# On the self-similar blowup for the 2D generalized SQG equation


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## Introduction

Consider the 2D generalized surface quasi-geostrophic (GSQG) equation

$$\begin{cases} \theta_t + \mathbf{u}\cdot\nabla\theta = 0 \\ \mathbf{u} = \Lambda^{\beta-1}\mathcal{R}^{\perp}\theta = \Lambda^{\beta-1}(-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \end{cases} \tag{1}$$

where $\beta \in (0,2)$ is a fixed parameter, $\theta(x,t) : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ is a scalar function that represents the potential temperature of the fluid, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$ is the velocity field, $\Lambda^\beta$ is a Fractional Laplacian of order $\beta$ and each $\mathcal{R}_j$ is the Riesz transform given by

$$\mathcal{R}_j\theta(x) = P.V.\int_{\mathbb{R}^2}\frac{x_j - y_j}{|x-y|^3}\theta(y)dy, \quad j \in \{1,2\}, \quad \text{and} \quad \Lambda^\beta\theta(x) = P.V.\int_{\mathbb{R}^2}\frac{\theta(x)-\theta(y)}{|x-y|^{2+\beta}}dy.$$

We say that $\theta \in \mathcal{C}([0,T); H^s) \cap L^\infty([0,T); L^2)$ for some $s > 1+\beta$ and $T > 0$, is a locally self-similar solution to the GSQG equation in a ball $B_\rho(x_0) \subset \mathbb{R}^2$, if it satisfies

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{1+\alpha-\beta}{1+\alpha}}}\Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right), \quad (x,t) \in B_\rho(x_0)\times(0,T), \tag{2}$$

where $\alpha > -1$ and $\Theta \in C^1_{loc}(\mathbb{R}^2)$ is called a self-similar profile of $\theta$.

## Main Result

### Theorem

*Fix $\beta \in (1,2)$. Suppose $\theta \in \mathcal{C}([0,T); H^s) \cap L^\infty(0,T; L^2)$, with $s > 1+\beta$, is a solution to the generalized SQG equation (1) that is locally self-similar in a ball $B_\rho(x_0) \subset \mathbb{R}^2$, with scaling parameter $\alpha > -1$ and profile $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$. Fix also $p \geq 2$, and suppose that for some $r > p$, $\gamma_1 \in [0, r(\beta-1)+2)$, and $\gamma_0 \in [0, \gamma_1 + r]$, it holds*

$$\int_{|y|\leq L}|\Theta(y)|^r dy \lesssim L^{\gamma_0} \quad \text{and} \quad \int_{|y|\leq L}|\nabla\Theta(y)|^r dy \lesssim L^{\gamma_1} \tag{3}$$

*for all $L$ sufficiently large. Then either $\Theta \equiv 0$, or the index $\alpha$ admitting nontrivial profiles belongs to the interval $\left[\beta - 1 + \frac{2-\gamma_0}{r}, \beta - 1 + \frac{2}{p}\right]$, and for each such $\alpha$ the corresponding profile $\Theta$ satisfies*

$$\int_{|y|\leq L}|\Theta(y)|^p dy \sim L^{2-p(1+\alpha-\beta)}, \tag{4}$$

*for all $L$ sufficiently large.*

The case $0 < \beta \leq 1$ was also proved in this work and we recover the result proved in [4] when $\beta = 1$.

### Theorem

*Fix $\beta \in (1,2)$. Suppose $\theta \in \mathcal{C}([0,T); H^s) \cap L^\infty(0,T; L^2)$, with $s > 1+\beta$, is a solution to the generalized SQG equation (1) that is locally self-similar in a ball $B_\rho(x_0) \subset \mathbb{R}^2$, with scaling parameter $\alpha > -1$ and profile $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$. Then, the following statements hold:*

(i) *If there exist some $\sigma_0 > 0$ and $\sigma_1 > 0$ such that $|\Theta(y)| \lesssim |y|^{-\sigma_0}$ and $|\nabla\Theta(y)| \lesssim |y|^{-\sigma_1}$ for all $|y| \gg 1$, then $\Theta \equiv 0$ in $\mathbb{R}^2$.*

(ii) *Suppose $|\Theta(y)| \gtrsim 1$ and that there exists a real number $0 \leq \sigma_1 < \beta - 1$ such that $|\nabla\Theta(y)| \lesssim |y|^{\sigma_1}$ for all $|y| \gg 1$, then the values of $\alpha$ admitting nontrivial profiles belong to the interval $[\beta - 2 - \sigma_1, \beta - 1]$ and for each such $\alpha$ the corresponding profile $\Theta$ satisfies*

$$\int_{|y|\leq L}|\Theta(y)|^p dy \sim L^{2-p(1+\alpha-\beta)}$$

*for all $L$ sufficiently large and $p \geq 2$.*

## Proof of Theorem 1

We start by showing that $\Theta \equiv 0$ on $\mathbb{R}^2$, for all $\alpha > \beta + \frac{2}{p} - 1$. Fix $t \in [0,T)$ and denote $L = \rho(T-t)^{\frac{-1}{1+\alpha}}$. Invoking the local self-similarity (2), it follows that

$$\int_{|x|\leq\rho}|\theta(x,t)|^p dx = \frac{1}{(T-t)^{\frac{p(1+\alpha-\beta)}{1+\alpha}}}\int_{|x|\leq\rho}\left|\Theta\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}\right)\right|^p dx = L^{p(1+\alpha-\beta)-2}\int_{|y|\leq L}|\Theta(y)|^p dy.$$

Since $H^s(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$ for $s > 1+\beta$, we get

$$\int_{|y|\leq L}|\Theta(y)|^p dy \leq CL^{2-p(1+\alpha-\beta)}. \tag{5}$$

Hence, if $\alpha > \beta - 1 + \frac{2}{p}$, we may take the limit as $t \to T$ in (5) and conclude that $\Theta \equiv 0$ on $\mathbb{R}^2$.

In the next step, we also prove that for all $-1 < \alpha < \beta - 1 + \frac{2-\gamma_0}{r}$ the corresponding profile $\Theta \equiv 0$ on $\mathbb{R}^2$. Let $\phi_{\frac{\rho}{4}}, \phi_\rho \in \mathcal{C}^\infty(\mathbb{R}^2)$ be cut-off functions with $0 \leq \phi_{\frac{\rho}{4}}, \phi_\rho \leq 1$, $\phi_{\frac{\rho}{4}} \equiv 1$ in $B_{\rho/8}$, $\phi_{\frac{\rho}{4}} \equiv 0$ in $B^c_{\rho/4}$, and $\phi_\rho \equiv 1$ in $B_{\rho/2}$, $\phi_\rho \equiv 0$ in $B^c_\rho$. Since $\theta \in \mathcal{C}([0,T); H^s(\mathbb{R}^2))$ for some $s > 1+\beta$, yields

$$\int_{\mathbb{R}^2}|\theta(x,t_2)|^p\phi_{\frac{\rho}{4}}(x)dx - \int_{\mathbb{R}^2}|\theta(x,t_1)|^p\phi_{\frac{\rho}{4}}(x)dx = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}(\mathbf{u}(x,t)\cdot\nabla\phi_{\frac{\rho}{4}}(x))|\theta(x,t)|^p dxdt, \tag{6}$$

where $t_1, t_2 \in (0,T)$ are fixed. By invoking the local self-similarity of $\theta$, it follows that

$$\int_{\mathbb{R}^2}|\theta(x,t_i)|^p\phi_{\frac{\rho}{4}}(x)dx = l_i^{p(1+\alpha-\beta)-2}\int_{|y|\leq\frac{\rho}{4}l_i}|\Theta(y)|^p\phi_{\frac{\rho}{4}}(yl_i^{-1})dy, \quad l_i = (T-t_i)^{-\frac{1}{1+\alpha}}, \ i = 1,2. \tag{7}$$

Decomposing the velocity field $\mathbf{u}$ in a self-similarity region and outside of it, we can conclude that

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^2}|\theta(x,t)|^p(\mathbf{u}(x,t)\cdot\nabla\phi_{\frac{\rho}{4}}(x))dxdt\right| \lesssim \int_{\frac{\rho}{8}l_1\leq|y|\leq\frac{\rho}{4}l_2}\frac{|\tilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy + \int_{\frac{\rho}{8}l_1\leq|y|\leq\frac{\rho}{4}l_2}\frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy,$$

where

$$\tilde{V}^{(1)}(y) := \int_{t_1}^{t_2}\left|\int_{\mathbb{R}^2}\frac{1}{|y-z|^\beta}\nabla^\perp\Theta(z)\phi_\rho(z(T-t)^{\frac{1}{1+\alpha}})dz\right|\mathbb{1}_{B(y)}(t)dt. \tag{8}$$

and

$$B(y) := \left\{t \in [t_1,t_2] : \frac{\rho}{8}\frac{1}{|y|} \leq (T-t)^{\frac{1}{1+\alpha}} \leq \frac{\rho}{4}\frac{1}{|y|}\right\}.$$

Plugging this back into (6) and recalling (7), we get

$$\begin{aligned}&\left|l_2^{p(1+\alpha-\beta)-2}\int_{\mathbb{R}^2}|\Theta(y)|^p\phi_{\frac{\rho}{4}}(yl_2^{-1})dy - l_1^{p(1+\alpha-\beta)-2}\int_{\mathbb{R}^2}|\Theta(y)|^p\phi_{\frac{\rho}{4}}(yl_1^{-1})dy\right| \\ &\leq \int_{\frac{\rho}{8}l_1\leq|y|\leq\frac{\rho}{4}l_2}\frac{|\tilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy + \int_{\frac{\rho}{8}l_1\leq|y|\leq\frac{\rho}{4}l_2}\frac{|\Theta(x)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy.\end{aligned}$$

Taking the limit as $l_2 \to \infty$ and recalling that $-1 < \alpha < \beta - 1 + \frac{2-\gamma_0}{r}$, we get

$$\frac{1}{L^{2-p(1+\alpha-\beta)}}\int_{|y|\leq L}|\Theta(y)|^p dy \leq c\int_{|y|\geq L}\frac{|\tilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy + c\int_{|y|\geq L}\frac{|\Theta(x)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy, \tag{9}$$

where $\frac{\rho}{8}l_1 = L$. Next, we will use the following Lemma:

### Lemma

*Let $\Theta \in \mathcal{C}^1_{loc}(\mathbb{R}^2)$ and $T > 0$. Suppose that for some $r > p$ and $\gamma > 0$, it holds*

$$\int_{|y|\leq L}|\nabla\Theta|^r dy \lesssim L^\gamma. \quad \textit{Then} \quad \int_{L\leq|y|\leq 2L}|\tilde{V}^{(1)}(y)|^r dy \lesssim L^{\gamma+r(1-\alpha-\beta)}, \quad L \gg 1.$$

Applying the dyadic decomposition together with Holder's inequality and Lemma, we obtain

$$\begin{aligned}\int_{|y|\geq L}\frac{|\tilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy &\leq \sum_{k=0}^{\infty}\frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}}\left(\int_{|y|\sim 2^kL}|\Theta(y)|^r dy\right)^{\frac{p}{r}}\left(\int_{|y|\sim 2^kL}|\tilde{V}^{(1)}(y)|^r dy\right)^{\frac{1}{r}}(2^kL)^{2\left(1-\frac{p+1}{r}\right)} \\ &\leq cL^{p(1+\alpha-\beta)-2-\beta+3+\frac{\gamma_1-2}{r}+\frac{(\gamma_0-2)p}{r}}.\end{aligned} \tag{10}$$

For the second term on the right-hand side of (9), we have that

$$\begin{aligned}\int_{|y|\geq L}\frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy &\leq c\sum_{k=0}^{\infty}\frac{1}{(2^kL)^{3+\alpha-p(1+\alpha-\beta)}}\left(\int_{|y|\sim 2^kL}|\Theta(y)|^r dy\right)^{\frac{p}{r}}(2^kL)^{2\left(1-\frac{p}{r}\right)} \\ &\leq cL^{p(1+\alpha-\beta)-2+1-\alpha+\frac{(\gamma_0-2)p}{r}},\end{aligned} \tag{11}$$

Combining (10) and (11) with (9), we conclude that

$$\int_{|y|\leq L}|\Theta(y)|^p dy \leq cL^{3-\beta+\frac{(\gamma_1-2)}{r}+\frac{(\gamma_0-2)p}{r}} + cL^{1-\alpha+\frac{(\gamma_0-2)p}{r}} \leq cL^{a_0}, \tag{12}$$

where $a_0 := \max\left\{1-\alpha+\frac{(\gamma_0-2)p}{r}, 3-\beta+\frac{(\gamma_1-2)}{r}+\frac{(\gamma_0-2)p}{r}\right\}$. Note that, if $a_0 < 0$, the proof is finished. Otherwise, if $a_0 \geq 0$, we obtain by interpolation ($p < q < r$) that

$$\int_{|y|\leq L}|\Theta(y)|^q dy \leq \left(\int_{|y|\leq L}|\Theta(y)|^p dy\right)^\delta\left(\int_{|y|\leq L}|\Theta(y)|^r dy\right)^{1-\delta} \leq CL^{a_0\delta+(1-\delta)\gamma_0}, \tag{13}$$

where $\delta := \frac{r-q}{r-p} \in (0,1)$. Next, proceeding similarly as in (10) and (11), we obtain

$$\int_{|y|\geq L}\frac{|\tilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy \leq L^{p(1+\alpha-\beta)-2+a_0-a_1} \text{ and } \int_{|y|\geq L}\frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy \leq L^{p(1+\alpha-\beta)-2+a_0-(1+\alpha)}.$$

where $a_1 > 0$. Plugging this back into (9), we deduce that

$$\int_{|y|\leq L}|\Theta(y)|^p dy \leq cL^{a_0-a_1} + cL^{a_0-(1+\alpha)} \leq cL^{a_0-b_0}, \quad \text{where } b_0 := \min\{a_1, 1+\alpha\}.$$

Now we repeat this process until

$$\int_{|y|\leq L}|\Theta(y)|^p dy \leq cL^\sigma, \tag{14}$$

for some $\sigma < 0$. Therefore, we conclude that $\Theta \equiv 0$ on $\mathbb{R}^2$ if $-1 < \alpha < \beta - 1 + \frac{2-\gamma_0}{r}$.

Finally, we prove that if $\beta - 1 + \frac{2-\gamma_0}{r} < \alpha < \beta - 1 + \frac{2}{p}$, then either $\equiv 0$ or

$$c_1L^{2-p(1+\alpha-\beta)} \leq \int_{|y|\leq L}|\Theta(y)|^p dy \leq c_2L^{2-p(1+\alpha-\beta)}, \quad c_1, c_2 > 0. \tag{15}$$

Assume $\Theta \not\equiv 0$, then in view of (5) it suffices to prove the lower bound of (15). Supposing that the lower bound does not hold, then there exists a sequence $L_i > 0, i \in \mathbb{N}$, such that

$$\frac{1}{L_i^{2-p(1+\alpha-\beta)}}\int_{|y|\leq L}|\Theta(y)|^p dy \to 0, \text{ as } L_i \to \infty.$$

Setting $l_2 = L_i$, $\frac{\rho}{8}l_1 = L$ and taking $l_2 \to \infty$ in (9), we obtain

$$\frac{1}{L^{2-p(1+\alpha-\beta)}}\int_{|y|\leq L}|\Theta(y)|^p dy \leq c\int_{|y|\geq L}\frac{|\tilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy + c\int_{|y|\geq L}\frac{|\Theta(x)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy.$$

Now we proceed with similar arguments as in the previous case, namely by applying a dyadic decomposition together with bootstrapping method, and then arrive at the contradiction that $\Theta \equiv 0$.

### References


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