Forcing, Category Theory and the Independence of the Continuum Hypothesis

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Abstract

In 1931, Kurt Gödel [4] proved his incompleteness theorems showing that any logical system with the minimum desirable properties to formalize mathematics is incomplete. In other words, there are mathematical formulas in the system which we can neither prove nor prove their negation. However, it was only in 1963 that one mathematical conjecture, known as the Continuum Hypothesis (CH), was proved to be formally undecidable from the axiomatic system most used by modern Mathematics, the ZFC system [3]. To show this independence, Paul Cohen [3] developed a technique called forcing to build a model for ZFC that violates the CH. Since then, forcing has been developed and applied to several areas of Mathematics, including Algebra, Analysis and Topology. Going to the Topos Theory, it is possible to give an alternative proof of the independence of the CH where the model gives rise to a category called Cohen topos, corresponding to the category of the double negation sheaves $Sh(\mathbb{P}_\kappa, \neg\neg)$. The resulting topos works like an intuitionistic model for ZFC, where the CH fails [1].

Introduction

Cantor's diagonal argument shows that there is **no** surjective function from the set of natural numbers $\mathbb{N}$ to the set of real numbers $\mathbb{R}$. In 1878, Jorge Cantor stated the so-called Continuum Hypothesis (CH). The CH claims that there is no infinite (cardinal number) strictly between the natural numbers (ω) and the real numbers (2^ω). David Hilbert, in 1900, presented at the International Congress of Mathematics in France the truth of the CH as the first of the 23 mathematical problems that would guide the development of mathematics in the 20th century. We will present two versions of Cohen's work to provide the independence of CH from ZFC, the set-theoretic and the sheaf-theoretic approaches, forcing the negation of the CH. Even though the proofs seem different, the mathematical content of them is the same [1].

Forcing $\neg CH$ through transitive models

The argument starts from a **countable transitive model** M for ZFC. Take $\kappa \in M$ a cardinal number such that $(\omega < 2^\omega < \kappa)^M$. Our goal is to extend M to another model $M \subseteq M[G]$ for ZFC where we have a monomorphism $(f : \kappa \rightarrowtail 2^\omega)^{M[G]}$ so that $(\omega < \kappa < 2^\omega)^{M[G]}$. Then, the CH fails in $M[G]$. Even though we do not have such a monomorphism in M, we have finite approximations of it that we can use to build this function in the extension. We call the Cohen poset the set $\mathbb{P}_\kappa = \{p : B \subseteq (\kappa \times \omega) \to 2\}$ of functions with finite domain, equipped with the pre-order $\leq := \supseteq$ given by the inverse inclusion. Now we have to collect the right elements of $\mathbb{P}_\kappa$ that give us non-contradiction information about our desired function f. To this end, we use a **generic filter** (over M) $G \subseteq \mathbb{P}_\kappa$. Basically, G satisfies two properties. (1) - Given $p, q \in G$, exists $r \in G$ such that $r \leq p, q$, and (2) - if $s \in G$ and $t \in \mathbb{P}_\kappa$ is such that $s \leq t$, then $z \in G$. . The property of G being generic is that G intersects specific subsets of $\mathbb{P}_\kappa$ called **dense subsets** so that when we take the union $\cup G$ of G, it results in a function $UG : \kappa \times \omega \to 2$, that will do the work of our desired function f and will be in $M[G]$. Then, for each $k \in \kappa$, we have a **different real number** $x_k := \cup G(k, \_) : \omega \to 2$, so 2^ω will have at least cardinality κ in $M[G]$. To define $M[G]$, we use the notion of $\mathbb{P}_\kappa - names$. Intuitively, it is a way to talk about the elements of $M[G]$ inside of M.

By recursion, a $\mathbb{P}_\kappa - name$ is a set τ of ordered pairs of the form (σ, p), where σ is a $\mathbb{P}_\kappa - name$ and $p \in \mathbb{P}_\kappa$. Given τ a $\mathbb{P}_\kappa$-name and a filter G, by recursion, we define $val(\tau, G) = \tau_G = \{val(\sigma, G) : \exists q \in G((\sigma, q) \in \tau\}$. To conclude, define $M[G] = \{\tau_G : (\tau \text{ is a } \mathbb{P}\text{– name})^M\}$. Then, $M[G]$ is a model for ZFC which satisfies the negation of the CH [2].

A topos where the CH fails

One can say that the properties that a category must satisfy to be called elementary topos are generalizations of possible constructions using set-theory arguments inspired by the category $\mathcal{S}ets$ of sets and functions. In particular, we have the notions of **natural number** and **powerset**, the last is defined using a notion of **subobject classifier** Ω, that works like the set $\mathbf{2}$ in $\mathcal{S}ets$. Then, under specific conditions, it is possible to give a categorical version of Cohen's proof, showing a topos constructed using the Cohen poset $\mathbb{P}_\kappa$ which, in some way, the CH fails. We define the Cohen topos as the full subcategory of $\mathcal{S}ets^{\mathbb{P}_\kappa^{op}}$ of sheaves for the double-negation topology $Sh(\mathbb{P}_\kappa, \neg\neg)$. Here, the goal is the same as using transitive models. We want to construct a monomorphism that shows that, in $Sh(\mathbb{P}_\kappa, \neg\neg)$, we have something between the natural numbers and its powerset. We can summarize the argument as follows. Let $\Delta : \mathcal{S}ets \to \mathcal{S}ets^{\mathbb{P}_\kappa^{op}}$ be the constant functor and let $a : \mathcal{S}ets^{\mathbb{P}_\kappa{}^{op}} \to Sh(\mathbb{P}_\kappa, \neg\neg)$ be the sheafification functor. Define $A : \mathbb{P}_\kappa^{op} \to \mathcal{S}ets$ such that $A(p) = \{(k, p) \in \kappa \times \omega : p(k, n) = 0\}$. In particular, A is a closed subfunctor of $\Delta(\kappa \times \omega)$ for the double negation topology. Let Ω be the subobject classifier of $\mathcal{S}ets$. The subobject classifier of the Cohen topos, denoted by $\Omega_{\neg\neg} \cong a\Delta(2)$, is the equalizer of $\neg\neg, id_\Omega : \Omega \to \Omega$. Then, the characteristic morphism $Ch(A)$ factors through $\Omega_{\neg\neg}$

$$\begin{array}{ccccc} \Omega_{\neg\neg} & \longrightarrow & \Omega & \overset{\neg\neg}{\underset{id_\Omega}{\rightrightarrows}} & \Omega \\ \uparrow{\scriptstyle \exists! f} & \nearrow{\scriptstyle ch(A)} & & & \\ \Delta(\kappa) \times \Delta(\omega) & & & & \end{array}.$$

Denote by $g : \Delta(\kappa) \to \Omega_{\neg\neg}^{\Delta(\Omega)}$ the transpose of the morphism f above, then m as defined bellow is a monomorphism $m := a(g) : a\Delta(\kappa) \rightarrowtail \Omega_{\neg\neg}^{a\Delta(\omega)}$. Now, using the notation $a(\Delta)(S) = \widehat{S}$ for a given set S and that this composition preserves strict monomorphisms, by de monomorphism m and by $\omega < 2^\omega < \kappa$ in $\mathcal{S}ets$, we have $\widehat{\omega} < \widehat{2^\omega} < \widehat{2}^{\widehat{\omega}}$ which violates the CH.

References

[1] Mac Lane, S. Moerdijk, I. **Sheaves in Geometry and Logic: A First Introduction to Topos Theory**, Universitex, Springer NewYork, 1994.

[2] Kunen, K. **Set Theory. Studies in Logic: Mathematical Logic and Foundations**. Individual author and College Publications 2011.

[3] Cohen, P. **The independence of the continuum hypothesis I**. Proceedings of the National Academy of Sciences USA 50 (1963), p. 1143-1148.

[4] Gödel, K. **Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I**. Monatshefte für Mathematik und Physik 38 (1931), p. 173-98.

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