Asymmetric affine Poincaré-Sobolev-Wirtinger inequalities on $BV(\Omega)$

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Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, with a Lipschitz boundary $\partial\Omega$ and $1 \le p \le \frac{n}{n-1}$, the Poincaré-Sobolev-Wirtinger inequality in $BV(\Omega)$ states that there exists $c = c(\Omega, p) > 0$ such that for every $v \in BV(\Omega)$, with $\int_\Omega v\,dx = 0$, the following is satisfied

$$c\|v\|_{L^p(\Omega)} \le |Dv|(\Omega). \quad (1)$$

In [2], it was studied an asymmetric version of (1) for $p = 1$, putting weights on the positive and negative part of the function. They showed that for each $r > 0$ there exists constants $\mu = \mu(r,\Omega) > 0$ and $\nu = \nu(r,\Omega) > 0$, with $\nu = r\mu$, such that for every $v \in BV(\Omega)$ with

$$\mu \int_\Omega v^+ dx - \nu \int_\Omega v^- dx = 0,$$

we have

$$\mu \int_\Omega v^+ dx + \nu \int_\Omega v^- dx \le |Dv|(\Omega).$$

In this work together with Marcos Montenegro, we study an asymmetric version of (1) replacing the classical total variation energy with the Zhang's affine energy associated with the class of bounded variation functions, $\mathcal{E}_\Omega$ that is defined as

$$(\mathcal{E}_\Omega(v))^{-n} := a_n \int_{\mathbb{S}^{n-1}} \left(\int_\Omega |\sigma_v(x)\cdot\xi|\, d(|Dv|)(x) \right)^{-n} d\xi,$$

where $a_n = (2\omega_{n-1})^n (n\omega_n)^{-(n+1)}$ and $\sigma_v : \Omega \to \mathbb{R}^n$ represents the Radon-Nikodym derivative of Dv with respect to its total variation $|Dv|$ on Ω. Recent studies on this energy can be seen in [1],[3] and [4].

We consider functions v such that $v^+ \ne 0$ and $v^- \ne 0$ in $L^p(\Omega)$, so it makes sense to talk about the ratio

$$r = \frac{\int_\Omega (v^+)^p dx}{\int_\Omega (v^-)^p dx} > 0.$$

As in [2] we aim to show that there exist constants $\mu = \mu(r,\Omega) > 0$ and $\nu = \nu(r,\Omega) > 0$, with $\nu = r\mu$, but this time for every $v \in BV(\Omega)$ with

$$\mu \int_\Omega (v^+)^p dx - \nu \int_\Omega (v^-)^p dx = 0,$$

we have

$$\mu \int_\Omega (v^+)^p dx + \nu \int_\Omega (v^-)^p dx \le (\mathcal{E}_{\mathbb{R}^n}(\bar{v}))^p, \quad (2)$$

where $\bar{v}$ is the extension of v outside of Ω such that $v = 0$ outside of Ω. Given a parameter $r > 0$ and the following definitions:

$$C_r := \left\{ v \in BV(\Omega) : \|v^+\|_p^p = \frac{1}{2}; \|v^-\|_p^p = \frac{1}{2r} \right\}$$

and $\mathcal{E}(v) := (\mathcal{E}_{\mathbb{R}^n}(\bar{v}))^p$, we can make the following characterization of the optimal constants μ and ν:

$$\mu(r) = \inf_{v \in C_r} \mathcal{E}(v) \text{ and } \nu(r) = \inf_{v \in -r^{\frac{1}{p}} C_r} \mathcal{E}(v). \quad (3)$$

We turn now to study the curve $\Gamma = \Gamma(\Omega)$ given by the functions $\mu(r) = \mu(r,\Omega)$ and $\nu(r) = \nu(r,\Omega)$, i.e.,

$$\Gamma = \{\gamma(r) = (\mu(r), \nu(r)) : r \in (0,+\infty)\}.$$

Main Results

Γ has the following properties for $n \ge 2$:

If $1 \le p < \frac{n}{n-1}$:

(i) The infimum in (3) are attained;

(ii) Γ is symmetric with respect to the diagonal;

(iii) Γ is strictly decreasing, meaning μ is strictly decreasing and ν is strictly increasing;

(iv) The curve Γ never crosses any of the axis, i.e. $\mu(r) > 0$ and $\nu(r) > 0$ for all $r > 0$;

(v) Γ is continuous, i.e., $r \mapsto \gamma(r)$ is continuous;

(vi) Γ has positive horizontal and vertical asymptotes.

If $p = \frac{n}{n-1}$, the items (ii),(iii),(iv) and (vi) are maintained and the functions μ and ν are upper semicontinuous.

If $p > \frac{n}{n-1}$ then the curves degenerates to a point, i.e. $\Gamma = \{(0,0)\}$.

If $n = 1$, we consider $\Omega = (-T, T)$ and we have $\mathcal{E}_\Omega(v) = |Dv|(\Omega)$, so it is natural to study the curve replacing $\mathcal{E}(v)$ with $(\mathcal{E}_\Omega(v))^p$ in (3), in this case we have that the curve is the graph of the function $f : \left(\frac{1}{4T}, +\infty\right) \to \left(\frac{1}{4T}, +\infty\right)$ defined as:

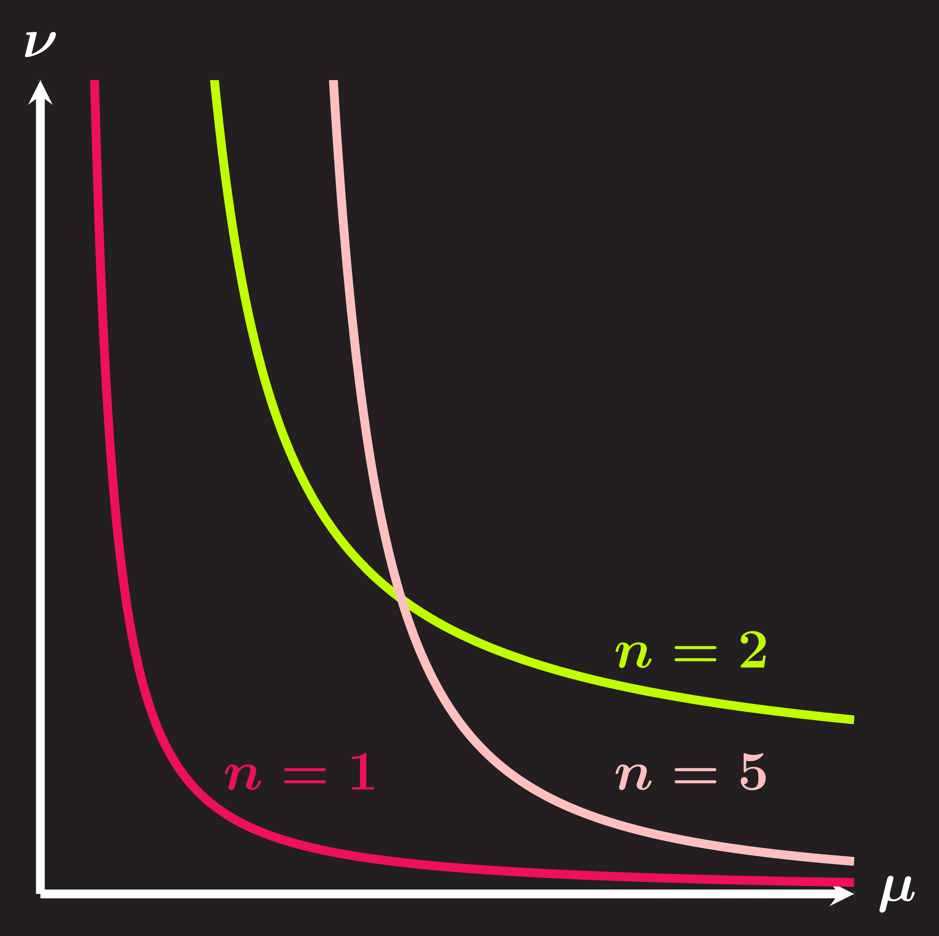
$$f(x) = \frac{x}{4T\left[x^{\frac{1}{p+1}} - \left(\frac{1}{4T}\right)^{\frac{1}{p+1}}\right]^{p+1}}.$$

The minimizers in this case are explicit and can be derived in an analogous way as done in [2]. If $n = 1$ and $v \in C_r$ such that $(\mathcal{E}_\Omega(v))^p = \mu(r)$, then $v = \varphi_r$ or $v = \varphi_r(2T - \cdot)$, where

$$\varphi_r(x) := \begin{cases} \frac{1}{(2(1+a_r)T)^{\frac{1}{p}}} & \text{for } x \in (-T, a_r T) \\ -\frac{1}{(2r(1-a_r)T)^{\frac{1}{p}}} & \text{for } x \in (a_r T, T) \end{cases},$$

with

$$a_r := \frac{r^{\frac{1}{p+1}} - 1}{r^{\frac{1}{p+1}} + 1}.$$



The curves for $n = 1, 2$ and 5

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Acknowledgments

I would like to express my sincere gratitude to CAPES for their generous financial support, which made this research possible. I would also like to express my sincere gratitude to IMPA for their support and for providing a stimulating environment that fosters intellectual growth and collaboration and a special thanks to my dedicated PhD advisor, Marcos Montenegro, for his invaluable guidance, support, and expertise throughout my research journey.