# The existence of limit cycles for perturbations of differential equations with invariant manifolds: an approach via Chebyshev systems

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### Abstract

In this work, we study bifurcations of limit cycles for polynomial and rational perturbations of differential equations with an invariant manifold filled with periodic orbits. Using averaging theory and the theory of Chebyshev systems, we obtain upper bounds for the maximum number of limit cycles that bifurcate from the periodic orbits on the invariant manifold. Furthermore, we show that these bounds are attained.


### Introduction

One of the main problems in the theory of differential equations is determining the number of limit cycles that bifurcate from a center. In this work, we are interested in a generalization of this problem: If a differential equation has an invariant manifold filled by periodic orbits, will small perturbations of this system produce any limit cycles? In this case, how many?

## 1 Averaging theory

Consider the differential system

$$\dot{\mathrm{x}} = F_0(t,\mathrm{x}) + \varepsilon F_1(t,\mathrm{x}) + \varepsilon^2 F_2(t,\mathrm{x},\varepsilon), \qquad (1)$$

where $\mathrm{x} \in \mathbb{R}^n$, $\varepsilon > 0$ is sufficiently small, and $F_0$, $F_1$ and $F_2$ are $C^k$ functions that are $T$-periodic on the first variable.

If $\phi(t,\mathrm{z})$ is a solution of $(1)|_{\varepsilon=0}$ such that $\phi(0,\mathrm{z}) = \mathrm{z}$, denote by $M_{\mathrm{z}}(t)$ the fundamental matrix of the linearization $\dot{\mathrm{y}} = D_2F_0(t,\phi(t,\mathrm{z}))\mathrm{y}$.

**Theorem 1.** *Let $V \subset \mathbb{R}^m$ be open and bounded. Consider $\mathcal{Z} := \{\mathrm{z}_\alpha = (\alpha,\beta(\alpha)) : \alpha \in \overline{V}\}$, where $\beta : \overline{V} \to \mathbb{R}^{n-m}$ (resp. $\mathbb{R}^m$) is a $C^k$ function. Assume that*

1. *The solution of $(1)|_{\varepsilon=0}$ through each point of $\mathcal{Z}$ is $T$-periodic;*
2. *For each $\alpha \in \overline{V}$, $M_{\mathrm{z}_\alpha}^{-1}(0) - M_{\mathrm{z}_\alpha}^{-1}(T)$ has a 0 upper right submatrix and a nonsingular lower right submatrix (or vice-versa).*

*If $\xi$ is the projection onto the first $m$ (resp. last $n-m$) coordinates, define*

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M_{\mathrm{z}_\alpha}^{-1}(t)F_1(t,\phi(t,\mathrm{z}_\alpha))dt\right). \qquad (2)$$

*Then, for each $\alpha_0 \in V$ such that $\mathcal{F}(\alpha_0) = 0$ and $\det\left(\frac{\partial\mathcal{F}}{\partial\alpha}(\alpha_0)\right) \neq 0$, for $\varepsilon \neq 0$ sufficiently small, there is a unique periodic solution $\phi_1(t,\varepsilon)$ of (1) passing through $\mathrm{z}_{\alpha_0}$ such that $\lim_{\varepsilon\to 0}\phi_1(t,\varepsilon) = \phi(t,z_{\alpha_0})$ and it is hyperbolic.*

## 2 Chebyshev systems

**Definition 1.** *A family of functions $\Phi = \{\phi_1, ..., \phi_k\}$ defined on an interval $I$ is said a Chebyshev system if any nontrivial linear combination of the elements of $\Phi$ admits at most $k-1$ zeros in $I$.*

**Example.** *The following sets are Chebyshev systems:*

- $\{1, x, x^2, \dots, x^n\}$ *for every* $n \in \mathbb{N}$;
- $\{\cos(kx), \sin(kx)\}_{k=0}^n$ *for every* $n \in \mathbb{N}$;
- $\{x^i e^{\alpha_j x} : i \in \{0,\dots,n_j\}\}_{j=1}^L$ *for finite increasing sequences* $(\alpha_j)_{j=1}^L \subset \mathbb{R}$ *and* $(n_j)_{j=1}^L \subset \mathbb{N}$;
- $\{x^{\alpha_i+\ell} : \ell \in \{0,\dots,n_i\}\}_{i=1}^L$ *for finite sequences* $(\alpha_j)_{j=1}^L \subset \mathbb{R}_+$ *and* $(n_j)_{j=1}^L \subset \mathbb{N}$;
- $\{x^i \log(x)^j : 0 \le i \le d-1-2j\}_{j=0}^{\lfloor(d-1)/2\rfloor}$ *for* $d \ge 3$.

## 3 Bifurcation of limit cycles from surfaces of revolution

Consider the differential system

$$\begin{cases} \dot{x} = -y + \varepsilon P(x,y,z); \\ \dot{y} = x + \varepsilon Q(x,y,z); \\ \dot{z} = \lambda F(x,y,z) + \varepsilon R(x,y,z), \end{cases} \qquad (3)$$

where $\lambda \in \mathbb{R}\setminus\{0\}$, $\varepsilon > 0$ is sufficiently small, $P$, $Q$ and $R$ are polynomials of degree at most $d$ and $F(x,y,z) = x^2 + y^2 - f(z)$.

See that $S_F = F^{-1}(0)$ is an invariant surface of $(3)|_{\varepsilon=0}$.

**Theorem 2** ([3]). *If $f$ is a polynomial of degree $s$, then, for $\varepsilon \neq 0$ sufficiently small, at most*

$$D = \begin{cases} d-1 & , \text{ if } s \in \{0,1,2\}; \\ d-1+(s-2)\left\lfloor\frac{d-1}{2}\right\rfloor & , \text{ if } s \ge 3 \end{cases}$$

*limit cycles bifurcate from the periodic orbits of the invariant surface $S_F$ of $(3)|_{\varepsilon=0}$. Moreover, this bound is attained.*

**Theorem 3** ([3]). *If $f \in \{z^{p/q}, e^z, \log(z)\}$, then, for $\varepsilon > 0$ sufficiently small, at most $D = \left(\left\lfloor\frac{d-1}{2}\right\rfloor+1\right)\left(\left\lfloor\frac{d}{2}\right\rfloor+1\right) - 1$ limit cycles bifurcate from the periodic orbits on $S_F$ of $(3)|_{\varepsilon=0}$. Moreover, this bound is attained.*

## 4 Bifurcation of limit cycles from an invariant torus

Consider the differential system

$$\begin{cases} \dot{x} &= (r(x,y)f(x,y,z) - z)\frac{x}{\sqrt{x^2+y^2}} + \epsilon P(x,y,z), \\ \dot{y} &= (r(x,y)f(x,y,z) - z)\frac{y}{\sqrt{x^2+y^2}} + \epsilon Q(x,y,z), \\ \dot{z} &= r(x,y) + zf(x,y,z) + \epsilon R(x,y,z) \end{cases} \qquad (4)$$

where $\varepsilon > 0$ is a small parameter, $r(x,y) = \sqrt{x^2+y^2} - 2$ and $f(x,y,z) = 1 - (\sqrt{x^2+y^2} - 2)^2 - z^2$.

See that $\mathbb{T} = f^{-1}(0)$ is an invariant torus for $(4)|_{\varepsilon=0}$.

**Theorem 4** ([2]). *If $P$, $Q$ and $R$ are polynomials of degree at most $d$, then at most $2(d+1)$ limit cycles bifurcate from the periodic orbits on the invariant torus $\mathbb{T}$ of $(4)|_{\varepsilon=0}$.*

**Theorem 5.** *Assume that $P(x,y,z) = \frac{p(x,y,z)}{a+bz}$ and $Q(x,y,z) = \frac{q(x,y,z)}{c+dz}$, where $p, q$ are polynomials of degree at most $d$, $|a| < |b|$ and $|c| < |d|$. Then, for $\varepsilon > 0$ sufficiently small, at most $2(d+1)$ limit cycles bifurcate from the periodic orbits on $\mathbb{T}$.*

**Theorem 6.** *If $P$ and $Q$ are suitable quotients of linear polynomials of $\mathbb{R}[x,y]$, then, for $\varepsilon > 0$ sufficiently small, there are polynomials $A, B, C, D, E \in \mathbb{R}[x,y]$ such that the limit cycles that bifurcate from the periodic orbits on $\mathbb{T}$ correspond to the simple zeros of*

$$\mathcal{F}(\phi) = \tilde{A}(\phi) + \tilde{B}(\phi)\tilde{C}(\phi)^{-1/2} + \tilde{D}(\phi)\tilde{E}(\phi)^{-1/2}.$$

### References


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