Hausdorff dimension and injective orthogonal projections in Hilbert spaces

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Abstract

This poster presents new findings regarding how Hausdorff dimension relates to the existence of injective embeddings of compact subsets of Euclidean finite dimensional spaces into lower-dimensional spaces. We first introduce a version of Mañe's theorem for finite dimensional spaces and orthogonal projections, showing that the proof in this case is more intuitive, and the hypothesis over the Hausdorff dimension is improved and becomes optimal. We also show that information on Hausdorff dimension is not enough to achieve Hölder continuity on the inverse of the injective projections, but the box-counting dimension can be used in this sense.

Introduction

Definition: An **orthogonal projection** P in $\mathbb{R}^n$ is a mapping that has the form $Px = \langle x, p_1\rangle p_1 + \cdots + \langle x, p_k\rangle p_k$ for any $x \in \mathbb{R}^n$, for some basis $\{p_1, \ldots, p_n\}$ and $k \leq n$.

Definition: The **Hausdorff dimension** of the set K is given by:

$$\dim_{\mathrm{H}}(K) = \inf\left\{d \geq 0 : \mathcal{H}^d(K) = 0\right\}$$

where $\mathcal{H}^d(K)$ is the **d-dimensional Hausdorff measure** of K (the "volume" of K in a d-dimensional Hilbert space).

Definition: The **box-counting dimension** of a compact set K is defined as

$$\dim_{\mathrm{B}}(K) = \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{\log\left(\frac{1}{\varepsilon}\right)}$$

where $N(\varepsilon)$ is the minimum number of boxes of size ε needed to cover the set A.

Results

First of all, we proved that if K is a compact set in $\mathbb{R}^n$, and $q \in \mathbb{S}^{n-1}$ then the orthogonal projection P_q of kernel $[q]$ given by $P_q x = x - \langle x, q\rangle q$ for all $x \in \mathbb{R}^n$ is injective in K if, and only if, $[q]$ only intersects $K - K$ at the origin. Moreover, the projection P_q is injective in K with a θ-Hölder continuous inverse $(P|_K)^{-1}$ if, and only if,

$$K - K \subset \{z_1 p_1 + \ldots z_{n-1}p_{n-1} + z_n q : z_n^2 \leq +\sqrt{C R_{n-1}^{2\theta} - R_{n-1}^2}\} \qquad (1)$$

where $R_{n-1} = \sqrt{z_1^2 + \cdots + z_{n-1}^2}$, for some $C > 0$.

Finally, the projection P_q is injective in $K - K$ with a Lipschitz inverse $(P|_K)^{-1}$ if, and only if, there is a cone centered at $[q]$ that only intersects $K - K$ at the origin.

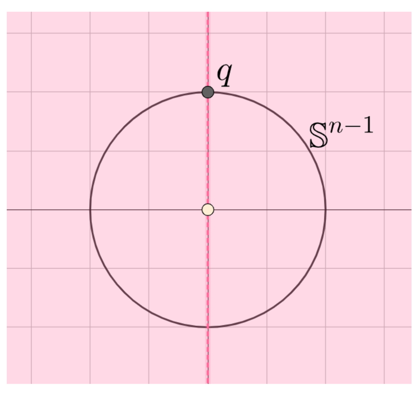

Figure 1: If $K - K$ is inside the light pink region, then P_q is 1-1 in K.

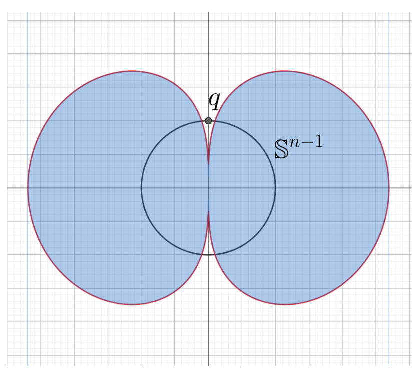

Figure 2: If $K - K$ is in the blue region, then P_q is 1-1 in K with Hölder inverse.

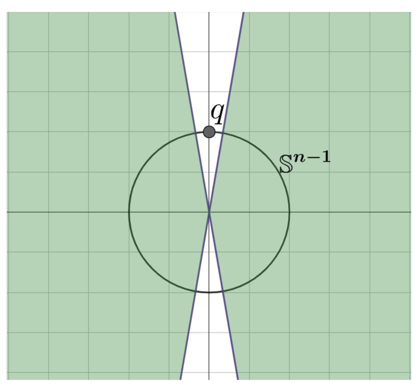

Figure 3: If $K - K$ is in the green region, then P_q is 1-1 in K with Lipschitz inverse.

Theorem: Let $K \subset \mathbb{R}^n$ be a compact set such that $\dim_H(K - K) < k \in \mathbb{N}$, then injectivity in K is a generic property (holds in a residual set) in the set of orthogonal projections in $\mathbb{R}^n$ of rank k.

Remark: This can be extended to infinite dimensional Hilbert spaces, providing an improvement in the hypothesis $\dim_{\mathrm{H}}(K - K) < k - 1$ in Mañé's original theorem.

Idea of the proof: We will give an idea of how to prove that there exists an orthogonal projection of rank k injective in K. We prove for $k = n - 1$, and the general case can be proved by induction. Notice that if ψ is the normalization in $\mathbb{R}^n$, and

$$A_j = \{z \in K - K : \|z\| \geq 1/j\},$$

$$\psi((K - K) \setminus \{0\}) = \bigcup_{j=1}^{\infty} \psi(A_j).$$

Then, using the countable stability of Hausdorff dimension, and the fact that ψ is Lipschitz in A_j, we have:

$$\dim_{\mathrm{H}}(\psi((K - K) \setminus \{0\})) = \sup_{j\in\mathbb{N}} \dim_{\mathrm{H}}(\psi(A_j))$$

$$\leq \sup_{j\in\mathbb{N}} \dim_{\mathrm{H}}(A_j) \leq \dim_{\mathrm{H}}(K - K) < n - 1.$$

Then, $\psi((K - K) \setminus \{0\}) \neq \mathbb{S}^{n-1}$. There exists $q \in \mathbb{S}^{n-1}$ such that $q \notin \psi((K - K) \setminus \{0\})$, so that $[q]$ only intersects $K - K$ at the origin, and P_q is injective in K.

Example: The hypothesis $\dim_H(K - K) < k$ cannot be weakened. Indeed, consider the following example. By countable stability, $\dim_{\mathrm{H}}(K - K) = 1$, but there is no line $[q]$ that only intersects $K - K$ at the origin.

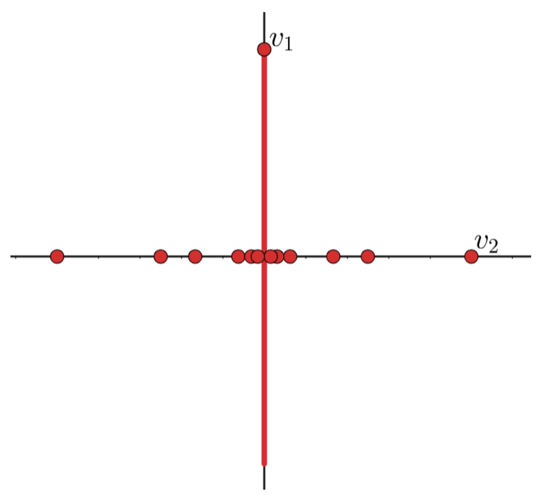

Figure 4: The set K

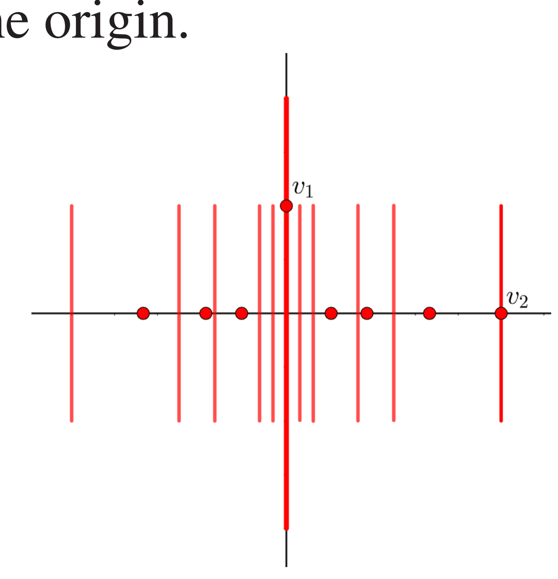

Figure 5: The set K-K

Example: We can find an example of a compact set K such that $K - K$ has zero Hausdorff dimension but no orthogonal projection of rank $n - 1$ is injective in K with Hölder inverse. We can also find an example such that $K - K$ has zero box-counting dimension but no orthogonal projection of rank $n - 1$ is injective in K with Lipschitz inverse.

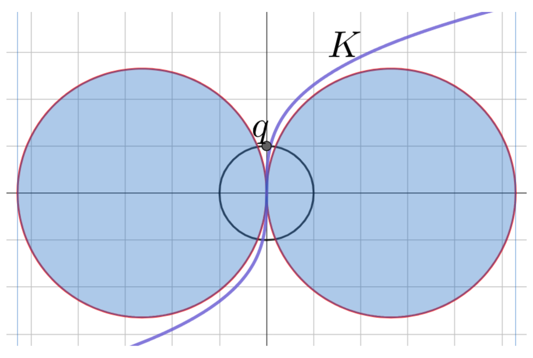

Figure 6: Idea for the Hölder example: K can be taken to be a countable set going to zero in the blue curve.

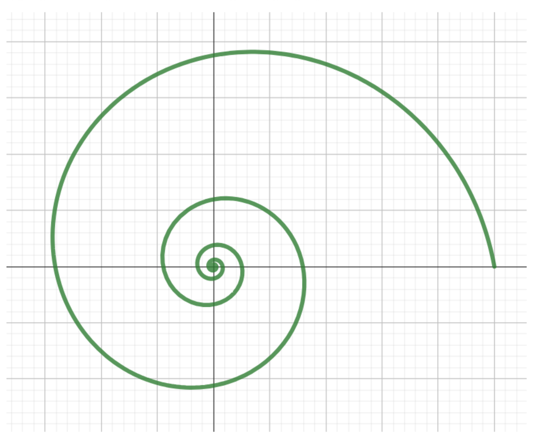

Figure 7: We can chose a countable set K in the green curve so that $\dim_{\mathrm{B}}(K) = 0$ and no orthogonal projection has Lipschitz inverse.

Conclusion

Results:

- $\dim_{\mathrm{H}}(K - K) < k$ implies that there exist orthogonal projections of rank k injective in K.
- Hausdorff dimension is not enough for Hölder inverse. Box-counting dimension can give Hölder inverse, but not Lipschitz inverse.

Open problems:

- Is there a computable condition on $K - K$ that implies the existence of projections with Lipschitz inverse $(P|_K)^{-1}$?
- Is there a condition on $K - K$ not related to box-counting dimension that implies the existence of orthogonal projections injective in K with Hölder continuous inverse?

References

[1] Ricardo Mañé. On the dimension of the compact invariant sets of certain non-linear maps. Springer Berlin Heidelberg, 1981.

[2] James C. Robinson. *Dimensions, Embeddings, and Attractors*. Cambridge University Press, 2010.

Agradecimentos

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