Classification of continuous flows

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Abstract

We give an overview on the topic of classification of continuous flows, going through Markus, Neumann, and López and Buendía's works.

Continuous flows

A *flow* on a set X is a group action of the additive group $(\mathbb{R}, +)$ on X. In another words, a flow φ on a set X is a mapping $\varphi : X \times \mathbb{R} \to X$ such that, for all $x \in X$ and $s, t \in \mathbb{R}$, we have

(i) $\varphi(x, 0) = x$;

(ii) $\varphi(\varphi(x, t), s) = \varphi(x, t + s)$.

If, instead of $X \times \mathbb{R}$, we consider the domain

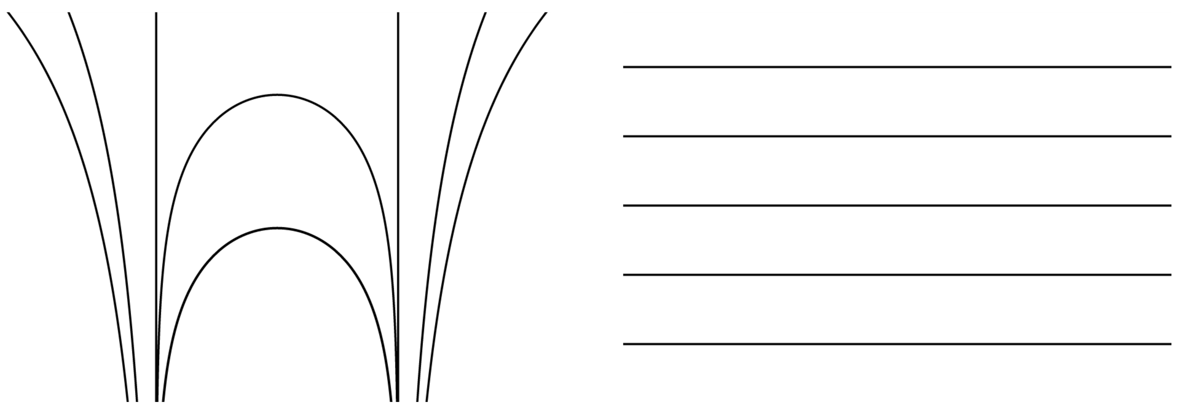
$$D = \{(x, t) \mid x \in X,\ t \in I_x\},$$

where I_x is an interval depending on x and containing 0, then $\varphi : D \to X$ is called a *local flow*. This often is the case when considering flows generated by vector fields.

On our work, usually we talk about *continuous* flows on a connected topological manifold M (second-countable, Hausdorff and locally Euclidean) without boundary; it is not necessarily compact nor orientable. The adjective continuous simply means that φ is continuous.

Topological equivalence and parallelism

Two continuous flows (M_1, φ_1) and (M_2, φ_2) are *topologically equivalent* if there exists a homeomorphism $h : M_1 \to M_2$ taking orbits onto orbits preserving sense, and that h is a *topological equivalence* between (M_1, φ_1) and (M_2, φ_2).



$$\begin{cases} \dot{x} = x^2 - 1 \\ \dot{y} = x \end{cases} \qquad \begin{cases} \dot{x} = 1 \\ \dot{y} = 0 \end{cases}$$

Let $U \subset M$ be a φ-invariant region. We call U *parallel* when the restriction (U, φ) is equivalent to one of the following:

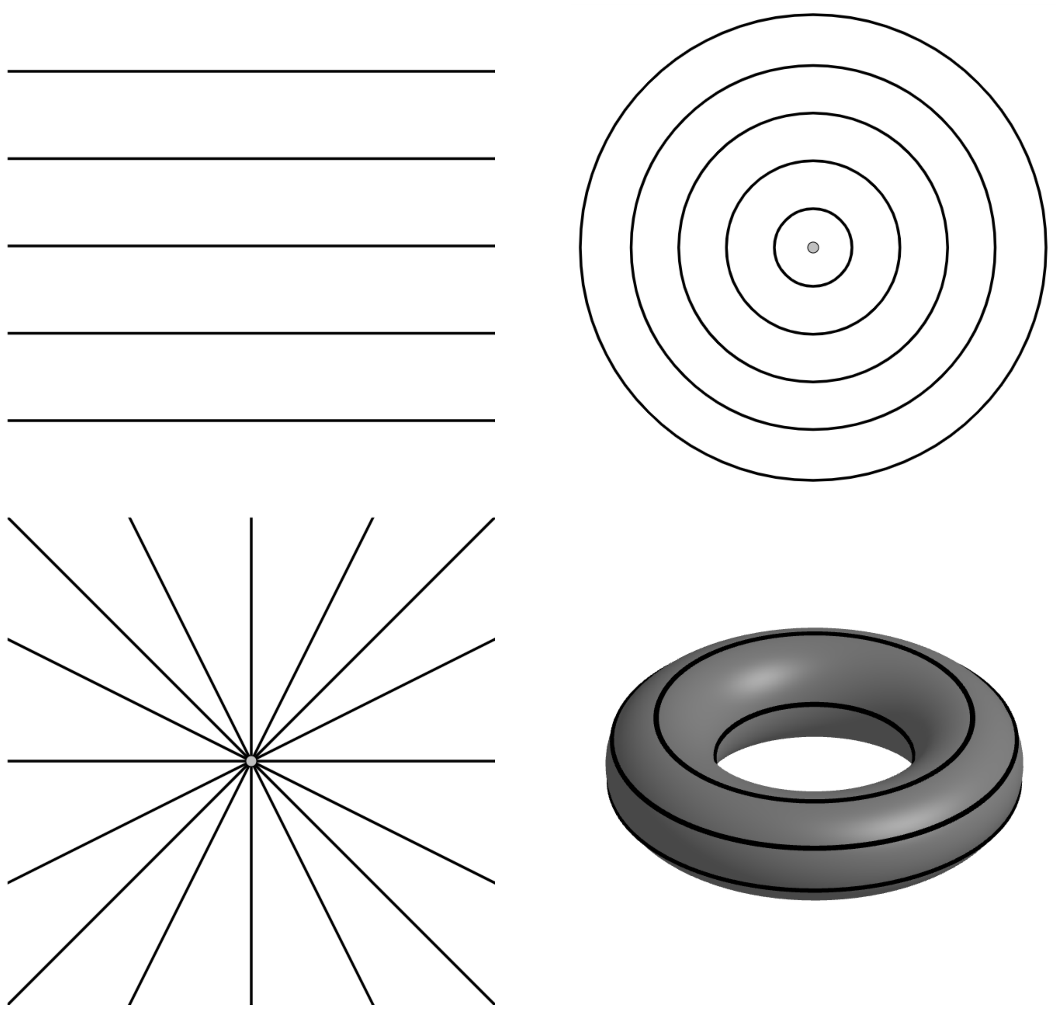
(i) $\mathbb{R}^2$ with flow defined by $y' = 0$;

(ii) $\mathbb{R}^2 - \{0\}$ with flow defined (in polar coordinates) by $dr/dt = 0$, $d\theta/dt = 1$;

(iii) $\mathbb{R}^2 - \{0\}$ with flow defined by $dr/dt = r$, $d\theta/dt = 0$;

(iv) $S^1 \times S^1$ with the flow induced by (i) above, under the covering map which associates (x, y) with $(x + n, y + m)$, where $m, n \in \mathbb{Z}$.

We distinguish these as *strip*, *annular*, *spiral* (or *radial*) and *toral*, respectively.



Markus' paper

In 1954, Lawrence Markus published a paper in which, among other results, he established some new concepts in order to generalize the idea of *separatrix* and to characterize vector fields in the plane based on the *separatrix configuration*. He tried to separate the flow into "simple" regions, and then classify the vector fields based on this. Roughly speaking, an *ordinary* orbit is one within a parallel region well behaved (in the sense that the α and ω-limit sets do not "change too much"); a separatrix is an orbit which is not ordinary. The separatrix configuration of a vector field is, then, the union of all separatrices together with a representative orbit from each canonical region.

Markus then "proved" that the separatrix configuration of a vector field with neither accumulation of critical points nor of separatrices (such accumulation on an orbit is called a *limit separatrix*) characterize the vector field completely: two such vector fields in the plane are equivalent if, and only if, they have equivalent separatrix configuration (i.e., there exists an automorphism of the plane taking one separatrix configuration onto the other).

The theorem does not work, even for simple examples. The problem comes from the very definition of separatrix.

The correction

After that, in 1975, Dean Arnold Neumann generalized Markus' result for continuous flows on two dimensional manifolds with limit separatrices. Unfortunately, since he was using a notion of separatrix that has already misguided Markus, Neumann's result is wrong as well.

In 2018, López and Buendía pointed the problems and suitable corrections. The main flaw and its correction is exactly what the intuition tells us: to guarantee the ordinariness of an orbit, one may be able to take an "arbitrarily small" parallel regions containing the desired orbit.

Definition. Let γ be an orbit of (M, φ). Consider the following properties about a strong strip U with border orbits γ_1 and γ_2:

(i) $\alpha(\mu) = \alpha(\gamma)$ and $\omega(\mu) = \omega(\gamma)$ for every orbit $\mu \subset U \cup \gamma_1 \cup \gamma_2$;

(ii) for every strong transversal T to U with endpoints p and q, the boundary of the regions which T separates U into can be written as $\partial U_T^- = T \cup \gamma_p^- \cup \gamma_q^- \cup \alpha(\gamma)$ and $\partial U_T^+ = T \cup \gamma_p^+ \cup \gamma_q^+ \cup \omega(\gamma)$.

We say that γ is *ordinary* if it is neighbored by an annular region or a strong strip with properties (i) and (ii). An orbit that is not ordinary is called a *separatrix*.

Finally, the theorem, with this simple correction, works fine; furthermore, it can be extended for more flows.

Theorem. *Let M be a 2-manifold and suppose that φ_1 and φ_2 are continuous flows on M whose set of essential singular points is discrete. Then φ_1 and φ_2 are equivalent if and only if they have equivalent separatrix configurations.*

The goal of my Master's studies was to understand, organize and correct this whole topic. More details and examples can be found in my dissertation.

References

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Acknowledgement

We would like to thank São Paulo Research Foundation (FAPESP) for the financial support, grant #2021/08895-3.