Structural stability for C^2- diffeomorphisms

Sávio Silva Santana & Paulo Varandas

Universidade Federal da Bahia - Instituto de Matemática e Estatística

santana.savio@ufba.br

impa

Instituto de Matemática Pura e Aplicada

Abstract

This work has as goal to present a proof of structural stability for C^2 diffeomorphisms which are axiom A and satisfy the strong transversality condition defined on a smooth, compact, boundaryless manifold.

Results and discussion

We say that $f, g \in \text{Diff}^r(M)$, $r > 1$, are *topologically conjugated* if there is a homeomorphism $\phi : M \circlearrowleft$ such that $g = \phi \circ f \circ \phi^{-1}$. A C^r map f is C^m-*structurally stable* $(1 \leq m \leq r)$ if there is a neighborhood U of f in the C^m topology such that for every map $g \in U$ is topologically conjugated f.

Definition 1. *A point $x \in M$ is nonwandering if for any neighborhood U of x, there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of nonwandering points of f.*

Definition 2. *A map $f \in \text{Diff}^r(M)$ satisfies Axiom A if $\Omega(f)$ is a hyperbolic and $\Omega(f) = \overline{Per(f)}$.*

Definition 3. *We say that f satisfies the strong transversality condition if*

$$E^s_x + E^u_x = T_x M,$$

for all $x \in M$, where $E^\sigma_x = E^\sigma \cap T_x M$.

Theorem 1. *(Robbin,71) A diffeomorphism $f \in \text{Diff}^2(\text{M})$ that satisfies axiom A and the strong transversality condition is C^1-structurally stable .*

Robbin's condition

Consider a homeomorphism $\phi : M \circlearrowleft$ satisfying $g \circ \phi = \phi \circ f$, for g C^1-close to f, we can rewrite this equation as follows: $f^{-1} \circ g \circ \phi = f^{-1} \circ \phi \circ f$. Since $f^{-1} \circ g$ is close to the identity in the C^1 topology, there is $Y \in \mathfrak{X}^1(M)$ such that $f^{-1} \circ g = \exp(Y)$, where $\exp : TM \longrightarrow M$ is the exponential map. As we want ϕ close to the identity, $\phi = \exp(X)$, for some $X \in \mathfrak{X}^0(M)$. So $g \circ \phi = \phi \circ f$ assumes the following form

$$(Id - f^\#)X = R_Y(X), \qquad (1)$$

where Id is the identity operator, $f^\#$ is the linear operator induced by f defined in the Banach space $\mathfrak{X}^0(M)$ and R_Y is a nonlinear operator with small lipschitz constant.

Assuming that $(Id - f^\#)$ has a right inverse J, we can rewrite (1),

$$X = JR_Y(X). \qquad (2)$$

Thus, the problem of finding a function ϕ that satisfies $g \circ \phi = \phi \circ f$ becomes the problem of finding a fixed point for a certain operator and choosing J appropriately such that $\phi = \exp(X)$ is a homeomorphism.

Definition 4. *We say that a vector field $X \in \mathfrak{X}^0(M)$ is d_f-Lipschitz if for every letter (φ, U) in M, there is a $K \geq 0$ such that*

$$|X_\varphi(x) - X_\varphi(y)| \leq K d_f(x, y),$$

where $X_\varphi : U \to \mathbb{R}^n$ such that $d\varphi_x X_x = (\varphi(x), X_\varphi(x))$. We denote by $\mathfrak{X}_f(M)$ the set of all fields d_f-lipschitz in M.

Definition 5. *Let $f : M \circlearrowleft$ be a C^2 diffeomorphism. We say that f satisfies Robbin's condition if there is a continuous linear operator $J : \mathfrak{X}^0(M) \to \mathfrak{X}^0(M)$ such that J is the right inverse of operator $(Id - f^\#)$ and $\mathfrak{X}_f(M)$ is $J-$invariant.*

Theorem 2. *If $f \in \text{Diff}^2(\text{M})$ satisfies Robbin's condition, then f is structurally stable.*

Locally hyperbolic

Definition 6. *Let $N \subseteq M$ be an open set and $E \subseteq TN$ be a continuous subbundle. We say that E is d_f-Lipschitz if for every $x \in N$ and any neighborhood U of x, there are vector fields $X_1, ..., X_k \in \mathfrak{X}_f(M)$, such that $X_1(y), ..., X_k(y)$ is a basis of E_y, for all $y \in U$.*

Definition 7. *We say that f is locally hyperbolic if there are open subsets $Z_1, ..., Z_l, W_1, ..., W_l \subset M$, subbundles E^u_i, E^s_i of TZ^f_i, where $i = 1, ..., l$, a Riemannian metric $\|\cdot\|$ and $0 < \lambda < 1$ such that for $i, j = 1, ..., l$, $\sigma = s, u$*

1. $\overline{W}_i \subseteq Z_i$;
2. $M = \cup^l_i W^f_i$;
3. W_i *is not revisited by* f;
4. $Z_i \cap Z_j = \emptyset$, *for* $i \neq j$;
5. E^σ_i *is* d_f*- lipschitz;*
6. E^σ_i *is* df *invariant;*
7. $TZ^f_i = E^s_i \oplus E^u_i$;
8. $\|df_x \cdot v\| \leq \lambda\|v\|$, *for* $v \in E^s_{i_x}$, *where* $x \in Z_i$,
   $\|df^{-1}_x \cdot v\| \leq \lambda\|v\|$, *for* $v \in E^u_{i_x}$, *where* $x \in Z_i$;
9. $E^s_{i_x} \subseteq E^s_{j_x}$ *and* $E^u_{j_x} \subseteq E^u_{i_x}$ *to* $x \in Z^{f+}_i \cap Z^{f-}_j$.

Theorem 3. *Let $f : M \to M$ be a C^2 diffeomorphism. If f is locally hyperbolic, then f satisfies Robbin's condition.*

Let $\theta_1, \ldots, \theta_l$ be a partition of unity subordinate to the coverage $W^f_1, ..., W^f_l$ of M. Let $X \in \mathfrak{X}^0(M)$. For $i = 1, \ldots, l$, we write

$$\theta_i X = X_{i,s} + X_{i,u}.$$

We now define the field

$$J(X) = \sum_{i=1}^{l} \big(J_{i,u}(X_{i,u}) + J_{i,s}(X_{i,s})\big). \qquad (3)$$

Since $J_{i,s}(X)$ and $J_{i,u}(X)$ converge uniformly, we have that $J(X)$ converges uniformly and $(Id - f^\#)J(X) = X$.

Theorem 4. *If f satisfies Axiom A and the strong transversality condition, then f is locally hyperbolic.*

Since f satisfies Axiom A, by Smale's Spectral Decomposition Theorem it is possible to write $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_l$ where Ω_i is compact and f-invariant. Using the strong transversality condition it is possible to consider Z_i neighborhoods of Ω_i, thus the local hyperbolicity conditions are satisfied.

Conclusion

Given a C^2 diffeomorphism satisfying *Axiom A* with *strong transversality*, it is *locally hyperbolic*. This property ensures the *Robbin's condition*, which, in turn, guarantees C^1 *structural stability.*

Reference

[1] ROBBIN, J. W. *A structural stability theorem*. Annals of Mathematics, John Hopkins University Press, v. 94, p. 447–493, 11 1971.

[2] ANDRONOV, A.; PONTRYAGIN, L. *Sistèmes grossier*. Dokady Akademi Nauk, p. 247–250, 1937.

[3] SMALE, S.; PALIS, J. *Structural stability theorems*. In: Proc. of Symposia in Pure Mathematics (Global Analysis) XIV. [S.l.]: AMS, 1970.

Acknowledgment

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior –Brasil (CAPES) –Finance Code 001.