# KCC Theory Applied to Geodesic Sprays


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### Introduction

Let $(x^1, \dots, x^n) = (x)$, $\left(\frac{dx^1}{dt}, \dots, \frac{dx^n}{dt}\right) = (\dot{x})$ be $2n$ coordinates in an open connected subset $\Omega$ of the Euclidean $(2n)$-dimensional space $\mathbb{R}^n \times \mathbb{R}^n$. For our purpose, suppose that we have

$$\frac{d^2x^i}{dt^2} + 2g^i(x, \dot{x}) = 0, \qquad i = 1, \dots, n, \tag{1}$$

where each $g^i$ is $C^\infty$ in some neighborhoof of initial conditions $((x_0), (\dot{x}_0)) \in \Omega$. We call (1) a spray. The intrinsic geometry properties of (1) under non-singular tranformations of the type:

$$\begin{cases} \tilde{x}^i = f^i(x^1, \dots, x^n), & i = 1, \dots, n, \\ \bar{t} = t \end{cases} \tag{2}$$

are given by the five KCC-differential invariants, which are:

$$\frac{\mathbb{D}\xi^i}{dt} = \epsilon^i = \frac{1}{2} g^i_{;r} \dot{x}^r - g^i \tag{3}$$

defining the first KCC-invariant of (1), the contravariant vector field on $\Omega$, $\epsilon^i$, which represents an 'external force'. The tensor $\mathcal{P}^i_j$ given by

$$\mathcal{P}^i_j = -g^i_{,j} - \frac{1}{2} g^r g^i_{;r;j} + \frac{1}{2} \dot{x}^r g^i_{,r;j} + \frac{1}{4} g^i_{;r} g^r_{;j} \tag{4}$$

is the second KCC-invariant of (1). The third, fourth and fifth invariants are:

$$\begin{cases} \mathcal{R}^i_{jk} = \frac{1}{3}(\mathcal{P}^i_{j;k} - \mathcal{P}^i_{k;j}) \\ \mathcal{B}^i_{jkl} = \mathcal{R}^i_{jk;l} \\ \mathcal{D}^i_{jkl} = g^i_{;j;k;l} \end{cases} \tag{5}$$

where the semi-colon indicates partial differentiation with respect to $\dot{x}^r$ and the comma indicates partial differentiation with respect to $x^r$, [1].

**Theorem 1** ([2]). *Two system of the form (1) on $\Omega$ are equivalent relative to (2) if and only if the five KCC-invariants are equivalent. In particular, there exist coordinates $(\bar{x})$ for which $g^i(\bar{x}, \dot{\bar{x}}, t)$ all vanish if and only if all KCC-invariants are zero.*

**Definition 1.** *Let $\gamma(t) = \left(x^i(t)\right) \in U \subset \Omega$ be a path of (1). If any other path with initial conditions close enough at $t = t_0$ remains close to $\gamma(t)$ for all $t > t_0$, we say that $\gamma(t)$ is a trajectory Jacobi stable. We define (1) to be Jacobi stable if all its solutions are Jacobi stable. Otherwise, we say that (1) is Jacobi unstable.*

**Theorem 2** ([3]). *The trajectories of (1) are Jacobi stable if and only if the real part of the eigenvalues of the tensor $\mathcal{P}^i_j$ are strictly negative everywhere, and Jacobi unstable, otherwise.*

**Observation 1** ([4]). *If $F : T\Omega \to R$ is a two-dimensional Finsler metric, its Euler-Lagrange equations generate a spray (1) and we can rewrite the third KCC-Invariant in terms of the Berwald's Gaussian curvature $\mathcal{K}$*

$$\mathcal{R}^i_{jk} = F\mathcal{K}m^i\left(l_j m_k - l_k m_j\right) \tag{6}$$

**Proposition 1.** *If the curvature $\mathcal{K}$ is bigger then zero everywhere, then trajectories oscillate back and forth, crossing the reference trajectory. In this case, we say (1) is Jacobi stable. If $\mathcal{K} \leq 0$ everywhere, trajectories diverge and system (1) is Jacobi unstable.*

### Objective

The aim of this work is to compare the sprays obtained by the Finsler metric:

$$F(x, \dot{x}) = F(x, y) = e^{\psi\left(x^1, x^2\right)} \frac{\left(y^2\right)^{1+(1/\lambda)}}{\left(y^1\right)^{1/\lambda}}$$

where $\psi(x^1, x^2) = -\alpha_1 x^1 + (\lambda + 1)\alpha_2 x^2 + \Psi(x^1, x^2)$, and in the case when $\Psi \equiv 0$ and $\Psi = \nu_3 x^1 x^2$. The respectively geodesics are:

$$\begin{aligned} dy^1/ds + \lambda\alpha_1 (y^1)^2 = 0 \\ dy^2/ds + \lambda\alpha_2 \left(y^2\right)^2 = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} dy^1/ds + \lambda\left(\alpha_1 - \nu_3 x^2\right)\left(y^1\right)^2 = 0 \\ dy^2/ds + \lambda\left(\alpha_2 + \frac{\nu_3}{\lambda + 1} x^1\right)\left(y^2\right)^2 = 0. \end{aligned} \tag{8}$$

In the first case, the curvature $\mathcal{K} = 0$, while the $\nu_3 \neq 0$ case is:

$$\mathcal{K} = \nu_3 \cdot \frac{\lambda^2}{\lambda + 1}\left(\frac{y^1}{y^2}\right)^{1+2\lambda} e^{-2\left[-\alpha_1 x^1 + (\lambda+1)x^2 + \nu_3 x^1 x^2\right]} \tag{9}$$

### Results

**Proposition 2.** *The geodesics (7) are always unstable, and the the geodesics (8) are stable if $\nu_3 > 0$ in (9).*

**Proposition 3.** *The sprays (7) and (8) are not equivalent. Moreover, there is no coordinates $(\bar{x})$ such that*

$$\begin{aligned} d\bar{y}^1/ds + \lambda\left(\alpha_1 - \nu_3 \bar{x}^2\right)\left(\bar{y}^1\right)^2 = 0 \\ d\bar{y}^2/ds + \lambda\left(\alpha_2 + \frac{\nu_3}{\lambda + 1}\bar{x}^1\right)\left(\bar{y}^2\right)^2 = 0. \end{aligned}$$

*although the coordinates $\tilde{x}^1 = \frac{1}{\lambda\alpha_1} e^{\lambda\alpha_1 x^1}$, $\tilde{x}^2 = \frac{1}{\lambda\alpha_2} e^{\lambda\alpha_2 x^2}$ makes*

$$\begin{aligned} d\tilde{y}^1/ds + \lambda\alpha_1\left(\tilde{y}^1\right)^2 = 0 \\ d\tilde{y}^2/ds + \lambda\alpha_2\left(\tilde{y}^2\right)^2 = 0 \end{aligned}$$

*Proof.* The 5 KCC-invariants of the spray (7) are all zero. The second KCC-invariant of the spray (8) can not be zero, because this system is Jacobi stable. □

### Conclusion

- The Finsler metric $F$ has an important meaning in the ecological applications. This metric represents the cost of production between two species, and the perfect one is when these species have their interaction represented by the geodesics of this functional $F$.
- KCC-Theory is fundamental to analyze two different sprays and say when they are equivalent, it is important because sometimes one of them is easier to obtain informations in many biological applications.

### References


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