Heintze-Karcher and Jellett type theorems in conformally flat spaces

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Abstract

In this work, we consider a class of conformally flat spaces that includes the space forms as well as the doubled Schwarzschild space, and, we will prove an extension of Jellett's theorem. Next, we build an example where the extension of Jellett's theorem does not hold. We also prove a Heintze-Karcher type inequality and we presented a 1-parametric family of conformally flat spaces, all distinct from space forms, where it holds a Heintze-Karcher type inequality.

Introduction

In the middle of the eighteenth century Jellett [7] pioneered that a star-shaped constant mean curvature surface $M \subset \mathbb{R}^3$ is a round sphere. Around one century after, Hopf [5] presented a generalization of this result showing that an immersed constant mean curvature surface $M \subset \mathbb{R}^3$ with the topology of a sphere is also a round sphere. In the eighties of the last century Hsiang, Teng and Yu [6] presented a new class of spherical immersed hypersurfaces of constant mean curvature in $\mathbb{R}^{2n}$ that are not round spheres. On the other hand, for hypersurfaces contained in the Euclidean sphere $\mathbb{S}^{n+1}$ we have a lot of examples of compact hypersurfaces with constant mean curvature which are not round spheres (Clifford Torus). However, keeping in mind Jellett's idea, Barros and Sousa, in [2], obtained a similar result in the Euclidean sphere $\mathbb{S}^{n+1}$, deducing that star-shaped hypersurfaces with constant mean curvature are round spheres.

We explore in this work star-shaped hypersurfaces in conformally flat spaces, whose conformal factor is radial and satisfies an appropriate geometric constraint. For this end, let $\mathfrak{B}^{n+1} = (\mathbb{B}^{n+1}_a, \overline{g})$ be a Euclidean ball with radius $a \leq \infty$, centred at the origin $O \in \mathbb{R}^{n+1}$ endowed with a metric $\overline{g} = e^{2h}g$ conformal to the canonical Euclidean metric g. We assume that $\mathbb{B}^{n+1}_\infty = \mathbb{R}^{n+1}$. Moreover, we will suppose that $h : \mathbb{B}^{n+1}_a \to \mathbb{R}$ is a radial function given by $h(x) = u(|x|^2)$ for some smooth function u defined on $I = [0, a^2)$, satisfying

$$u''(t) - \big(u'(t)\big)^2 \geq 0. \qquad (1)$$

Theorem 1. *Let $M^n \subset \mathfrak{B}^{n+1}$ be a compact star-shaped hypersurface of constant mean curvature. If u satisfies (1), then M^n is totally umbilical.*

In 1956 Alexandrov [1] proved that a compact embedded constant mean curvature hypersurface $M^n \subset \mathbb{R}^{n+1}$ is also a round sphere. Latter, Montiel and Ros [8] gave an alternative proof of Alexandrov's theorem based on ideas of Heintze and Karcher [4]. More recently, in order to study Alexandrov type rigidity phenomena in general relativity, Brendle [3] obtained a Heintze-Karcher type inequality in a large class of warped product spaces, including the Schwarzschild space.

Before to announce the next result, we remember that if $\overline{g} = e^{2h}g$ as well as V is the position vector field in $\mathbb{R}^{n+1}$, then we have

$$\mathcal{L}_V\overline{g} = 2\sigma\overline{g}, \quad \sigma(x) = 1 + 2u'(|x|^2) \mid x \mid^2.$$

Theorem 2. *Let $\Omega \subset \mathfrak{B}^{n+1}$ be a compact domain with smooth boundary M. Let H be the normalized mean curvature of M, then we have*

$$(n+1)\int_\Omega \sigma \, d\Omega \leq \int_M \frac{\sigma}{H} dM, \qquad (2)$$

provided that $u''(t) - \big(u'(t)\big)^2 = \lambda(t)$, for $t \in [0, a^2)$, $H > 0$, $\sigma > 0$ and $\lambda(t)$ satisfies

$$\lambda(t) \geq 0 \text{ and } t\lambda'(t) + 2\lambda(t) \geq 0.$$

Moreover, the equality in (2) holds if and only if M^n is totally umbilical.

Class of examples

Example 1. *It is well known that this class of manifolds includes the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ e the punctured sphere $\mathbb{S}^{n+1} \setminus \{p\}$.*

Example 2. *Given $a > 0$ consider the function $u(t) = e^{-at}$, $t \geq 0$. Taking into account that $u''(t) - \big(u'(t)\big)^2 > 0$, for all $t > 0$, we obtain a family of complete conformally flat spaces $\mathfrak{B}^{n+1}_a = (\mathbb{R}^{n+1}, \overline{g} = e^{2u(|x|^2)}g)$, all distinct from space forms, where by Theorem 1 holds an extension of Jellett's theorem.*

Example 3. *Consider $\overline{M} = \mathbb{R}^n \setminus \{0\}$ equipped with the metric*

$$\overline{g}_{ij} = \left(1 + |x|^{2-n}\right)^{\frac{4}{n-2}} g_{ij}.$$

The manifold $(\overline{M}, \overline{g})$ is called the doubled Schwarzschild space. Considering the function $u : (0, \infty) \to \mathbb{R}$ defined by

$$u(t) = \frac{2}{n-2}\ln\left(1 + t^{\frac{2-n}{2}}\right),$$

it is easy to see that $\overline{g} = e^{2u(|x|^2)}g$ and $u''(t) > \big(u'(t)\big)^2$.

Example 4. *Given $a > 0$, let $\lambda : [0, \frac{2}{a}) \to \mathbb{R}$ be defined by $\lambda(t) = a^2(e^{-at} - e^{-2at})$. Note that $\lambda(t)$ satisfies the hypotheses of Theorem 2 and the function $u(t) = e^{-at}$ solves the equation $u''(t) - \big(u'(t)\big)^2 = \lambda(t)$. With this, we have a 1-parametric family of conformally flat spaces, all distinct from space forms, where it holds a Heintze-Karcher type inequality.*

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