Discrete versions of four-vertex-type theorems for spherical curves

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Abstract

We state and prove discrete analogs of four-vertex-type theorems for spherical curves. The original smooth case goes back to the work of Beniamino Segre ([4]) and states that a closed and smooth spherical curve not contained in any closed hemisphere and without self-intersections admits at least four geodesic inflections. Recently Mohammad Ghomi ([1]) extended this result for spherical curves with self-intersections or cusps. For the discrete setting, we define an inflection of a spherical polygon as an edge whose generated great circle separates the preceding and the following vertices. Our theorems then state that spherical polygons which are not in any closed hemisphere must have at least 4 inflections if they have no self-intersections, or at least admit a lower bound of 4 on the number of inflections plus the number of self-intersections counted with multiplicity.

Definitions and main result

Definition 0.1. A **spherical poligonal line** $Q = [u_1, u_2, ..., u_n]$ is a curve on the sphere obtained by concatenation of spherical segments, i.e., of geodesic segments of the sphere with the least length. For this we assume that consecutive vertices u_i and u_{i+1} are not antipodal of each other. A **spherical polygon** $Q = [u_1, u_2, ..., u_n]$ is a closed spherical poligonal line (i.e., we also consider the spherical segment $[u_n, u_1]$).

Definition 0.2. Given a spherical polygon $Q \subset \mathbb{S}^2$, a **(spherical) inflection** of Q is a pair $\{u_i, u_{i+1}\}$ such that u_{i-1} and u_{i+2} are in different hemispheres determined by the spherical line spanned by $\{u_i, u_{i+1}\}$.

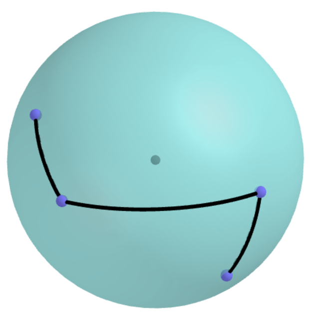

Figura 1: Inflection on the sphere

Definition 0.3. A set of points $Q = \{u_1, ..., u_n\} \subset \mathbb{S}^2$ ($n \geq 4$), not in the same spherical line, is said to be **balanced** or in **balanced position** if its points are not in the same closed hemisphere. A point u_i of a balanced set is said to be **essential** if the set $\{u_1, ..., \hat{u}_i, ..., u_n\}$ is not balanced. Otherwise u_i is **nonessential**. For a spherical polygon $Q = [u_1, .., u_n]$, the same definitions apply to Q considered as a set of vertices.

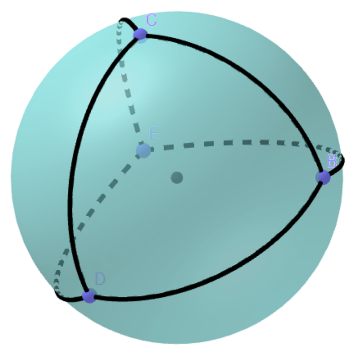

Figura 2: Four points in balanced position

Theorem 0.1. *Let $Q = [u_1, ..., u_n] \in \mathbb{S}^2$ ($n \geq 4$) be a spherical polygon in balanced position and without self-intersections. Then Q has at least four spherical inflections.*

Lemma 0.1. *Let $Q = \{u_1, u_2, ..., u_n\}$ be a finite set of points on the sphere $\mathbb{S}^2$, with $n \geq 5$, not all of them on the same hemisphere. Then the set X of nonessential vertices has at least $n - 3$ elements.*

Definition 0.4. A spherical polygon $Q = [u_1, ..., u_n]$ is **simple** if it does not have self-intersections. A vertex u_i is said to be **good** if the spherical polygon $Q - u_i$ (obtained by deleting spherical segments $[u_{i-1}, u_i]$ and $[u_i, u_{i+1}]$ and adding spherical segment $[u_{i-1}, u_{i+1}]$ to the remaining polygonal line) is simple. Otherwise u_i is said to be **bad**.

Lemma 0.2. *Let $Q = [u_1, u_2, ..., u_n]$ be a balanced, simple spherical polygon, ($n \geq 4$). Then the set Y of good vertices has at least 4 elements.*

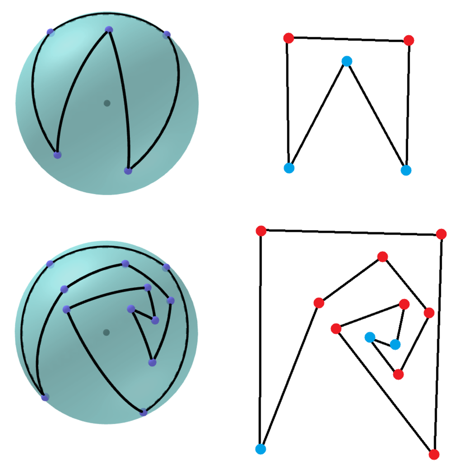

Figura 3: The balanced position hypothesis is necessary for the previous lemma to be true

Lemma 0.3. *Given a balanced, simple spherical polygon Q, let u_i be a good, nonessential vertex of Q. Then the number of spherical inflections of Q is greater or equal to the number of spherical inflections of the resulting spherical polygon $Q - u_i$.*

Applications to other Spherical Polygons

Theorem 0.2. *(Discrete Tennis Ball Theorem) If a spherical, simple polygon $Q = [u_1, ..., u_n]$ ($n \geq 4$) divides the sphere into two regions with the same area, then Q has at least 4 spherical inflections.*

Definition 0.5. For a set $X \subset \mathbb{R}^d$, define $-X$ as $-X = \{-x; x \in X\}$. We say that X is **centrally symmetric** if $-X = X$.

Theorem 0.3. *A simple, centrally symmetric polygon Q with at least $2n$ points ($2n \geq 6$) has at least 6 inflections.*

Theorem 0.4. *Given a spherical polygon P, denote by S the number of its self-intersections and by I the number of its inflections. If P is balanced, then the following inequality holds:*

$$2S + I \geq 4.$$

Furthermore, if P is centrally symmetric, then

$$2S + I \geq 6.$$

Forthcoming Research

We are working on the discrete version of another result by Ghomi ([1]), in which pairs of antipodal points of the curve are also considered. In this case, an inequality such as the first one in Theorem 0.4 also holds.

Our next goal is to study discrete analogs of other classes of convex curves. Among these, two are of interest: *locally convex curves* ([2]) and *strictly convex curves* ([3]).

Referências

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