The Contact Angle of Surfaces in the Special Linear Group

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Abstract

In this paper we establish the equation for the Gaussian curvature of a constant mean curvature surface in the special linear group $Sl(2,\mathbb{R})$. Using the Gauss equation we prove that constant mean curvature surfaces in $Sl(2,\mathbb{R})$ with constant contact angle have constant Gaussian curvature. Also, we provide a congruence theorem for constant mean curvature surfaces of $Sl(2,\mathbb{R})$. Finally , we give an example of a minimal surface in $Sl(2,\mathbb{R})$ with non constant contact angle.

Introduction

Surfaces making constant angles with certain directions are interesting and they are intensively studied by several authors in different ambient spaces. An interesting characterization of constant angle surfaces in the special linear group $Sl(2,\mathbb{R})$ was showed by Montaldo, Onnis and Passamani, see [5]. The study in Bianchi-Cartan-Vranceanu spaces was completed in [5] and in [6]. Also, in [2] and [3], Dillen and others have studied constant angle surfaces in product spaces $S^2\times\mathbb{R}$ and $H^2\times\mathbb{R}$, namely those surfaces for which the unit normal makes a constant angle with the tangent direction to $\mathbb{R}$. Recently, Munteanu, Fastenakels and van der Veken, in [10], extended the notion of constant angle surfaces in $S^2\times\mathbb{R}$ and $H^2\times\mathbb{R}$ to general Bianchi–Cartan–Vranceanu spaces and they showed that these surfaces have constant Gaussian curvature, also they gave a complete local classification in the Heisenberg group.

Results

In [7] we introduced the notion of contact angle, which can be considered as a new geometric invariant useful for investigating the geometry of immersed surfaces in S^3. Also in [7], we derived formulae for the Gaussian curvature and the Laplacian of an immersed minimal surface in S^3, and we gave a characterization of the Clifford torus as the only minimal surface in S^3 with constant contact angle. In [8], we construct a family of minimal tori in S^5 with constant contact angle and constant holomorphic angle. These tori are parametrized by the following circle equation

$$a^2+\left(b-\frac{\cos\beta}{1+\sin^2\beta}\right)^2=2\frac{\sin^4\beta}{(1+\sin^2\beta)^2}. \tag{1}$$

I assume that β is the contact angle.
In particular, when $a=0$, we recover the examples found by Kenmotsu. These examples are defined for $0<\beta<\frac{\pi}{2}$. Also, when $b=0$, we find a new family of minimal tori in S^5, and these tori are defined for $\frac{\pi}{4}<\beta<\frac{\pi}{2}$. For $\beta=\frac{\pi}{2}$, we give an alternative proof of this classification of a Theorem from Blair, and Yamaguchi, Kon and Miyahara. For Legendrian minimal surfaces in S^5 with constant Gaussian curvature. Also in [9] we provide a congruence theorem for minimal surfaces in S^5 with constant contact angle using Gauss-Codazzi-Ricci equations. More precisely, we prove that Gauss-Codazzi-Ricci equations for minimal surfaces in S^5 with constant contact angle satisfy an equation for the Laplacian of the holomorphic angle.

Definition 1. *We denote $d\beta(e_1)=\beta_1$, $d\beta(e_2)=\beta_2$, $dH(e_1)=H_1$ and $dH(e_2)=H_2$.*

In this notes, we show that the Gaussian curvature K of a constant mean curvature surface of $Sl(2,\mathbb{R})$. with contact angle β is given by:

$$K=1-|\nabla\beta+(\cosh^2(\beta)+\sinh^2(\beta))e_1|^2-2H\beta_2. \tag{2}$$

Using the equation (2), we have proved the following theorem.

Theorem 1. *The Gaussian curvature for constant mean curvature surfaces in $Sl(2,\mathbb{R})$ with constant contact angle is constant.*

Main Result

More in general, we have the following congruence result.

Theorem 2. *Consider S a Riemannian surface, e a vector field on S, and $\beta : S\to]0,\frac{\pi}{2}[$ a function over S that satisfies the following equation*

$$\Delta(\beta)=-\tanh(\beta)|\nabla\beta+2(\cosh(\beta)^2+\sinh(\beta)^2)e|^2+2H_2$$

then there exist one, up to isometries of S into $Sl(2,\mathbb{R})$, immersion of mean curvature H , such that, e is the characteristic vector fied, and β is the contact angle of this immersion.

A particular case for $0<\beta<\frac{\pi}{2}$ and supposing that we have a compact surface with constant mean curvature H produce the following result.

Corollary 1. *In particular, when H is constant and for $0<\beta<\frac{\pi}{2}$, we have that compact surfaces in the special linear group $Sl(2,\mathbb{R})$ have constant contact angle β.*

Remark 1. *For non-compact surfaces, there is an example that is the hyperbolic space H^2, for this case the contact angle is $\beta=arccosh(x_2)$.*

Remark 2. *For minimal surfaces in product spaces $S^2\times\mathbb{R}$ and $H^2\times\mathbb{R}$, the author proves the following theorem bellow (see [1]).*

Theorem 3. *(***Benoit Daniel***)*
Let Σ be a minimal surface in $M^2(c)\times\mathbb{R}$. Then its angle function $\nu:\to[-1,1]$ satisfies
(M1) $\|\nabla\nu\|^2=-(1-\nu^2)(K-c\nu^2)$,
(M2) $\Delta\nu-2K\nu+c(1+\nu^2)\nu=0$,
where K denotes the intrinsic curvature of Σ.
Conversely, let Σ be a real analytic simply connected Riemannian surface and $\nu:\to[-1,1]$ a smooth function satisfying (M1) and (M2) where K is the curvature of Σ. Then there exists an isometric minimal immersion $f:\Sigma\to M^2(c)\times\mathbb{R}$ whose angle function is ν. Moreover, if $g:\Sigma\to M^2(c)\times\mathbb{R}$ is another isometric minimal immersion whose angle function is ν, then f and g are associate.

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