A characterization of pseudo-parallel Lorentzian surfaces in 4-dimensional pseudo-Riemannian space forms

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Abstract

We give a characterization of pseudo-parallel Lorentzian surfaces with non-flat normal bundle in 4-dimensional pseudo-Riemannian space forms as super-extremal surfaces, i.e., λ-isotropic surfaces with vanishing mean curvature vector field. We also characterize this kind of surfaces using the concept of hyperbola of curvature. We show that any pseudo-parallel Lorentzian surface with non-flat normal bundle and constant pseudo-parallelism function in codimension two is, locally, a piece of a Lorentzian surface of the Veronese type, extending an analogous result by Asperti-Lobos-Mercuri for pseudo-parallel surfaces in 4-dimensional Riemannian space forms.

Introduction

An isometric immersion $f : M \to \widetilde{M}$ between pseudo-Riemannian manifolds, with second fundamental form α, is:

- parallel if $\overline{\nabla}_X \cdot \alpha = 0$;
- semi-parallel if $\overline{R}(X,Y) \cdot \alpha = 0$;
- pseudo-parallel if $\overline{R}(X,Y) \cdot \alpha = \psi(X \wedge Y) \cdot \alpha$,

for some smooth real-valued function ψ on M and any tangent vector fields X, Y on M, where $\overline{R} = R \oplus R^{\perp}$ is the curvature tensor corresponding to the Van der Waerden-Bortolotti connection $\overline{\nabla} = \nabla \oplus \nabla^{\perp}$ of f and $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$.

Pseudo-parallel submanifolds were introduced by Asperti-Lobos-Mercuri as a generalization of semi-parallel submanifolds and as an extrinsic analogue of pseudo-symmetric manifolds. In [1], authors proved that ψ-pseudo-parallel surfaces with $R^{\perp} \neq 0$ in 4-dimensional Riemannian space forms are superminimal, i.e., minimal and λ-isotropic. Also, they classified such surfaces in codimension 2 with constant ψ.

Preliminaries and notations

Let $\mathbb{E}^N_s$ be the N-dimensional pseudo-Euclidean space with the metric of index s given by

$$\langle x, y\rangle = \langle (x_1,\dots,x_N),(y_1,\dots,y_N)\rangle = -\sum_{i=1}^{s} x_i y_i + \sum_{i=s+1}^{N} x_i y_i.$$

We use $\mathbb{Q}^m_s(c)$ to refer the m-dimensional pseudo-Riemannian space form with constant sectional curvature c and metric of index s, such that:

$$\mathbb{Q}^m_s(c) = \begin{cases} \mathbb{H}^m_s(c) \subset \mathbb{E}^{m+1}_{s+1}, & \text{if } c < 0, \quad \text{(pseudo-hyperbolic space)} \\ \mathbb{E}^m_s, & \text{if } c = 0, \\ \mathbb{S}^m_s(c) \subset \mathbb{E}^{m+1}_s, & \text{if } c > 0. \quad \text{(pseudo-sphere)} \end{cases}$$

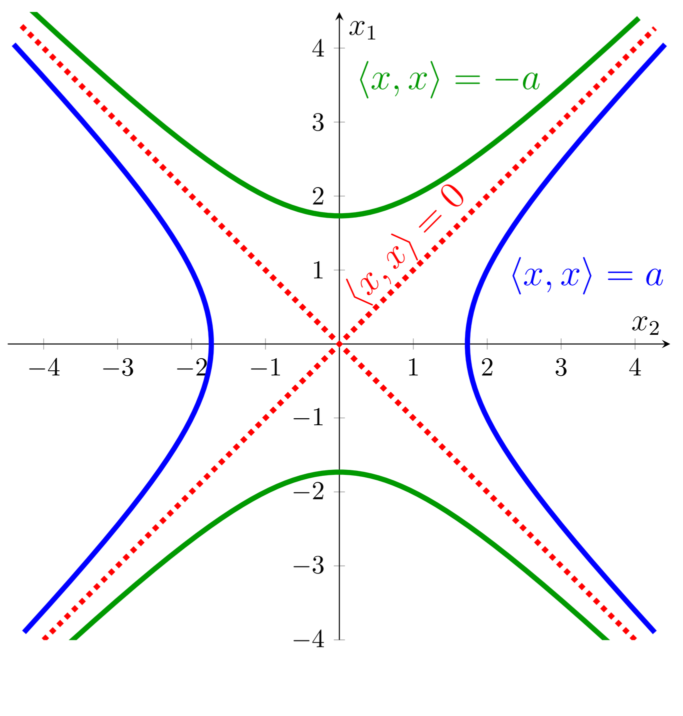

Figure 1: Equilateral hyperbolas in Lorentz-Minkowsky plane $\mathbb{E}^2_1$.

Let $f : M^2_1 \to \mathbb{Q}^m_s(c)$ be an isometric immersion from a Lorentzian surface M^2_1. By Fundamental Equations we get

$$K = c - \langle \alpha_{11}, \alpha_{22}\rangle + \langle \alpha_{12}, \alpha_{12}\rangle, \tag{1}$$
$$R^{\perp}(e_1,e_2)\xi = ((\alpha_{11} + \alpha_{22}) \wedge \alpha_{12})\xi, \tag{2}$$

and for the pseudo-parallelism condition we get the relations:

$$R^{\perp}(e_1,e_2)\alpha_{11} = R^{\perp}(e_1,e_2)\alpha_{22} = 2(\psi - K)\alpha_{12}, \tag{3}$$
$$R^{\perp}(e_1,e_2)\alpha_{12} = (\psi - K)(\alpha_{11} + \alpha_{22}), \tag{4}$$

where $\{e_1, e_2\}$ is an orthonormal frame for M^2_1, $\alpha_{ij} = \alpha(e_i, e_j)$, ξ is any normal vector to M^2_1 and K is the Gaussian curvature of M^2_1.

An isometric immersion $f : M^2_1 \to \mathbb{Q}^m_s(c)$ is λ-*isotropic* if $\langle \alpha(X,X), \alpha(X,X)\rangle = \lambda(x)$, $\forall X \in T_x M^2_1$ with $\|X\| = \sqrt{|\langle X, X\rangle|} = 1$, $\forall x \in M^2_1$, where $\lambda : M^2_1 \to \mathbb{R}$ is a smooth function.

Results

A theorem of characterization (see [3]). *An isometric immersion $f : M^2_1 \to \mathbb{Q}^4_s(c)$ with non-flat normal bundle on any open subset of M^2_1 is ψ-pseudo-parallel if and only if it is λ-isotropic. For such an immersion we have that $s = 2$, $H = 0$ and $\lambda = \frac{1}{2}(c - K) = K - \psi \neq 0$, where K is the Gaussian curvature of M^2_1, H is the mean curvature vector field of f and λ is a smooth real-valued function on M^2_1.*

Corollary 1 (see [3]). Let $f : M^2_1 \to \mathbb{Q}^4_s(c)$ be an isometric immersion. f is ψ-pseudo-parallel with $R^{\perp} \neq 0$ if and only if $s = 2$ and for each $x \in M^2_1$, the set

$$\mathcal{H}_x = \{\langle X, X\rangle \alpha(X,X) : X \in T_x M^2_1 \text{ with } \langle X, X\rangle = \pm 1\}$$

is a non-degenerate *equilateral* hyperbola with center at 0 in the normal space to M^2_1 at x. In this case, $H = 0$ and the constant radius squared of the hyperbola is $\lambda(x) = K - \psi$.

Example 1: Lorentzian surfaces of the Veronese type. The isometric immersion $f : \mathbb{S}^2_1(1) \to \mathbb{S}^4_2(3)$ defined by

$$f(x,y,z) = \left(xy, xz, yz, \frac{1}{2\sqrt{3}}(2x^2 + y^2 + z^2), \frac{1}{2}(y^2 - z^2)\right),$$

corresponds to the Veronese immersion in Riemannian geometry. f is λ-isotropic with $\lambda = 1$ and ψ-pseudo-parallel with $\psi = 0$; in fact, f is a parallel immersion with $R^{\perp} \neq 0$.

Hasegawa in [2] proved that f is an *isotropic with negative spin* immersion. Also, f is *extremal*, i.e., $H = 0$.

Extremal and isotropic with negative spin immersions of the Veronese type from $\mathbb{Q}^2_1(\hat{c})$ to $\mathbb{Q}^4_2(3\hat{c})$, $\hat{c} \neq 0$, can be obtained from f.

Proposition (Hasegawa [2]). Let $f : M^2_1 \to \mathbb{Q}^4_2(c)$ be an extremal and isotropic with negative spin immersion of constant Gaussian curvature K. If $K \neq c$, then $c = 3K \neq 0$ and f is an open set of the Veronese type surface given in Example 1.

Corollary 2 (see [3]). Let $f : M^2_1 \to \mathbb{Q}^4_s(c)$ be an isometric immersion with $R^{\perp} \neq 0$. If f is ψ-pseudo-parallel with constant ψ, then $s = 2$, $c = 3K \neq 0$, $\psi = 0$ and f is an open set of the Veronese type surface in Example 1.

Conjecture: There are pseudo-parallel Lorentzian surfaces with $R^{\perp} \neq 0$ in $\mathbb{Q}^4_2(c)$ which are not semi-parallel.

References

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