Total mean curvature surfaces in the product space $\mathbb{S}^n \times \mathbb{R}$ and applications.

Sylvia Ferreira

UFRPE

sylvia.ferreira@ufrpe.br

Instituto de Matemática Pura e Aplicada

Introduction

An interesting line of research is to study which submanifolds are critical points of certain functional. In this scenario, we can highlight the work from [1, 2], being the last one concerning about the *the total mean curvature functional*, for a submanifold σ^m in the Euclidean space, $\mathcal{H}$ given by

$$\mathcal{H}(\Sigma) = \int_\Sigma H^m d\Sigma, \tag{1}$$

and H is the mean curvature function of the submanifold. Our main result is inspired by the functional (1) and our aim is to study the $\mathcal{H} - surfaces$ in the product space $\mathbb{S}^n \times \mathbb{R}$, i.e, surfaces which are critical points of the $\mathcal{H}$ functional in order to obtain an integral inequality relating the total umbilicity tensor $|\phi|$ and the Euler- Lagrange characteristic of the surface. As a consequence we characterizes those in what the equality holds. The results presented here are a part of [6].

1 Set up

Let Σ^m be a submanifold isometrically immersed in the product space $\mathbb{S}^n \times \mathbb{R}$. We denote by ∂_t the parallel and unitary vector field associated to this product and the second fundamental form of the imersion by σ, with A_ξ being the Weingarten Operator in the normal direction ξ. Since $\partial_t \in \mathfrak{X}(\mathbb{S}^n \times \mathbb{R})$, it can be decomposed along Σ^m as $\partial_t = T + N$, where $T := \partial_t^\top$ and $N := \partial_t^\perp$ denote, respectively, the tangent and normal part of the vector field ∂_t on the tangent and normal bundle of the submanifold Σ^m in $\mathbb{S}^n \times \mathbb{R}$. Let us denote by h the mean curvature vector field of Σ^m in $\mathbb{S}^n \times \mathbb{R}$, and by H its norm, i.e, $\langle h, h\rangle = H^2$.

Proposition 1. *Let $x : \Sigma^m \to \mathbb{S}^n \times \mathbb{R}$ be an isometrically immersed closed submanifold. Then x is a stationary point of $\mathcal{H}$ if and only if*

$$H^{m-2}\left\{\Delta^\perp h + \left((m - |T|^2) - mH^2\right) h - m\langle N, h\rangle N\right\} + \left(\sum_{\alpha,\beta} H^\alpha \mathrm{tr}(A_\alpha A_\beta) e_\beta\right) = 0, \text{for } m > 2 \text{ and} \tag{2}$$

$$\Delta^\perp h + (2 - |T|^2 - 2H^2)h - 2\langle N, h\rangle N + \sum_{\alpha,\beta} H^\alpha tr(A_\alpha A_\beta) e_\beta = 0 \tag{3}$$

in the case where $m = 2$, where $m + 1 \leq \alpha, \beta \leq n + 1$.

Main Result

Before proving our main result, we need the following proposition.

Proposition 2. *Let Σ^2 be an $\mathcal{H}$-surface in the product space $\mathbb{S}^n \times \mathbb{R}$. Then, we have*

$$\int_\Sigma \left(|\nabla^\perp \sigma|^2 + 2\sum_\alpha \mathrm{tr}(A_\alpha \circ \mathrm{Hess}\, H^\alpha)\right) d\Sigma \geq \int_\Sigma \left(2\langle N, h\rangle^2 - (2 - |T|^2 + |\phi|^2)H^2\right) d\Sigma. \tag{4}$$

Theorem 1. *Let Σ^2 be a compact $\mathcal{H}$-surface in the product space $\mathbb{S}^n \times \mathbb{R}$. Then*

$$\int_\Sigma |\phi|^2 \left(1 - 5|T|^2 - \frac{3}{2}|\phi|^2\right) d\Sigma - \int_\Sigma \left\{2(|\phi_h| + 1)|T|^2 + 2\right\} d\Sigma \leq 4\pi\chi(\Sigma). \tag{5}$$

In particular, the equality holds if and only if Σ^2 is isometric to either

(i) a slice $\mathbb{S}^2 \times \{t_0\}$, or

(ii) a totally geodesic 2-sphere or a Clifford torus in $\mathbb{S}^3 \times \{t_0\}$,

(iii) or a Veronese surface in $\mathbb{S}^4 \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. With a straighforward computation the [3, Proposition 1] can be written as follow

$$\frac{1}{2}\Delta|\sigma|^2 \geq |\nabla^\perp \sigma|^2 + 2\sum_\alpha \mathrm{tr}(A_\alpha \circ \mathrm{Hess}\, H^\alpha) + 2|\phi_N|^2 - 2|\phi_h||T|^2 + \left(2 - 5|T|^2 + 2H^2 - \frac{3}{2}|\phi|^2\right)|\phi|^2.$$

Taking the integrals and using the divergence theorem, it follows from Proposition 2 that,

$$0 \geq \int_\Sigma \left\{2(|\phi_N|^2 + \langle N, h\rangle^2) + (|T|^2 + |\phi|^2)H^2\right\} d\Sigma + \int_\Sigma \left\{\left(2 - 5|T|^2 - \frac{3}{2}|\phi|^2\right) - 2H^2 - 2|\phi_h||T|^2\right\} d\Sigma.$$

Hence

$$\int_\Sigma \left\{\left(2 - 5|T|^2 - \frac{3}{2}|\phi|^2\right)|\phi|^2 - 2H^2 - 2|\phi_h||T|^2\right\} d\Sigma \leq 0.$$

Then, the Gauss-Bonnet theorem implies

$$\int_\Sigma \left\{\left(1 - 5|T|^2 - \frac{3}{2}|\phi|^2\right)|\phi|^2\right\} d\Sigma - \int_\Sigma \left\{2(|\phi_h| + 1)|T|^2 + 2\right\} d\Sigma \leq 4\pi\chi(\Sigma). \tag{6}$$

Finaly, if the equality holds in (6), all inequalities obtained along of the proof becomes equalities. In particular it follows that $|\phi_N| = \langle N, h\rangle = 0$ and either $|T| = |\phi| = 0$ or $H = 0$. In the first case , Σ^2 is a $\mathcal{H}-$surface satisfying the assumptions of [6, Corollary 3.3] so it is totally geodesic. Therefore, either it is isometric to a slice $\mathbb{S}^2 \times \{t_0\}$ in the case $n = 2$, or to a totally geodesic sphere $\mathbb{S}^2$ in a certain $\mathbb{S}^3 \times \{t_0\}$. For the second case, since $H = 0$, we must have that Σ^2 is a parallel surface of $\mathbb{S}^2 \times \mathbb{R}$. On the one hand, since $|\phi_N| = \langle N, h\rangle = 0$ it implies that $A_N = 0$. Consequently it is not hard to see from the Codazzi equation that $T = 0$, so Σ^2 is a minimal surface in a slice of $\mathbb{S}^n \times \mathbb{R}$. For the case where Σ^2 can be isometrically immersed in a certain $\mathbb{S}^3 \times \{t_0\}$, by [4] we have that Σ^2 is isometric to a Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ in $\mathbb{S}^3 \times \{t_0\}$ for some $t_0 \in \mathbb{R}$. In other case, observe that for $|\phi|^2 = |\sigma|^2$, the equality in (1) becomes

$$\int_\Sigma |\sigma|^2 \left(\frac{3}{2}|\sigma|^2 - 2\right) d\Sigma = 0. \tag{7}$$

Therefore, from [5, Theorem 1], Σ^2 is isometric to a Veronese surface in $\mathbb{S}^4 \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. □

Referências

[1] B-Y. Chen. On the total curvature of immersed manifolds i. an inequality of fenchel-borsuk-willmore. *Amer. J. Math.*, pages 148–162, 1971.

[2] B-Y. Chen. Some conformal invariants of submanifolds and their applications. *Boll. Un. Mat. Ital.*, pages 380–385, 1974.

[3] F.R. dos Santos and S.F. da Silva. On complete submanifolds with parallel normalized mean curvature in product spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, 2022.

[4] H.B. Lawson Jr. Local rigidity theorems for minimal hypersurfaces. *Ann. of Math.*, -:187–197, 1969.

[5] A.M. Li and J.M. Li. An intrinsic rigidity theorem for minimal submanifolds in a sphere. *Arch. Math.*, pages 582–594, 1992.

[6] A. L. Albujer S. F. Da Silva and F.R. Dos Santos. Total mean curvature surfaces in the product space and applications. *Proceedings of the Edingburgh Mathematical Society*, pages 1–20, 2023.