On Centralizers of Elements in Infinite Groups

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Abstract

Let G be a group and denote by $\mathrm{Cent}(G)$ the set of all centralizers of elements of G. We say that G is n-centralizer when $|\mathrm{Cent}(G)| = n$. A natural question is if we fixed $|\mathrm{Cent}(G)|$, is it possible to obtain a caracterization of G. This question has already been answered for finite groups and certain values of n. The techniques used for such groups use strongly the finitness of the group and, therefore, it is interesting to try to obtain the same results for infinite groups. In this work, we will introduce the concept of isoclinism, which is an equivalence relation between groups that is a little weaker than isomorphism. Based on Zarrin's result [8], the main application of this tool will be to extend to infinite groups the valid results for finite n-centralizers groups.

Introduction

Our aim in this work is to generalize for infinite groups the following result

Theorem 1. *Let G be a finite n-centralizer group. Then*

1. $\dfrac{G}{Z(G)} \cong C_2 \times C_2 \iff n = 4.$
2. $\dfrac{G}{Z(G)} \cong C_3 \times C_3$ *or* $S_3 \iff n = 5.$
3. $\dfrac{G}{Z(G)} \cong D_8, A_4, C_2^3$ *or* C_2^4, *if* $n = 6.$
4. $\dfrac{G}{Z(G)} \cong C_5 \times C_5, D_{10},$ *or* $\langle x, y : x^5 = y^4 = 1, y^{-1}xy = x^3\rangle \iff n = 7.$
5. $\dfrac{G}{Z(G)} \cong D_{12}, A_4$ *or* C_2^3, *if* $n = 8.$

Proof. The proof of this result can be founded in [5] (for $n = 4, 5$), [4] (for $n = 6$) and in [1] (for $n = 7, 8$). □

Isoclinic Groups

In 1940, Hall [6] seeking classification results for p-groups, introduced a weaker concept than isomorphism. This concept was called isoclinism. In general, two groups G and H are isoclinic when there is an isomorphism between the quotient groups $\dfrac{G}{Z(G)}$ and $\dfrac{H}{Z(H)}$ that induces an isomorphism in its derived subgroups. More precisely

Definition 1. *The groups G and H are said to be isoclinic if there are two isomorphisms $\beta : \dfrac{G}{Z(G)} \to \dfrac{H}{Z(H)}$ and $\gamma : G' \to H'$ such that, if $\beta(g_1Z(G)) = h_1Z(H)$ and $\beta(g_2Z(G)) = h_2Z(G)$, then $\gamma([g_1, g_2]) = [h_1, h_2]$.*

The great gain of working with isoclinism are the following three results

Lemma 1. *For every group G there exists a group K isoclinic to G such that $Z(K) \leq K'$.*

Lemma 2. *For any two isoclinic groups G and H, it holds that $|\mathrm{Cent}(G)| = |\mathrm{Cent}(H)|$.*

Proposition 1. *Let n be a positive integer and G be a n-centralizer group. There exists a finite group K such that K is isoclinic to G and $|\mathrm{Cent}(K)| = |\mathrm{Cent}(G)|$.*

Before proving Proposition 1, it is necessary to mention the validity of the following equivalences

Theorem 2. *For any group G, the following statements are equivalent.*

1. *G has finitely many centralizers.*
2. *G is a centre-by-finite group.*
3. *G has finitely many pairwise noncommuting elements*

Proof. (of Proposition 1)

As $|\mathrm{Cent}(G)| = n$, we have that $\dfrac{G}{Z(G)}$ is finite. From Lemma 1, there exists a group K such that G is isoclinic to K and $Z(K) \subset K'$. Thus, $\dfrac{G}{Z(G)} \cong \dfrac{K}{Z(K)}$ and therefore K is centre-by-finite. From Schur's Theorem, K' is finite and in particular $Z(K)$ is finite, from which it follows that K is finite. Finally, from Lemma 2, $|\mathrm{Cent}(K)| = |\mathrm{Cent}(G)|$. □

Main Result

Now, we are ready to do our main result.

Theorem 3 (Zarrin [8])**.** *The Theorem 1 can be generalized for infinite groups.*

Proof. Initially, assume $\dfrac{G}{Z(G)} \cong A$, where A is any group among those listed in items 1, 2 or 4 of Theorem 1. Since A is finite, G is centre-by-finite. Thus, by Theorem 2, there exists $n \in \mathbb{N}$ such that $|\mathrm{Cent}(G)| = n$. Hence, the Lemma 1 guarantees the existence of a finite group K isoclinic to G and is worth $|\mathrm{Cent}(K)| = n$. From isoclinism, $\dfrac{K}{Z(K)} \cong \dfrac{G}{Z(G)} \cong A$ and, therefore, from Theorem 1, we have $n = 4, 5$ or 7.

Now let G be a n-centralizer group with $n \in \{4, 5, 6, 7, 8\}$. By Proposition 1, there exists a finite group K isoclinic to G and also n-centralized. In particular, $\dfrac{G}{Z(G)} \cong \dfrac{K}{Z(K)}$ and, therefore, the result follows from items 1 to 5 of the Theorem 1. □

Conclusion

In this work we classify the n-centralizers groups for $n \in \{4, 5, 6, 7, 8\}$. However, it is interesting to point out that the 9-centralizers [3] and 10-centralizers [2] groups have already been completely classified, and, partially, the 11-centralizers groups [7].

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Acknowledgements

We thank CAPES and UnB for funding the research that resulted in this work.