Production Cost Functionals in Adaptation Models to Environmental Changes

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Introduction

We model the dynamics of biological phenomena by simplifying them into basic mechanisms such as: growth of the populations N^i, interaction between species, external influences and productivity.

We use Volterra Hamilton (VH) systems,

$$\begin{cases} \dfrac{dx^i}{dt} = k_{(i)} N^i \\ \dfrac{dN^i}{dt} = -G^i_{jk} N^j N^k + r^i_j N^j + e^i \end{cases}, \tag{1}$$

for $i, j, k = 1, \cdots, n$, combined with the production dynamics $x^i(t) = k_{(i)} \int_0^t N^i(\tau) d\tau + x(0)$ of Volterra [3] and the population growth dynamics of Hutchinson [2].

We consider that a cost is generated from productivity, and that such cost is given as a function of the size of the population and the production, hence, we introduce Finsler metrics $F(x^i, N^i)$ to calculate such cost.

Definition 1. *A Finsler metric is a function $F : TM \to \mathbb{R}$ such that:*

- *(i) F is C^∞ in $TM - \mathbb{O}$ and continuous in the null section $\mathbb{O}$;*
- *(ii) F is positive definite at $TM - \mathbb{O}$;*
- *(iii) F is p-homogeneous of degree 1 in the second variable;*
- *(iv) The matrix with coefficients $g_{ij}(x, y) = \dfrac{1}{2} \dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite at $TM - \mathbb{O}$.*

From the homogeneity of F, the cost is proportional. Furthermore, if a dynamic is represented by the Euler-Lagrange equations of F, then the process is optimized.

Starting with a constant environment, that is, $e^i = 0$, by parameter change, the system (1) becomes a spray

$$\frac{d^2x^i}{ds^2} + G^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{2}$$

with $y := \dfrac{dx}{ds}$.

In biological processes of development or evolution, which have a predetermined sequence or not, each stage is approached through a VH system (1).

The passage to a next stage is understood as the introduction of a projective change

$$G^i(x, y) \to \bar{G}^i(x, \xi) = G^i(x, \xi) + \psi(x, \xi) \xi^i,$$

which turns the spray (2) into projective spray

$$\frac{d^2x^i}{dp^2} + \bar{G}^i_{jk} \frac{dx^j}{dp} \frac{dx^k}{dp} = 0, \tag{3}$$

for $\bar{G}^i_{jk}(x, \xi) := G^i_{jk}(x, y) + \delta^i_j \psi_k + \delta^i_k \psi_j + y^i \psi_{jk}$, $\psi_j := \dfrac{\partial \psi(x, y)}{\partial y^j}$, $\psi_{jk} := \dfrac{\partial \psi_j}{\partial y^k}$, $\xi := \dfrac{dx}{dp}$ and $\psi(x, \xi)$ is function p-homogeneus of degree 1 in ξ.

Objective

The structure of VH systems ensure that the growth of a specie is influenced by interaction with other species, and so arises a question: are there other relations between the basic mechanisms we are considering? We seek to response that question!

Results

Theorem 1 (Adaptation Theorem)**.** *Consider a local spray (2) and a Finsler norm F such that $\dfrac{dF}{ds} = 0$ along (2). A conform norm $\bar{F} = e^{\phi_k(x) x^k} F$ satisfy $\dfrac{d\bar{F}}{dp} = 0$ along of the projective spray (3) if, and only if, the function of the projective transformation is*

$$\psi(x, y) = \frac{1}{2} \phi_k(x) y^k.$$

In the Adaptation Theorem, if (2) is the geodesic spray of a Berwald space (M, F) and $\phi_k(x) = \sigma_k$ with σ_k constants, the projective spray solutions are autoparallel curves of a Wagner space $(M, \bar{F})$ with Wagner connection

$$\bar{W}\Gamma = \left(N^i_j + y^i \sigma_j, \Gamma^i_{jk} + \delta^i_j \sigma_k, C^i_{jk} \right),$$

where $N^i_j, \Gamma^i_{jk}, C^i_{jk}$ are they coefficients of the Cartan connection of (M, F).

For dynamics between two species, the Finsler Gate derived from the classifications of Antonelli and Matsumoto [1] present Finsler metrics apropprietad:

$$F_1 = \frac{(y^2)^{\frac{\lambda+1}{\lambda}}}{(y^1)^{\frac{1}{\lambda}}} e^{-\alpha_1 x^1 + (\lambda+1)\alpha_2 x^2 + \nu_3 x^1 x^2},$$
$$F_2 = y^1 e^{\frac{y^2}{y^1} + (\beta_1+\beta_2)x^1 + \beta_1 x^2 + \nu_3 (x^2)^2},$$
$$F_3 = \sqrt{(y^1)^2 + (y^2)^2}\, e^{\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2} \arctan\left(\frac{y^1}{y^2}\right) + \frac{(\gamma_1)^2+(\gamma_2)^2}{\gamma_1+\gamma_2}(x^1+x^2) + \nu_1 (x^1)^2 + \nu_2 (x^2)^2}, \tag{4}$$

with $\lambda > 0$ e $\alpha_i, \beta_i, \gamma_i, \nu_i \in \mathbb{R}, i = 1, 2$.

Conclusion

- The Adaptation Theorem ensure that the incorporation of an external influences of the gradient type is equivalent the passage from present stage to a new stage of the process;
- The Finsler metrics obtained by Finsler Gate represent famous dynamics in the literature.

References

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