Images of multilinear graded polynomials on upper triangular matrices

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Abstract

In 2022, Gargate and de Mello classified the images of multilinear polynomials on upper triangular matrices. The aim of this poster is to present a graded version of Gargate and de Mello's result. More precisely we provide a classification of the images of multilinear graded polynomials on upper triangular matrices endowed with certain $\mathbb{Z}_q$-gradings. We also consider the cases of 2×2 and 3×3 upper triangular matrices, and we show that the image of multilinear graded polynomials on these algebras are always a homogeneous vector space.

Introduction

Images of polynomials on algebras appear in many branches of *Ring Theory*, for instance in algebras with polynomial identities (*PI-algebras*). Nowadays, the main problem concerning such objects is perhaps the following conjecture:

Conjecture 1 (L'vov-Kaplansky)**.** The image of a multilinear polynomial on the full matrix algebra $M_n(F)$ is a vector space.

The conjecture above is equivalent to say that the image is one of the four options: $\{0\}$, F (the space of *scalar matrices*), $sl_n(F)$ (the space of *traceless matrices*) or $M_n(F)$.

Despite several tries in having a complete solution of the conjecture above (sucessfully done in small cases), variations of it have been considered in the past few years. For instance, we mention

Theorem 2 (Gargate, de Mello - 2022)**.** *Let F be an infinite field. Then the image of a multilinear polynomial on the upper triangular matrix algebra $UT_n(F)$ is either $UT_n(F)$ or some power of J (the Jacobson radical of UT_n).*

In next we introduce the main definitions and tools for presenting a graded version of Gargate and de Mello's theorem.

Preliminares

We start by recalling that a *grading* on an algebra $\mathcal{A}$ by a group G is a decomposition $\mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$ of $\mathcal{A}$ into a direct sum of vector spaces such that $\mathcal{A}_g\mathcal{A}_h \subset \mathcal{A}_{gh}$, for all $g,h\in G$.

- $\mathcal{A}_g$ is called *homogeneous component* of degree g;
- $a\in\mathcal{A}_g \leftrightarrow deg(a) = g$.

A subspace $\mathcal{V}\subset\mathcal{A}$ is said *homogeneous* if it inherits the grading from $\mathcal{A}$.

Example 3. Let $X = \{x_i^{(g)} | g\in G, i = 1,2,\dots\}$ and set $F\langle X|G\rangle$ as the free associative algebra generated by X. This algebra has a G-grading defined by setting the homogeneous component of degree g as the subspace spanned by monomials of the form

$$x_1^{(g_1)}\cdots x_m^{(g_m)} \text{ where } g_1\cdots g_m = g.$$

The elements from $F\langle X|G\rangle$ are called *G-graded polynomials*.

Example 4. $f = f(x_1^{(g_1)},\dots,x_m^{(g_m)}) \in F\langle X|G\rangle$ is said multilinear if

$$f = \sum_{\sigma\in S_m}\alpha_\sigma x_{\sigma(1)}^{(g_{\sigma(1)})}\cdots x_{\sigma(m)}^{(g_{\sigma(m)})} \quad (\alpha_\sigma\in F).$$

Example 5. A sequence $(g_1,\dots,g_n)\in G^n$ defines a G-grading on UT_n by taking $deg(e_{ij}) = g_i^{-1}g_j$, where e_{ij} is the standard matrix unit. This grading is called *elementary* G-grading induced by the sequence $(g_1,\dots,g_n)$.

Let $q\le n$. Taking $G=\mathbb{Z}_q$ and

$$(\bar{0},\bar{1},\dots,\overline{q-2},\overline{q-1},\dots,\overline{q-1})$$

in the previous example we have

$$UT_n = \bigoplus_{\bar{l}\in\mathbb{Z}_q}\mathcal{A}_{\bar{l}} \qquad (1)$$

where each $\mathcal{A}_{\bar{l}}$ can be regarded as

$$\begin{pmatrix} A & C\\ 0 & B\end{pmatrix},$$

$A\in UT_{q-1}, B\in UT_{n-q+1}$ and $C\in M_{q-1,n-q+1}$. Precisely,

- $\mathcal{A}_{\bar{0}}$ - A diagonal matrix, B arbitrary and $C=0$;
- $\mathcal{A}_{\bar{l}}$ - A is $(l+1)$-diagonal matrix, $B=0$ and C is $(q-l)$-row matrix.

Definition 6. Let $f = f(x_1^{(g_1)},\dots,x_m^{(g_m)}) \in F\langle X|G\rangle$ and let $\mathcal{A}$ be an F-algebra. The *image* of f on $\mathcal{A}$ is defined as

$$f(\mathcal{A}) = \{f(a_1,\dots,a_m) | a_i\in\mathcal{A}_{g_i}\}.$$

Main results

For $r = 1,\dots,n-q$, we set the following homogeneous subspaces from $\mathcal{A}_{\bar{l}}$:

$$\mathcal{B}_{\bar{l},r} = span\{e_{q-l,j} | j = q+r,\dots,n\}$$

Theorem 7. *Let F be an infinite field. Then the image of a multilinear f on UT_n endowed with the grading* (1) *is one of the following homogeneous subspaces: $\{0\}, J^r, \mathcal{B}_{\bar{l},r}$ or $\mathcal{A}_{\bar{l}}$, where J stands for the Jacobson radical of the neutral component.*

Corollary 8. *Let $char(F) = 0$ and let $f\in F\langle X\rangle$ be a (non-graded) multilinear polynomial. Assume that*

$$f(x_1^{(0)},\dots,x_{m-1}^{(0)},x_m^{(i)})$$

is not an identity for UT_n with the $\mathbb{Z}$-natural grading, $i = 1,\dots,n-1$. Then $f(M_n(F))\supset sl_n(F)$.

Example 9. Take the polynomial

$$f = \sum_{\sigma\in S_{m-1}}\alpha_\sigma[x_m, x_{\sigma(1)},\dots,x_{\sigma(m-1)}]$$

where $\sum\alpha_\sigma\neq 0$. Then $f(M_n(F)) = sl_n(F)$. In particular, we recover the old result from Shoda about commutators in matrix algebras.

By a well known result from Valenti and Zaicev, *gradings on UT_n are essentially elementary*. Hence the following result tells us that in dealing with images of graded polynomials on UT_n, one should only care about the elementary gradings.

Proposition 10. *Let $f\in F\langle X|G\rangle$. If $f(\mathcal{A})$ is a homogeneous vector space, then the same holds for any graded homomorphic image of $\mathcal{A}$.*

As a consequence we have

Theorem 11. *The image of a multilinear graded polynomial on UT_2 is always a homogeneous vector space. Analogous conclusion for the graded algebra UT_3. In case UT_3 is endowed with the trivial grading we additionally require $|F|\ge 3$.*

References

[1] I. Gargate, T. de Mello, *Images of multilinear polynomials on $n\times n$ upper triangular matrices over infinite fields*, Israel J. Math. **252** (2022), 337–354.

[2] P. Fagundes, P. Koshlukov. *Images of multilinear graded polynomials on upper triangular matrix algebras*, accepted in Canad. J. Math.

Funding

The first author was supported by FAPESP Grant No. 2019/16994-1. The second author was supported by FAPESP Grant No. 2018/ 23690-6 and CNPq Grant No. 302238/2019-0.