independence complex of a ternary graph is contractible, or the dimension of the sphere when it is not.

Using Theorem 5, we proved the following:

Theorem 6 (S. Faridi, T. Holleben (2022))**.** Let G be a ternary graph.

1. If $\mathrm{Ind}(G)$ is not contractible, then it is homotopy equivalent to $S^{\mathrm{depth}(\mathcal{F})-1}$, where $\mathcal{F}$ is any maximal filtration of G. Moreover, for every maximal filtration $\mathcal{F}$ the following are invariants of G:
   (a) $\mathrm{pd}(G) := \mathrm{del}(\mathcal{F}) + \sum_{v_i \in N(\mathcal{F})} \deg v_i$, where $\deg v_i$ is the degree of v_i as a vertex of G_i.
   (b) $\mathrm{depth}(G) := \mathrm{depth}(\mathcal{F})$
2. If $\mathrm{Ind}(G)$ is contractible, we set the following notation:
   (a) $\mathrm{pd}(G) := \max\{\mathrm{pd}(H) | \mathrm{Ind}(H)$ is not contractible$\}$, where H runs over induced subgraphs of G
   (b) $\mathrm{depth}(G) := |V(G)| - \mathrm{pd}(G)$

Applications to Commutative Algebra

Applying Theorem 3 to the theorems above we get the following result:

Theorem 7. Let $R = k[x_1, \dots, x_n]$ and $I(G)$ the edge ideal of a ternary graph. For every squarefree monomial m let $G[m] = G[\{x_i \mid x_i | m\}]$, then

$$b_{i,m}(R/I(G)) = \begin{cases} 1 & \text{if } \mathrm{sign}(G[m]) = -1 \text{ and } i = \mathrm{pd}(G[m]) \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$$\mathrm{pd}(G) = \mathrm{pd}(R/I(G)) \text{ and } \mathrm{depth}(G) = \mathrm{depth}(R/I(G))$$

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