On a Hurwitz problem

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Abstract

In [1], A. Hurwitz characterises the existence of a branched covering $\phi : M \to N$ of degree d between closed surfaces, when N and the *branch datum* are fixed. This existence problem is not completely solved when $N = \mathbb{S}^2$. In this work we study a relation between this case and the case $N = \mathbb{R}P^2$.

1 Introduction

A branched covering $\phi : M \to N$ of degree d between surfaces determines a finite collection $\mathcal{D}$ of partitions of d, *the branch datum*, in correspondence with the *branch point set* $B_\phi \subset N$. If $\#B_\phi = k$, we denote the branched covering by (M, ϕ, N, k, d).
The *total defect* of (M, ϕ, N, k, d) is a positive integer $\nu(\mathcal{D})$ characterised by the properties:

$$(*) \quad \nu(\mathcal{D}) = d\chi(N) - \chi(M) \equiv 0 \mod 2.$$

If $N \neq \mathbb{S}^2$, given a collection $\mathcal{D}$ of k partitions of d satisfying $(*)$, there exists ϕ such that $\mathcal{D}$ is realisable as branch datum of (M, ϕ, N, k, d). This is not the case for $N = \mathbb{S}^2$.
In [3], A. Edmonds, R. Kulkarni and R. Stong exhibit, for this case and for any non-prime integer d, an *exceptional collection*, i.e. a collection of partitions that satisfies $(*)$ but that is not realisable as branch datum over $\mathbb{S}^2$.
The completely solved cases are only those where $\mathcal{D}$ contains partitions like $[d]$ or $[d - r, r]$, see [2], [3], [6] and [8].

The case $[d - (r + s), r, s] \in \mathcal{D}$ is partially solved, but only for $r = s = 1$:

1. if d is even then $\{[2, \dots, 2], [4, 2, \dots, 2], [d - 2, 1, 1]\}$ is exceptional, in [5].
2. if d is odd then $\mathcal{D}$ is realizable, in [7].

This last case is interesting because the result in [7] is a consequence of a realisation result on the projective plane $\mathbb{R}P^2$.

2 Related problems

If $N = \mathbb{S}^2$, in [3], the authors reduced the problem to the case of $\mathcal{D}$ with three partitions. Moreover, in [4], H. Zheng pointed out that the most difficult case to realise is $M = N = \mathbb{S}^2$. We will focus on this case.
Let a, b, c be positive integers ≥ 3 and let A, B, C be partitions of d with a, b, c summands, respectively. Suppose $\mathcal{D} = \{A, B, C\}$ satisfies $(*)$. The number $\nu(\mathcal{D})$ represents the "defect" of the branched covering to be an unbranched covering, i.e. $\nu(\mathcal{D}) = 3d - (a + b + c)$. So by $(*)$ we have:

$$\text{(I)} \quad a + b + c = d + 2,$$

equivalently

$$\text{(II)} \quad a + b = d + 2 - c.$$

Note that (II) is equivalent to $(*)$ for the case $N = \mathbb{R}P^2$, $M = M_c$ the non-orientable surface of genus c and $\mathcal{D}' = \{A', B'\}$, for partitions A', B' with a, b summands, respectively (for $N = \mathbb{R}P^2$ the general problem is reduce to the case of $\mathcal{D}$ with two partitions (see [3])). In other words, a solution of a problem of type $(\mathbb{S}^2, \phi, \mathbb{S}^2, d, 3)$ could be a solution of a problem of type $(M_c, \psi, \mathbb{R}P^2, d, 2)$, and vice versa. But we will see that this depends on the partition C.

3 Monodromy groups

Associated with (M, ϕ, N, k, d), there is a permutation group, *the monodromy group*, given by the image of the *Hurwitz representation*, in the symmetric group of degree d, Σ_d:

$$\rho_\phi : \pi_1(N - B_\phi, z) \to \Sigma_d$$

which for $x \in B_\phi$, sends the class $u_x \in \pi_1(N - B_\phi, z)$ to a permutation α_x of $\phi^{-1}(z) = \{z_1, \dots, z_d\}$ with cyclic structure defined by the partition D_x of d given by the local degrees of ϕ associated to x, we write $\alpha_x \in D_x$.

Hurwitz Theorem: Let N be a closed surface and $\mathcal{D} = \{D_1, \dots, D_t\}$ a collection of partitions of d satisfying $(*)$. Then $\mathcal{D}$ is realisable if and only if there are permutations $\alpha_i \in D_i$, for $i = 1, \dots, t$, such that:

1. If $N = \mathbb{S}^2$, then $\langle \alpha_1, \dots, \alpha_t | \prod_{i=1}^t \alpha_i = 1 \rangle$ is transitive.
2. If $N = \mathbb{R}P^2$, there is $\gamma \in \Sigma_d$, such that $\langle \alpha_1, \dots, \alpha_t, \gamma | \prod_{i=1}^t \alpha_i = \beta^2 \rangle$ is transitive. □

4 Results

We say that a partition is *a square partition* if it can be the cyclic structure of the square of a permutation. For example, for d odd:

- a partition of d whose summands are odd numbers,
- a partition whose even summands appear in pairs,
- $[d - 2r, r, r]$.

Let us consider the case in §2. By Hurwitz's theorem, case (I) is realisable if and only if there are permutations $\alpha \in A$, $\beta \in B$, $\gamma \in C$ such that $\langle \alpha, \beta, \gamma | \alpha\beta\gamma = 1 \rangle$ is transitive. Analogously, case (II) is realisable if and only if there are permutations $\alpha \in A'$, $\beta \in B'$, $\gamma \in \Sigma_d$ such that $\langle \alpha, \beta, \gamma | \alpha\beta = \gamma^2 \rangle$ is transitive.

Theorem

1. A realisation for (I) is a realisation for (II) if and only if $\mathcal{D}$ contains a square partition.
2. A realisation for (II) with $\langle \alpha, \beta \rangle$ transitive is a realisation for (I). □

This result suggests us to study, via $\mathbb{R}P^2$, the case of $\mathcal{D}$ with a square partition, for example, $[d - 2r, r, r]$, for d odd.

References

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