An AK-MS Conjecture for Simple Groups

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Summary

The main objective of this work was to study the classification of finite groups via the number of centralizers of subsets of two elements of the group. In particular, we study an AK-MS Conjecture for simple groups. At this poster I will be presenting a new result, that is a parcial response to the AK-MS Conjecture for some PSL groups.

Introduction

Based on the results of Ali Reza Ashrafi, Fatemeh Koorepazan-Moftakhar and Mohammad Ali Salahshour (AK-MS), in article [1], the motivation of this work was to study the importance of the number of centralizers of 2-element subsets in a finite group with a characterization for its structure. Particularly, we focus on Conjecture 5.4 [AK-MS], from which it was possible to obtain a new result. The GAP software was used in the calculations.

In this work, all groups present are finite. For a group G, we define

$$Cent(G) = \{C_G(x) \mid x \in G\},$$

$$2 - Cent(G) = \{C_G(\{x, y\}) \mid x, y \in G,\ x \neq y\},$$

where $C_G(\{x, y\}) = C_G(x) \cap C_G(y)$. G is called **n-centralizer** if $|Cent(G)| = n$. Also G is called **(2,n)-centralizer** if $|2 - Cent(G)| = n$.

Auxiliar Results

Theorem 1. *[AK-MS, 2020] Suppose G is a finite group.*

(1) There is no (2,4)-centralizer group.

(2) G is (2,5)-centralizer $\iff$ $G \cong S_3$ or $\frac{G}{Z(G)} \cong C_2 \times C_2$.

(3) G is (2,6) centralizer $\iff$ $G \cong A_8$ or $\frac{G}{Z(G)} \cong C_3 \times C_3$.

(4) G is (2,7)-centralizer $\iff$ $G \cong D_{10}$, R or G is 6-centralizer.

(5) G is (2,8)-centralizer $\iff$ G is 7-centralizer with non-trivial center.

(6) G is (2,9)-centralizer $\iff$ $G \cong D_{14}$, $Hol(C_7)$, a non-abelian group of order 21 or G is 8-centralizer with non-trivial center.

The main objective of this work was to study the following conjecture:

Conjecture 1. *[AK-MS, 2020] Suppose G and H are finite simple groups and $|2 - Cent(G)| = |2 - Cent(H)|$. Then $G \cong H$.*

Next, some fundamental results and definitions will be addressed to our new result.

Definition 1. *A group G is called **CA-group** when $C_G(x)$ is abelian, $\forall x \notin Z(G)$.*

Theorem 2. *[AK-MS, 2020]*

Let G be a CA-group. Then:

- *If $Z(G) = 1$, then $|2 - Cent(G)| = |Cent(G)|$.*
- *If $Z(G) \neq 1$, then $|2 - Cent(G)| = |Cent(G)| + 1$.*

Remark 1. *[J. Dutta, 2010] If $|Cent(PSL(2, q_1))| = |Cent(PSL(2, q_2))|$, then $q_1 = q_2$.*

Lemma 1. *[K. Khoramshahi, M. Zarrin, 2019] Let $nacent(G)$ be the set of all non-abelian centralizers of a group G. If $G = PSL(2, q)$, where $q > 5$ is a prime power. Then*

- *If $q \equiv 0 \ mod\ 4$, then $|nacent(G)| = 1$.*
- *If $q \equiv 1 \ mod\ 4$, then $|nacent(G)| = (q^2 + q + 2)/2$.*
- *If $q \equiv 2 \ mod\ 4$, then $|nacent(G)| = (q^2 - q + 2)/2$.*

Theorem 3. *[C. Jordan, 1838-1922] Let q be a prime power and $n > 1$ natural. So $PSL(n, q)$ is simple except for $(n, q) = (2, 2)$ or $2, 3)$.*

New Results

Main Theorem [Fernandes, Lima, 2022]. *Let $G = PSL(2, q_1)$ and $H = PSL(2, q_2)$, with $q_1, q_2 > 5$ and $q_1, q_2 \equiv 0 \ mod\ 4$ such that $|2 - Cent(G)| = |2 - Cent(H)|$. Then $G \cong H$.*

Proof. We have $G = PSL(2, q_1)$ with $q_1 > 5$ and $q_1 \equiv 0 \ mod\ 4$, so by Lemma 1, $|nacent(G)| = 1$. So G is a CA-group, and similarly H will be too. By Theorem 3, G and H are simple groups. Furthermore, $Z(G) = Z(H) = 1$, and by Theorem 2, we have $|2 - Cent(G)| = |Cent(G)|$ and $|2 - Cent(H)| = |Cent(H)|$. Then

$$|2 - Cent(G)| = |2 - Cent(H)| \implies |Cent(G)| = |Cent(H$$

By Remark 1,

$$|Cent(PSL(2, q_1))| = |Cent(PSL(2, q_2))| \implies q_1 = q_2.$$

$\therefore G \cong H.$

□

Conclusion

The study of centralizers of 2-element subsets is recent. The AK-MS paper brought the main properties of (2,n)-centralizer groups as well as a characterization of these groups for $n \leq 9$. By studying Conjecture 5.4 [AK-MS], it was possible to reach a new result that helps in the progress of the study of finite groups.

References

[1] A. R. Ashrafi, F. Koorepazan-Moftakhar, M. A. Salahshour. *Counting the number of centralizers of 2-element subsets in a finite group*, Communications in Algebra, (2020).

[2] J. Dutta. *A study of finite groups in terms of their centralizers*, M. Phil. thesis, North-Eastern Hill University (2010).

[3] K. Khoramshahi, M. Zarrin. *Groups with the same number of centralizers*, J. Algebra Appl, (2019).

[4] *GAP - Groups, Algorithms, Programming - a System for Computational Discrete Algebra* Disponível em: https://www.gap-system.org/. Acesso em: 19 de set. de 2022.

Acknowledgements

We thank Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and UnB for funding the research that resulted in this work.