Extra-special p-groups as groups of automorphisms

Maria Edna Gomes da Silva, Emerson Ferreira de Melo & Igor Lima

Universidade Federal de Jataí e Universidade de Brasília

maria.gomes@ufj.edu.br

Abstract

Let A be a group acting by automorphisms on a finite group G. In this work we consider that A is an extra-special p-group and we will present results regarding the nilpotency of the lower central series and derived series of the fixed points subgroups of the group G and similar results to Lie algebras. Furthermore, we prove results regarding supersoluble fixed points.

Introdução

Suppose that a finite group A acts by automorphisms on a finite group G. The action is coprime if the groups A and G have coprime orders. We denote by $C_G(a)$ the set $\{g \in G \mid g^a = g\}$, the centralizer of a in G (the fixed-point subgroup). Similarly, we denote by $C_G(A)$ the set $\{g \in G \mid g^a = g \text{ for all } a \in A\}$. It has been known for some time that centralizers of coprime automorphisms have strong influence on the structure of G.

In the study of the influence of $C_G(a)$ on the structure of a group, Ward showed that if A is an elementary Abelian p-group of rank 3 acting on a finite p'-group G such that $C_G(a)$ is nilpotent for any $a \in A^{\#}$, so the group G is nilpotent [6].

The first result has been extended in [4] for the case where A is not necessarily abelian. That is, it was shown that if A is a finite group of prime exponent p and order at least p^3 acting on a finite p'-group G in such a way that $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$, so G is nilpotent with class bounded only in terms of c and p .

Later, in [2] a generalization was obtained. Let p be a prime. If A is an elementary Abelian p-group of order p^k with $k \geq 3$ acting on a finite p'-group G such that $\gamma_{k-2}(C_G(a))$ is nilpotent of class at most c for any $a \in A^{\#}$, so $\gamma_{k-2}(G)$ is nilpotent with class limited only in terms of c, k and p. It was also proved that if for some integer d such that $2^d + 2 \leq k$, the dth group derived from $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$, so the dth derived group $G^{(d)}$ is nilpotent with class bounded only in terms of c , k and p.

In [1] the authors showed that if A is an elementary Abelian p-group of order, at least p^4 acts on a p'-group G. Assuming that $C_G(a)$ are supersolvable for all $a \in A^{\#}$. Then G is supersolvable.

Results for Lie Algebra

Theorem 0.1. Let p be a prime and E an extra-special p-group of exponent p and order p^{2n+1}. Suppose that E acts by automorphisms on Lie algebra L.

i) If $\gamma_n(C_L(a))$ is nilpotent of class at most c for any $a \in E^{\#}$, then $\gamma_n(L)$ is nilpotent with class bounded solely in terms of c, n and p.

ii) If, for some integer d such that $2^d \leq n$, the dth derived group of $C_L(a)$ is nilpotent of class at most c for any $a \in E^{\#}$, then the dth derived group $L^{(d)}$ is nilpotent with class bounded solely in terms of c, n and p.

Lemma 0.2. If $r \geq k-1$, then $\gamma_r(L) = \sum_{a \in A} N_a$ where each N_a is a subalgebra of $\gamma_{k-1}(C_L(a))$.

As usual, we denote by E' the commutator subgroup of E. Since E is extra-special, E' has order p and $E' = Z(E) = \Phi(E)$. Choose a generator φ of E' and set $H_0 = C_{\gamma_n(L)}(\varphi) = C_L(\varphi) \cap \gamma_n(L)$. The subring H_0 is E-invariant since φ commutes with any element of E.

Lemma 0.3. Let $B_1, \ldots, B_l$ be all elementary abelian subgroups of order p^{n+1} of E. Then $H_0 = \sum_i C_{\gamma_n(L)}(B_i)$

Lemma 0.4. H_0 is nilpotent of (c, n, p)-bounded class.

Lemma 0.5. There is a (c, n, p)-bounded number s such that

$$[L, \underbrace{H_0, \cdots, H_0}_{s}] = 0.$$

Corollary 0.6. The Lie algebra $\gamma_n(L)$ is soluble with (c, n, p)-bounded derived length.

Results for Groups

Theorem 0.7. Let p be a prime and E an extra-special p-group of exponent p and order p^{2n+1}. Suppose that E acts by automorphisms on a finite p'-group G.

i) If $\gamma_n(C_G(a))$ is nilpotent for any $a \in E^{\#}$, then $\gamma_n(G)$ is nilpotent.

ii) If, for some integer d such that $2^d \leq n$, the dth derived group of $C_G(a)$ is nilpotent for any $a \in E^{\#}$, then the dth derived group $G^{(d)}$ is nilpotent.

Lemma 0.8. Let A be an elementary abelian p-group of order p^k with $k \geq 2$ acting on a finite p'-group G and $A_1, \ldots, A_s$ the maximal subgroups of A. Let $r \geq 2$. Let Q be an A-invariant Sylow q-subgroup of $\gamma_{r-1}(G)$. Let $L_1, \ldots, L_t$ be all the subgroups of the form $\gamma_j(Q) \cap H$ where H is some γ-A-special subgroup of G of degree $r + j - 2$. Then $\gamma_j(Q) = \langle L_1, \ldots, L_t \rangle$.

Lemma 0.9. Let $B_1, \ldots, B_l$ be all elementary abelian subgroups of order p^{n+1} of E and V an E-invariant subgroup of $C_G(E')$. Then $V = \langle C_V(B_i) \mid 1 \leq i \leq l \rangle$

Theorem 0.10. Let p be a prime and E an extra-special p-group of the exponent p and order p^5. Suppose E acts by automorphisms on a finite p'-group G. If $C_G(a)$ is supersolvable for any $a \in E^{\#}$, then G is supersolvable.

Referências

[1] Hangyang Meng and Xiuyun Guo. (2018). Coprime actions with supersolvable fixed-point groups. Journal of Group Theory. 475-484. DOI:https://doi.org/10.1515/jgth-2018-0005.

[2] Acciarri, C., Shumyatsky, P. (2014). Centralizers of coprime automorphisms of finite groups. Annali di Matematica Pura ed Applicata. 193: 317-324. DOI:https://doi.org/10.1007/s10231-012-0274-x.

[3] De Melo, E. (2019). Nilpotent residual and Fitting subgroup of fixed points in finite groups. J. Group Theory. 22: 1059-1068. DOI:https://doi.org/10.1515/jgth-2019-0047.

[4] De Melo, E., Shumyatsky, P. (2017). Finite groups and their coprime automorphisms. Proc. Amer. Math. Society. 145: 3755-3760. DOI:https://doi.org/10.1090/proc/13550.

[5] de Melo,E. and Gomes,M.E. and Lima,Igor. (2022). Extra-special p-groups as groups of automorphisms. Communications in Algebra. DOI:https://doi.org/10.1080/00927872.2022.2102178.

[6] Ward, J. N. (1971). On finite groups admitting automorphisms with nilpotent fixed-point group. Bull. Aust. Math. 5: 281-282. DOI:https://doi.org/10.1017/S0004972700047171.

acknowledgment

I would like to thank the Federal University of Jataí, the University of Brasília and Capes for the financial support.