On the symmetric breaking and existence of radial solutions for the supercritical Hénon equation with Grushin operator

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Abstract

In this work, we use variational methods to find a nontrivial and nonnegative weak solution for an Hénon type equation involving the Grushin operator. This solution is obtained in the space of radial functions of $S^2_{\alpha,0}(B)$, where B is the unit ball. The growth of the nonlinearity includes a certain range of supercritical values. In addition, we prove that, under certain conditions, the ground state solution of this problem in the whole space $S^2_{\alpha,0}(B)$ is not radial for superquadratic and subcritical growth.

Introduction

The following nonlinear equation

$$-\Delta u = |x|^\alpha u^{p-1}$$

with $\alpha > 0$ was introduced by Michel Hénon as a model to study spherically symmetric clusters of stars. Some researchers have been studying existence, non existence of positive solutions, multiplicity for the Dirichlet problem

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1}, B \\ u > 0, B \\ u = 0, \partial B \end{cases} \tag{1}$$

where B is the unit ball in $\mathbb{R}^N$, $\alpha > 0$ and $p > 2$. Numerical methods suggested that the ground state solution is not radial. In fact, the authors in [2] presented conditions under which this phenomenon is really true. This is called symmetric breaking. For a system with Hénon equation and symmetric breaking we refer to [1].

Goals

1. Study the Hénon type equation

$$\begin{cases} -G_\alpha u(z) = |z|^\ell |u|^{p-2}u, B \\ u = 0, \partial B \end{cases} \tag{2}$$

where $G_\alpha u(z) = \Delta_x u(x,y) + |x|^{2\alpha}\Delta_y u(x,y)$ is the Grushin's operator.

2. Investigate symmetric breaking.

Results

1 Workplace

Consider $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, with $N > 2$ and the space

$$S^2_\alpha(B) := \{u \in L^2(B) : \nabla_x u, |x|^\alpha \nabla_y u \in L^2(B)\}$$

endowed with the norm

$$||u||_{S^2_\alpha(\Omega)} := \left(\int_\Omega (|u|^2 + |\nabla_\alpha u|^2)dx\right)^{1/2},$$

where $\nabla_\alpha u(z) = (\nabla_x u(z), |x|^\alpha \nabla_y u(z))$. We also define $S^2_{\alpha,0}(B) := \overline{C_0^\infty(B)}^{||\cdot||_{S^2_\alpha(B)}}$. In addition, let E be the space of radial functions $\mathcal{R}(B)$ in $S^2_{\alpha,0}(B)$.

2 Compactness result

Proposition 1. *Let $\alpha \geq 0$ be such that $\widetilde{N} = N + N_2\alpha > 2$. If $N_2 \geq 2$, then there exists a positive constant $C > 0$ such that*

$$|u(z)| \leq C\frac{||\nabla_\alpha u||_{L^2(B)}}{|z|^{\frac{\widetilde{N}-2}{2}}}, \forall z \in B, \tag{3}$$

for any $u \in \mathcal{R}(B) \cap C(\overline{B})$ such that $u \equiv 0$ on ∂B.

Proposition 2. *If $m \geq 0$, then $\Psi_m : E \to L^p(B)$ given by $\Psi_m(u) = |\cdot|^m u$ is compact for any $p \in [1, \widetilde{m})$, where*

$$\widetilde{m} = \begin{cases} \frac{2N}{\widetilde{N}-2-2m}, & se\ m < \frac{\widetilde{N}-2}{2} \\ +\infty, & se\ m \geq \frac{\widetilde{N}-2}{2} \end{cases}.$$

Proof. The ideia is to use the last Proposition to get

$$||\Psi_m(u)||_{L^p(B)} \leq C_2^{\frac{p-\beta}{p}} ||u||^{\frac{\beta}{p}}_{L^1(B)} ||\nabla_\alpha u||^{\frac{p-\beta}{p}}_{L^2(B)}$$

for any $u \in E$ and for some $\beta > 0$. The result follows from the compact immersion $S^2_{\gamma,0}(B) \overset{c}{\hookrightarrow} L^p(\Omega)$, for all $p \in [1, 2^*_\alpha)$, where $2^*_\alpha = \frac{2\widetilde{N}}{\widetilde{N}-2}$. □

3 Existence of weak solution

Theorem 1. *Problem (2) has a nontrivial and nonnegative weak solution in E for all $\alpha > 0$, $\ell > N_2\alpha$ and $2 < p < 2^*_\alpha + \frac{2(\ell - \alpha N_2)}{\widetilde{N}-2}$. In addition, for $2 < p < 2^*_\alpha$, problem (2) has a nontrivial nonnegative weak solution in $S^2_{\alpha,0}(B)$.*

4 Symmetry Breaking

Let

$$R(u) = \frac{\int_B |\nabla_\alpha u|^2 dz}{\left(\int_B |z|^\ell |u|^p dz\right)^{\frac{2}{p}}},$$

$$S_{\ell,p} = \inf_{\substack{u \in S^2_{\alpha,0}(B) \\ u \neq 0}} R(u) \text{ and } S^R_{\ell,p} = \inf_{\substack{u \in E \\ u \neq 0}} R(u).$$

Proposition 3. *For any $p \in (2, 2^*_\alpha)$, there exist:*

(a) *$C_p > 0$ such that $S^R_{\ell,p} \geq C_p \ell^{(1+\frac{2}{p})}$, for any $\ell > 0$.*

(b) *$D_p > 0$ and $\ell_0 > 0$ (not depending on p) such that $S_{\ell,p} \leq D_p \ell^{2-\widetilde{N}+2\frac{\widetilde{N}}{p}}$, for any $\ell \geq \ell_0$.*

(c) *$\ell^* > 0$ (depending on p) such that $S_{\ell,p} < S^R_{\ell,p}$, for any $\ell \geq \ell^*$.*

Theorem 2. *Given $p \in (2, 2^*_\alpha)$, there exists $\ell^* > 0$ such that the ground state solution of (2) is not radial provided $\ell \geq \ell^*$.*

Proof. Given $p \in (2, 2^*_\alpha)$, there exists $\ell^* > 0$ such that $S_{\ell,p} < S^R_{\ell,p}$, for any $\ell \geq \ell^*$. If u was a radial ground state solution of (2) with $\ell \geq \ell^*$, then we should have $S^R_{\ell,p} \leq R(u) = S_{\ell,p}$, which is absurd. □

Conclusion

- As G_α is invariant under rotation, the principle of symmetric criticality fails. So the weak solution of (2) obtained in E may not be a solution in $S^2_{\alpha,0}(B)$.
- As expected, the ground state solution of problem (2) cannot be radial, for large ℓ.

References

[1] Haiyang He. Symmetry breaking for ground-state solutions of hénon systems in a ball. *Glasgow Mathematical Journal*, 53(2):245–255, 2011.

[2] D. Smets, M. Willem, and J. Su. Non-radial ground states for the hénon equation. *Communications in Contemporary Mathematics*, 4(03):467–480, 2002.

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