Nim on Graphs with multiple tokens

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Abstract

The game of Nim on Graphs [1] is played by moving a single token on a weighted undirected graph. Each player must move the token to a neighboring vertex following an edge with positive weight and reduce the weight of that edge in at least one. Following the normal play convention, the player who is not able to move the token in his turn loses. When all edges have unit weight, the game is equivalent to undirected edge geography, which is PSPACE-complete [2]. In this work, we introduce a variation where there are multiple tokens and show winning strategies for some simple unit-weight graphs.

1 Introduction

The game of Nim on Graphs is a variation of the game of Nim with the following rules. The game is played on an undirected graph with integer weights on its edges. A token is placed on one of its vertices. Two players take turns moving the token along an edge with positive weight and reducing its weight to some strictly smaller integer. The player that finds himself unable to play in his turn loses. An edge that reaches a non-positive weight may be removed from the graph.

We propose a generalization by allowing for k tokens to be placed in a vertex each. A player is not allowed to make a move that would place two tokens in the same vertex, or to move more than one token in his turn. For this game, two variations can be considered: a partial version where each player has a distinct set of tokens that may be moved and an impartial version where both players can move any of the tokens. In this work, we are concerned with the impartial variation. It is worth noting that if the graph consists of multiple connected components, each component may be considered an independent subgame, such that a player may only play in one subgame per turn.

For any impartial combinatorial game, a position can be said to be a winning position or p-position if there's some strategy that may be followed by the next player to always win the game. Similarly, a position can be said to be a losing position or 0-position if no move from the next player can lead to a victory against an optimal player. If a player is able to determine whether any position is a p-position, this may be used to determine his optimal moves.

Such positions may be identified by means of the Sprague-Grundy theorem. This is done by attributing to each position of the game a Grundy number. Positions that have no possible moves receive the Grundy number 0. The Grundy number of any other position is the least non-negative integer that is not the Grundy number of any position that can be reached in a single movement from it. If a position can be described as a set of independent subgames, then its Grundy number can also be calculated as the XOR sum of the Grundy numbers of each subgame. Calculating these values can take a long time. In fact, when all edges have unit weight, Nim on Graphs is equivalent to undirected edge geography, which is PSPACE-complete.

Thus, we are concerned with finding the Grundy numbers and some of the winning strategies for simple graphs where all edges have unit weight.

2 Paths with a single token

While our main interest is for multiple tokens, calculating the Grundy numbers for paths with a single token is also necessary, as some moves may divide the tokens in separate connected components. That said, consider a path with a single token in one of its vertices. By removing the vertex containing the token, we would divide the graph in two paths with x and y vertices each. For such a graph, the Grundy numbers are as presented in Table 1. Note that the values of x and y are interchangeable.

x	y	Grundy numbers
0	Odd	1
Even, $x>0$	Odd	2
Even	Even	0
Odd	Odd	1

Table 1: Grundy numbers for paths with 1 token.

3 Paths with two tokens

Now we consider the case of paths containing two tokens. Let x and y be the amount of vertices in the paths from a token to one of the extremities of the graph, such that the path does not contain a token and z be the amount of vertices in the path between the two tokens, such that the graph contains $n = x + y + z + 2$ vertices. Then, the Grundy numbers of such graph are as presented in Table 2. Note that the values of x and y are interchangeable.

x	y	z	Grundy numbers
0	0	Even	0
0	Even	0	0
0	Even, $y>0$	Even, $z>0$	3
Even, $x>0$	Even	Even	0
0	0	Odd	1
0	Even, $y>0$	Odd	2
Even, $x>0$	Even, $y>0$	Odd	0
0	Odd	Even, $z>0$	0
Even, $x>0$	Odd	Even, $z>0$	3
Even	Odd	0	1
0	Odd	Odd	1
Even, $x>0$	Odd	1	2
Even, $x>0$	Odd	Odd, $z>1$	1
Odd	Odd	Even	0
Odd	Odd	Odd	1

Table 2: Grundy numbers for paths with 2 tokens.

4 Cycles with two tokens

Now we consider the case of cycles containing two tokens. Let x and y be the amount of vertices in the two possible paths between the two tokens. Then the Grundy numbers of such graph are as presented in Table 3. Note that the values of x and y are interchangeable.

x	y	Grundy numbers
Even	Even	0
0	1	0
0	Odd, $y>1$	1
Even, $x>0$	1	1
Even, $x>0$	Odd, $y>1$	1
1	1	1
1	Odd, $y>1$	0
Odd, $x>1$	Odd, $y>1$	0

Table 3: Grundy numbers for cycles with 2 tokens.

References

[1] Masahiko Fukuyama. A nim game played on graphs. *Theoretical Computer Science*, 304(1-3):387–399, 2003.

[2] Aviezri S Fraenkel, Edward R Scheinerman, and Daniel Ullman. Undirected edge geography. *Theoretical Computer Science*, 112(2):371–381, 1993.