Combinatorial Games in Graph Convexity

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Abstract

This work presents a continuation of a series of five papers [1, 2, 3, 4, 5] that ended in 2003 on geodesic convexity games on graphs. We extend these games to various graph convexities, obtain winning strategies and the first PSPACE-hardness results. We prove that the hull game's normal play and misère play for geodesic and monophonic convexities are PSPACE-Complete, even in graphs with a diameter two.

Introduction

A *graph convexity* $\mathcal{C}$ on a finite graph G is a family of subsets of $V = V(G)$ s.t. $\emptyset, V \in \mathcal{C}$, and $\mathcal{C}$ is closed under intersections. The $\mathcal{C}$-*convex hull* of $S \subseteq V$ is the smallest member $\mathrm{hull}_{\mathcal{C}}(S) \supseteq S$ of $\mathcal{C}$. If $\mathrm{hull}_{\mathcal{C}}(S) = V$, S is a $\mathcal{C}$-*hull set*. The $\mathcal{C}$-convex hull of S can be obtained by iteratively applying an interval function $\mathrm{I}_{\mathcal{C}}(\cdot)$ until obtaining a member of $\mathcal{C}$. The *geodesic* and *monophonic* convexities are defined by considering $\mathrm{I}_{\mathcal{C}}(S)$ to be S with any vertex on a path between two vertices of S which is minimum or induced, resp.

Definition 1. *Given a graph convexity $\mathcal{C}$ on G, we introduce 4 convexity games. In all of them, the set L of labeled vertices is initially empty and $f(L)$ and $g(L)$ depend on the game. Alice and Bob alternately label one unlabeled vertex v which is not in $f(L)$. The game ends when $g(L) = V(G)$.*
1. In the $\mathcal{C}$-hull-game: $f(L) = L$ and $g(L) = \mathrm{hull}_{\mathcal{C}}(L)$.
2. In the $\mathcal{C}$-interval-game: $f(L) = L$ and $g(L) = \mathrm{I}_{\mathcal{C}}(L)$.
3. In the closed $\mathcal{C}$-hull game: $f(L) = g(L) = \mathrm{hull}_{\mathcal{C}}(L)$.
4. In the closed $\mathcal{C}$-interval-game: $f(L) = g(L) = \mathrm{I}_{\mathcal{C}}(L)$.

Each game has 3 variants: *normal* (the last to play wins), *misère* (the last to play loses), and the *optimization* variant. In the optimization variant, it is given an additional parameter $k > 0$, and Alice wins if $|L| \leq k$ at the end.

Definition 2. *The optimization variants induce four new parameters:* game $\mathcal{C}$-hull number $\mathrm{ghn}_{\mathcal{C}}(G)$, game $\mathcal{C}$-interval number $\mathrm{gin}_{\mathcal{C}}(G)$, closed game $\mathcal{C}$-hull number $\mathrm{cghn}_{\mathcal{C}}(G)$ *and* closed game $\mathcal{C}$-interval number $\mathrm{cgin}_{\mathcal{C}}(G)$. *as the minimum k such that Alice has a winning strategy in the optimization variant of the corresponding convexity games on the graph G.*

From Zermelo's Theorem [6], one player has a winning strategy in each of these games and their variations, since they are finite perfect information games without draw. So, the decision problem is to decide if Alice has a winning strategy.

PSPACE-hardness of the monophonic and geodesic hull games and closed hull games

We prove that simplified versions of the normal and misère variants of the monophonic hull game are PSPACE-Complete.

In the *simplified hull game*, the input consists of a graph G and a vertex v, which is already labelled before the beginning of the game. We obtain reductions from the clique-forming game, in which Alice and Bob take turns selecting vertices which must induce a clique (the last to play wins). This problem is known to be PSPACE-complete [7].

Theorem 1. *The misère variant of the simplified monophonic hull game is PSPACE-complete even in diameter 2 graphs.*

Proof. **[sketch]** Let a not complete graph H be an instance of the clique-forming game. Let G be the graph of diameter 2 obtained from H by adding two new vertices u_1 and u_2 adjacent to all vertices of H. Also let u_1 be the vertex which is already labelled. We prove that Alice has a winning strategy in the clique-forming game on H if and only if she has a winning strategy in the avoidance variant of the simplified monophonic hull game on (G, u_1). If a player labels u_2, he/she loses immediately, since $\mathrm{hull}_m(\{u_1, u_2\}) = V(G)$ (the subscript "m" to indicate the monophonic convexity). Moreover, if a player labels a vertex v_j of H in the hull game and there is a non-adjacent labelled vertex v_i in H, then the player loses immediately, since $\mathrm{hull}_m(\{v_i, v_j\}) = V(G)$. So, we may assume that set L of labelled vertices form a clique in all turns, except the last one. We conclude the if Alice has a winning strategy in the clique-forming game on H, then Bob loses the avoidance hull game. Analogously, if Bob has a winning strategy in the clique-forming game on H. □

Theorem 2. *The normal variant of the simplified monophonic hull game is PSPACE-complete even in diameter 2 graphs.*

Proof. **[sketch]** Consider the same reduction of Theorem 1, but adding to G a new vertex w, which is isolated. If a player labels w (resp. u_2) when u_2 (resp. w) is not labelled, he/she loses immediately, since the opponent labels u_2 (resp. w) and wins the achievement game because $\mathrm{hull}_m(\{u_1, u_2, w\}) = V(G)$. Moreover, if a player labels a vertex v_j of H in the hull game and there is a non-adjacent labelled vertex v_i in H, then the player loses immediately, since the opponent labels w, winning the game because $\mathrm{hull}_m(\{v_i, v_j, w\}) = V(G)$. So, the set L of labelled vertices form a clique in all turns, except the last two. we conclude that Alice has a winning strategy in the achievement hull game on G if and only if she has a winning strategy in the clique-forming game on H. □

Notice that the monophonic and the geodesic convexities may be different in the constructed graphs. However, it is not difficult to check that the same arguments are valid for them.

Corollary 3. *The normal and misère variants of the hull game and the closed hull game on the monophonic and the geodesic convexities are PSPACE-complete, even in diameter 2 graphs.*

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