Lower bounds for the length of the second fundamental form via the first eigenvalue of the p-Laplacian

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Abstract

A suitable Bochner type formula for a certain divergence type operator was obtained. As an application, several integral inequalities involving the squared norm of the second fundamental form of a minimal compact (with or without boundary) submanifold in the sphere and the first eigenvalue of p-Laplacian are established. Finally, lower bounds of the second fundamental form of a such minimal submanifold are proven.

Introduction

In the last decades, the study of the first non-zero eigenvalue for elliptic operators has been of substantial interest from both physical and mathematical points of view. From the physical point of view, this constant determines the convergence rate of numerical schemes in numerical analysis, describes the energy of a particle in the ground state in quantum mechanics, and determines the decay rate of the heart flows in thermodynamics. On the other hand, from the mathematical point of view, for instance, Leung [2] showed that

THEOREM 1. Let M^n be a closed minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the Laplacian of M^n associated to λ. Then

$$\int_M (S+\lambda-n)|\nabla u|^2 dM \geq 0, \tag{1}$$

If equality holds, then

a) either M^n is totally geodesic and λ is the first non-zero eigenvalue of the 2-Laplacian,

b) or $n = 2$ and $m = 2q$ and M^n is isometric to $\mathbb{S}^2\left(\sqrt{q(q+1)/2}\right)$ and λ is the first non-zero eigenvalue of the Laplacian.

Just like the Laplacian, besides being of mathematical interest, the study of the p-Laplacian operator is also of interest in the theory of non-Newtonian fluids both for the case $p \geq 2$ (dilatant fluids) and the case $1 < p < 2$ (pseudo-plastic fluids). It is also of geometrical interest for $p \geq 2$, some of which are discussed in [5]. Given its physical and mathematical significance, numerous bounds have been established for the Dirichlet first eigenvalue of the Laplacian operator, and many results have extended to the nonlinear p-Laplacian during the last two decades.

Settings

In this work all manifolds are connected, otherwise the arguments must be done in each connected component. Let M^n be a Riemannian manifold.

DEFINITION 2. Let $u \in \mathcal{C}^2(M)$ be any function. For any $1 < p < +\infty$, we define the $p-Laplacian$ operator of u as

$$\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Moreover, if $p = \infty$, we define the $\infty - Laplacian$ as

$$\Delta_\infty u = |\nabla u|^{-2}\langle \operatorname{Hess} u(\nabla u), \nabla u\rangle.$$

DEFINITION 3. We say that u is an $eigenfunction$ of the Δ_p associated to λ if and only if $\Delta_p u = \lambda |u|^{p-2}u$.

When the boundary is non-empty, is natural to control the function u in the boundary. In this work, we are using the following problems:

$$\begin{array}{rrcll} & \Delta_p u &=& \lambda|u|^{p-2}u, & \text{in } M^n, \\ \textit{Neumann condition} & u_\eta &=& 0, & \text{on } \partial M, \\ \textit{Dirichlet condition} & u &=& 0, & \text{on } \partial M, \end{array}$$

Results

PROPOSITION 4. Let M^n be compact manifold with boundary ∂M. Then, for any function $u \in \mathcal{C}^2(M)$ with $\nabla u \neq 0$ on M^n, we have

$$\begin{aligned} &\int_M |\nabla u|^{2p-4}\left(|\operatorname{Hess} u|^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) dM \\ &= \int_M (\Delta_p u)^2 dM - 2(p-2)\int_M |\nabla u|^{2p-6}|\operatorname{Hess} u(\nabla u)|^2 \\ &\quad - \int_M (p-2)^2(\Delta_\infty u)^2 dM + \int_{\partial M} |\nabla u|^{2p-4} Q(u) d\sigma, \end{aligned}$$

where

$$\begin{aligned} Q(u) &= du(\eta)\left(\Delta^\partial u + (n-1)\mathcal{H}\, du(\eta)\right) \\ &\quad + \langle \mathcal{A}_\eta(\nabla^\partial u), \nabla^\partial u\rangle + \langle \nabla^\partial u, \nabla^\partial du(\eta)\rangle, \end{aligned}$$

and $d\sigma$ denotes the Riemannian volume element on ∂M.

THEOREM 5. Let M^n be a closed minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the p-Laplacian of M^n associated to λ. Then

$$\int_M (S + \alpha_{n,p}\lambda^{p/2} - n)|\nabla u|^{2p-2} dM \geq 0, \tag{2}$$

where $p \in [2, \infty)$ and

$$\alpha_{n,p} = \frac{n(p-1)^2 - 1}{(n-1)(p-1)^{2/p}}. \tag{3}$$

Moreover, if equality holds, then $p = 2$ and

a) either M^n is totally geodesic and λ is the first non-zero eigenvalue of the 2-Laplacian,

b) or $n = 2$ and $m = 2q$ and M^n is isometric to $\mathbb{S}^2\left(\sqrt{q(q+1)/2}\right)$ and λ is the first non-zero eigenvalue of the 2-Laplacian.

THEOREM 6. Let M^n be a compact minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the Neumann problem of the p-Laplacian of M^n associated to λ. In addition, assume that the boundary ∂M is convex. Then

$$\int_M \left(S + \alpha_{n,p}\lambda^{2/p} - n\right)|\nabla u|^{2p-2} dM \geq 0, \tag{4}$$

where $p \in [2, \infty)$ and $\alpha_{n,p}$ is defined in (3). Moreover, if equality holds, then $p = 2$ and M^n is isometric to the hemisphere $\mathbb{S}^n_+(1)$ and $\lambda = \lambda^N_{1,2}(\mathbb{S}^n_+)$

THEOREM 7. Let M^n be a compact minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the Dirichlet problem of the p-Laplacian of M^n associated to μ. In addition, assume that $\mathcal{H}$ is nonpositive. Then

$$\int_M \left(S + \alpha_{n,p}\lambda^{2/p} - n\right)|\nabla u|^{2p-2} dM \geq 0, \tag{5}$$

where $p \in [2, \infty)$ and $\alpha_{n,p}$ is defined in (3). Moreover, if equality holds and $S = const.$, then $p = 2$ and M^n is isometric to a closed hemisphere of the Euclidean sphere $\mathbb{S}^n(1)$ and $\lambda = \lambda_{1,2}(\mathbb{S}^n)$.

The demonstration of the three theorems consists of using the divergence theorem multiple times in conjunction with the Holder's theorem. The main point is use the proposition 4 and control the terms $\Delta_\infty u$ and $|\operatorname{Hess} u(\nabla u)|$. For the equality, we use the Obata's Theorem [3] in the closed case, the Escobar's Theorem [1] for Neumann condition and the Reilly's Theorem [4] for Dirichlet condition.

References

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