# Simple Binary Lie Superalgebras


## Marina Rasskazova

Universidade Federal ABC (Brasil)

marinarasskazova@yandex.ru


### Overview

A.Malcev in the paper [1] introduced the variety BL of binary Lie algebras ($BL-$algebras). By definition an algebra $B$ is $BL-$algebra if any two elements from $B$ generate a Lie subalgebra. It is clear that in every $BL-$algebra we have the identities $x^2 = J(x, y, xy) = 0.$

A.Gainov in the paper [2] was proved that those identities define the variety BL.

The main result about the structure of finite dimensional $BL-$algebras over C was obtained by A.Grishkov in 1980-86.

Recall that for given variety $\mathrm{V}$ we can define the corresponding variety of $super$ $\mathrm{V}-$algebras ($\mathrm{SV}-$algebras) in the following way.

An algebra $A$ is super $\mathrm{V}-$algebra if $A = A_0 \oplus A_1$ admits a $\mathrm{Z}_2-$gradation such that $G(A) = A_0 \otimes G_0 \oplus A_1 \otimes G_1$ is $\mathrm{V}-$algebra, where $G = G_0 \oplus G_1$ is any Grassman algebra.

In the case of $BL-$algebras we get the following definition.

**Definition 1.** *A $\mathrm{Z}_2-$graded algebra $B = B_0 \oplus B_1$ is called* $\boldsymbol{super}$ $\boldsymbol{BL-algebra}$ *($SBL-$algebra) if it satisfies the following graded identities.*

$xy = (-1)^{1+xy}yx,$

$(xy)z.t - x.(y(zt)) + (-1)^{xy}\{y.(xz.t) + y.(x.zt) - (y.xz)t)\}+$

$(-1)^{zt}\{x.(yt.z) - (xy)t.z) - (x.yt)z\} = 0,$

*here* $x, y, z, t \in B_0 \cup B_1$ *and* $(-1)^x = 1$, *if* $x \in B_0$;

$(-1)^x = -1$, *if* $x \in B_1$.

**Example 1 (I.Shestakov).** Let $SB = ka \oplus kx$, $a^2 = 0$, $x^2 = a$, $xa = -ax = x.$

This is unique simple $SBL-$algebra finite-dimensional over algebraically closed field $k$ of characteristic $0$ known until now.

Moreover, we have

**Conjecture 1.** Every simple $SBL-$algebra finite-dimensional over algebraically closed field $k$ of characteristic $0$ with non-trivial odd part is $2-$dimensional algebra from example of I.Shestakov.

### Introduction

We propose the following strategy for proving Conjecture in four steps:

**First Step.** Reduction to the case when $B_0$ is solvable.

**Second Step.** Reduction to the case when $B_0$ is nilpotent.

**Third Step.** Reduction to the case when $B_0$ is abelian.

**Forth, last Step.** To prove Conjecture.

We note that in Conjecture all conditions: basic field is of characteristic 0. it is $\boldsymbol{algebraically}$ closed. superalgebra has finite dimension are $\boldsymbol{nesessary}$. It we showed it in the paper [3], where we costructed three new examples simple $BL-$superalgebras ,which do not have these conditions.

### Results

All results were obtaned with coloboration A.Grishkov and I. Shestakov. Recently (not published yet) we finish the proof of the First Step of this program.

We proved the theorem.

**Theorem 1.** *Let $B = B_0 \oplus B_1$ be a simple finite dimension super binary-Lie algebra over an algebraically closed field $\mathbf{k}$ of characteristic 0. Then $B_0$ is a solvable binary Lie algebra, if $B_1 \neq 0$.*

For proof we used the main result of A. Grishkov (1980-86) about structure of a finite dimensional $BL-$algebras.

**Theorem 2.** *(Grishkov) Let $B$ be a finite dimensional $BL-$algebra over the field $\mathrm{C}$ of complex numbers and $G$ be a solvable radical of $B$ and $R(B) = 0$. Then there exists a factor Levi - semisimple Malcev subalgebra $L \subseteq B$ such that $B = L \oplus G$, $G \cap L = 0$. Moreover, $B = M + G_0$, where $M$ is a Malcev ideal of $B$, $G_0$ is solvable subalgebra of $B$.*

As Corollary of this Theorem we have two important facts:

(1) Any non solvable finite dimensional $BL-$algebra over $\mathrm{C}$ has $3-$dimensional simple Lie subalgebra.

(2) If $R(B) = 0$, then $B$ is a completely reducible $S-$module over every semisimple subalgebra $S \subset B$.

From those facts we hence the following "strategy" of the proof of the main Theorem.

**1.)** Choose a simple $3-$dimensional subalgebra $S$ in the even part $B_0$ of simple super binary Lie algebra $B = B_0 \oplus B_1$.

**2.)** Decompose $B = (\sum_i \oplus L_i) \oplus (\sum_j \oplus M^j)$ as $S-$module, where $L_i$ is an irreducible $n_i-$dimensional Lie $S-$module; $M^j$ is $2-$dimensional Malcev non-Lie $S-$module.

**3.)** Describe the products $L_i.L_j$, $L_iM^j$ and $M^i.M^j$ as $S-$modules.

We proved:

(i) $L_i.L_j \subseteq L_i \otimes L_j$, where $L_i \otimes L_j$ is a standard tensor product of Lie $S-$modules.

(ii) $L_iM^j = 0$, if $dimL_i \neq 3$. If $dimL_i = 3$ then $L_iM^j = 0$ or $L_iM^j \simeq M^j$.

(iii) $M^i.M^j = 0$ or $M^i.M^j \simeq S \oplus V$, where $S$ is adjoined $S-$module and $dimV = 1, VS = 0$. Moreover, $V \subseteq R(B)$, hence $V = 0$.

**4.)** We prove that ideal of $B = B_0 \oplus B_1$, generated by Lie $S-$submodule $V \subset B$ without $3-$dimensional irreducible $S-$modules, is a Lie $S-$module. There we have two possibilities.

a) Ideal $I$, described in the Step 4, is not $0$.

b) Any irreducible $S-$submodule $W$ of $B$ has dimension $3$.

**5.)** Ideal $I$, described in the Step 4, is not $0$. In this case we prove that $B$ is a Lie superalgebra.

**6.)** Any irreducible Lie $S-$submodule of $B$ has dimension $3$.

In this case we have that $B_0 = B_{00} \oplus B_{01}$, $B_1 = B_{10} \oplus B_{11}$, where $B_{00} \oplus B_{10}$ is a direct sum of irreducible $S-$modules of dimension $3$ and $B_{01} \oplus B_{11}$ is a direct sum of irreducible non-Lie Malcev $S-$modules of dimension $2$.

**REFERENCE**


1. A. Malcev *Analytic loops*, Mat.Sbornik, 1955

2. A. Gainov, *Identities for BL-rings*, Usp.mat., USSR, 1957, v.16, n.3, pp.141-146

3. A. Grishkov, M. Rasskazova, I. Shestakov *New examples of simple super binary Lie algebras* Algebra and Logic, 60, 6 (2021), 557-568.DOI: 10.33048/alglog.2021.60.603



#### Acknowledgement

The first author would like to thank FAPESP for the financial support of this work (process n. 2021/12820-9).