Some properties of the Bieri-Strebel invariant

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Introduction

A group G is said to be metabelian if there is an exact sequence of groups

$$A \hookrightarrow G \twoheadrightarrow Q,$$

where A and Q are abelian. In 1980, Bieri and Strebel defined a geometric invariant $\Sigma_A(Q)$ and with it they proved a necessary and sufficient condition for the metabelian group G to be finitely presented. In this work, we present the invariant Σ and some results which are usefull to calculate it.

The Bieri-Strebel invariant

Let Q be a finitely generated abelian group and R be a commutative ring with $1 \neq 0$. We call a character of Q a homomorphism $\chi : Q \to \mathbb{R}$, where $\mathbb{R}$ is considered an aditive group. Each character $\chi : Q \to \mathbb{R}$ can be extended to a valuation $v_\chi : RQ \to \mathbb{R}_\infty$ defining $v_\chi(0) = \infty$ and

$$v_\chi(\lambda) = \min\{\chi(q) \mid \lambda_q \neq 0\},$$

where $\lambda = \Sigma\, \lambda_q q \in RQ$.

Now, writing $Q \simeq \operatorname{tor} Q \oplus \mathbb{Z}^n$, note that

$$\operatorname{Hom}(Q, \mathbb{R}) \simeq \operatorname{Hom}(\mathbb{Z}^n, \mathbb{R}) \simeq \mathbb{R}^n.$$

We say two characters χ_1 and χ_2 are equivalent if, and only if, there's a positive real number r such that $\chi_1 = r\chi_2$ and we denote $\chi_1 \sim \chi_2$. Thus, we define the character sphere

$$S(Q) = \frac{\operatorname{Hom}(Q, \mathbb{R}) \setminus \{0\}}{\sim} \simeq S^{n-1},$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. We also denote by $[\chi]$ the equivalence class of the character χ.

For each character, we define the monoid

$$Q_\chi = \{q \in Q \mid \chi(q) \geq 0\}.$$

Note that if χ_1 and χ_2 are two equivalent characters, then their respective monoids coincide

$$Q_{\chi_1} = Q_{\chi_2}.$$

Now, let A be a RQ-module. We define the Bieri-Strebel invariant by

$$\Sigma_A(Q) := \{[\chi] \in S(Q) \mid A \text{ is fin. gen. over } RQ_\chi\}.$$

It can also be proved that

$$\Sigma_A(Q) = \bigcup_{\lambda \in C(A)} \{[\chi] \in S(Q) \mid v_\chi(\lambda) > 0\},$$

where $C(A)$ is the centralizer of A in RQ.

Finitely presented metabelian groups

Lets start this section with the definition of tame module.

Definition 1. We call a $\mathbb{Z}Q$-module A tame, if A is finitely generated and $\Sigma_A(Q) \cup -\Sigma_A(Q) = S(Q)$.

Equivalently, A is tame if, and only if, $\Sigma_A^c(Q) := S(Q) \setminus \Sigma_A(Q)$ contains no pair of antipodal points.

Proposition 1 (2.5(i),[2]). If the $\mathbb{Z}Q$-module A is tame, then every submodule of A, every homomorphic image of A and every product of finitely many copies of A are also tame.

Now let G be a finitely generated group and A and Q be two finitely generated abelian groups such that the short sequence

$$A \hookrightarrow G \twoheadrightarrow Q$$

is exact. Define an action of $Q \simeq G/A$ in A by conjugation and consider A as a $\mathbb{Z}Q$-module. The first main result proved by Bieri and Strebel is

Theorem 1 (5.1,[2]). Let G be a finitely generated group and let $A \triangleleft G$ be a normal subgroup such that both A and $Q = G/A$ are abelian. Then G is finitely presented if and only if A is tame as a $\mathbb{Z}Q$-module

Some properties of the invariant

Now, we'll explore three usefull results to calculate $\Sigma_A(Q)$ or $\Sigma_A^c(Q)$.

Theorem 2 (1.1, [3]). Let R be a Noetherian ring and A a finitely generated RQ-module. If $P_1, P_2, \ldots, P_m$ are the minimal prime ideals over $\operatorname{Ann}(A)$, then

$$\Sigma_A = \Sigma_{RQ/P_1} \cap \Sigma_{RQ/P_2} \cap \ldots \cap \Sigma_{RQ/P_m}.$$

We call $H \subset S(Q)$ a rational closed hemisphere if we can write it as $H = \{[\chi] \mid \chi(q) \geq 0\}$, for some $q \in Q$ of infinite order and we call $C \subseteq S(Q)$ a convex rational spherical polyhedron if we can write it as a finite intersection of rational closed hemispheres. And we say that $\Delta \subseteq S(Q)$ is a rational spherical polyhedron if it can be written as a finite union of convex rational spherical polyhedrons.

Theorem 3 (8.3, [1]). Σ_A^c is a rational spherical polyhedron.

Now, let B be an algebra over RQ given by the ring homomorphism $\kappa : RQ \to B$ and let $v : R \to \mathbb{R}_\infty$ be a valuation of R. We define the set $\Delta_B^v(Q) \subseteq \operatorname{Hom}(Q, \mathbb{R})$ as the set of all characters χ such that there is a valuation $w : B \to \mathbb{R}_\infty$ satisfying $w\kappa|_R = v$ and $w\kappa|_Q = \chi$.

Theorem 4 (8.1, [1]). Let A be a finitely generated RQ-module and $B = RQ/\operatorname{Ann}_{RQ}(A)$. Then

$$\Sigma_A^c = \bigcup_v [\Delta_B^v(Q)],$$

where v runs on all valuations $v : R \to \mathbb{R}_\infty$ such that $v(R) \geq 0$.

Example

Consider $A = \mathbb{Z}[x, x^{-1}, (x+1)^{-1}]$ and $Q = \langle x, y \rangle \simeq \mathbb{Z}^2$, where Q acts in A via conjugation, $x \in Q$ acts as multiplication by x and $y \in Q$ acts as multiplication by $x + 1$. Here $S(Q) \simeq S^1$.

We want to calculate

$$\Sigma_A^c = \bigcup_v [\Delta_A^v(Q)].$$

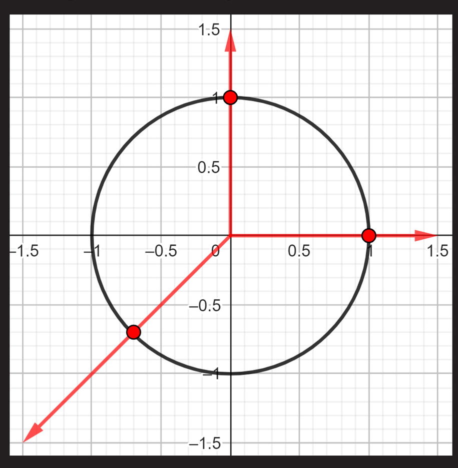
Now, note that, if $\chi \in \Delta_A^v(Q)$, then there is a valuation $w : A \to \mathbb{R}_\infty$ such that $\chi(y) = w\kappa(y)$. So

$$\chi(y) = w(x+1) \geq \min\{w(x), w(1)\} = \min\{\chi(x), 0\}.$$

Now, if $\chi(x) \neq 0$, then $\chi(y) = \min\{\chi(x), 0\}$, so we can divide the problem in three cases:

- If $\chi(x) = 0$, then $\chi(y) \geq 0$;
- If $\chi(x) > 0$, then $\chi(y) = 0$;
- If $\chi(x) < 0$, then $\chi(y) = \chi(x)$.

Therefore, $\Sigma_A^c(Q)$ is given by, at best three points:



References

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[2] Robert Bieri and Ralph Strebel. Valuations and finitely presented metabelian groups. *Proceedings of the London Mathematical Society*, 3(3):439–464, 1980.

[3] Robert Bieri and Ralph Strebel. A geometric invariant for modules over an abelian group. *J. Reine Angew. Math.*, 322:170–189, 1981.

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