# Codimension-one singularities of Constrained systems on $\mathbb{R}^3$


## Lucyjane de A. Silva & João C. R. Medrado

Universidade Federal de Jataí & Universidade Federal de Goiás

lucyjane.silva@ufj.edu.br & medrado@ufg.br



### Abstract

This paper is concerned with the dynamics near an impasse singularity of Constrained systems defined on $\mathbb{R}^3$. We present all the topological types and their respective normal forms of the codimension-one impasse singularities for a large class of Constrained systems on the three-dimensional space. We split the study of impasse equilibrium points into nonresonant and resonant cases. In the first case, the constrained system presents cubic impasse bifurcation, while for the other one occurs focus–node, saddle-node, or Hopf impasse bifurcations. The tangential impasse points present three types: Lips, Bec-to-bec, and Dove's tail singularities.


### Introdution

Let $A = A(x)$ be an $n \times n$ matrix-valued function, $n \geq 2$, and $F = F(x)$ a vector field defined on $\mathbb{R}^n$. Assuming that $A$ and $F$ are smooth, a *Constrained system*, or simply a CS–system, on $\mathbb{R}^n$ is a differential system of the form

$$A(x)\dot{x} = F(x), \tag{1}$$

where $x \in \mathbb{R}^n$. Constrained systems are characterized by the existence of *impasse hypersurface* $I_A = \{x \in \mathbb{R}^n : \delta_A = \det A(x) = 0\}$ whose points are called *impasse points*. Note that, outside of the impasse hypersurface the Constrained system can be rewritten as $\dot{x} = A^{-1}(x)F(x) = \delta_A^{-1}A^*(x)F(x)$, where $A^*$ denotes the adjoint matrix of $A$. Then, for every system in the form (1), we can define the vector field $\widetilde{X} = A^*(x)F(x)$ called the *regularization* of the CS–system.

An Impasse point $p$ of a Constrained system (1) is said *regular* if $\delta_A$ is a regular function at $p$, i.e, $D(\delta_A)(p) \neq 0$. Moreover, a regular impasse point $p$ will be called nonsingular if $\ker A(p)$ is transversal to $I_A$ and the vector $F(p)$ does not belong to the image of $A(p)$. Otherwise, if at least one of these conditions is violated, $p$ is called an *impasse singularity*. In this work we consider the Constrained systems (1) defined on $\mathbb{R}^3$ and classify the codimension–one singularity and provide the respective local one-parameter normal forms.

### Objectives

1. To classify all topological types of the codimension-one impasse singularities.
2. To present the normal forms of the impasse singularities in the one–parameter space.

### Main results

Consider the space of the germs at the impasse singularity $p \in \mathbb{R}^3$ of the $C^r-$Constrained systems, endowed with the $C^r-$topology, $r > 3$, $\Xi = \{X : A(x)X = F(x), x \in \mathbb{R}^3\}$, where $A$ is a smooth $3\times 3$ matrix valued function and $F$ is a smooth vector field. In what follows we denote $\Xi_0 \subset \Xi$ the set of all elements in $\Xi$ which are structurally stable at the impasse singularity $p$ (see [2]) and $\Xi_1 = \Xi \setminus \Xi_0$.

#### Codimension–one Impasse Singularities

Let $\Sigma_1(a) \subset \Xi$ be the subset of Constrained systems (1) such that $\widetilde{X}(0) \neq 0$, $\widetilde{X}\delta_A(0) = \widetilde{X}^2\delta_A(0) = 0$, and one of the following statements is true:

**I-Lips** $\widetilde{X}^3\delta_A(0) \neq 0$, $\text{Hess}(\widetilde{X}\delta_A\big|_{I_A}(0))$ is positive, and $\text{rank}\{D\delta_A(0), D\widetilde{X}\delta_A(0), \widetilde{X}^2\delta_A(0)\} = 2$,.

**I-bec-to-bec** $\widetilde{X}^3\delta_A(0) \neq 0$, $\text{Hess}(\widetilde{X}\delta_A\big|_{I_A}(0))$ is negative, and $\text{rank}\{D\delta_A(0), D\widetilde{X}\delta_A(0), \widetilde{X}^2\delta_A(0)\} = 2$.

**I-Dove's tail** $\widetilde{X}^3\delta_A(0) = 0$, $\widetilde{X}^4\delta_A(0) \neq 0$, and $0$ is a regular point of $\widetilde{X}\delta_A|_{I_A}$.

We denote $\Sigma_1(b) \subset \Xi$ the subset of Constrained systems (1), such that $\widetilde{X}(0) = 0$ and one of the following statements is true:

**I–Cubic** $0$ is nonresonant and one of the eigenspaces associated with nonzero eigenvalues of $D\widetilde{X}(0)$ has cubic contact with the impasse surface $I_A$.

**I–Focus node** $D\widetilde{X}(0)$ has one zero eigenvalue and two equal and nonzero real eigenvalues $\lambda$, $D\widetilde{X}(0)$ is non diagonalizable, and the eigenspace associated to $\lambda$ is transversal to the impasse surface.

**I–Saddle–node** $D\widetilde{X}(0)$ has two zero eigenvalues, $0$ is a hyperbolic singularity of the vector field $F|_{P_0}$, and $0$ is a saddle–node for the regularization $\widetilde{X}$ on $P_0$ whose the eigenspaces are transversal to the impasse surface, where $P_0$ is an invariant plane of $\mathbb{R}^3$.

**I–Hopf** $D\widetilde{X}(0)$ has one zero eigenvalue and a pair of imaginary eigenvalues, and the trace of $D\widetilde{X}(0)$ has a zero with non vanishing derivative along the singularity curve on $P_0$, i.e., $0$ is a weak focus of first order for the regularization $\widetilde{X}|_{P_0}$, where $P_0$ is an invariant plane of $\mathbb{R}^3$.

#### Normal Forms

Consider $\Sigma_1 = \Sigma_1(a) \cup \Sigma_1(b)$, then the following statements hold:

1. $\Sigma_1$ is a codimension–one submanifold of $\Xi$;
2. $\Sigma_1$ is open and dense in $\Xi_1$ in the topology induced from $\Xi$;
3. For a residual set of smooth curves $\gamma : \mathbb{R} \to \Xi$, $\gamma$ meets $\Xi_1$ transversally and $\gamma^{-1}(\Xi_2) = \emptyset$, $\Xi_2 = \Xi_1 \setminus \Sigma_1$;
4. the normal forms of any one–parameter families of vector fields in $\Sigma_1$ is topologically equivalent to one of the following germs:

   **I–Lips** $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $(x_1 + \mu x_3 + x_2^2 x_3 + x_3^3)\dot{x}_3 = 1$;

   **I–Bec-to-bec** $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $(x_1 + \mu x_3 - x_2^2 x_3 + x_3^3)\dot{x}_3 = 1$;

   **I–Dove's tail** $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $(x_1 + x_2 x_3 + \mu x_3^2 + \kappa x_3^4)\dot{x}_3 = 1$, where $\kappa = \pm 1$.

   **I–Cubic** $\dot{x}_1 = 0$, $\dot{x}_2 = \lambda + x_1$, $(x_2 + \mu x_3 + x_1 x_3^2 + x_3^3)\dot{x}_3 = x_3$;

   **I–Focus node** $\dot{x}_1 = 0$, $\dot{x}_2 = 1$, $(x_2 + x_3)\dot{x}_3 = \mu x_2 + x_3$;

   **I–Saddle node** $\dot{x}_1 = 0$, $\dot{x}_2 = x_2$, $(x_2 + x_3 - \mu)\dot{x}_3 = \pm x_3$;

   **I–Hopf** $\dot{x}_1 = 1$, $\dot{x}_2 = 0$, $x_3\dot{x}_3 = -x_1 + \mu x_3 \pm x_3^3$.

### Conclusion

We provided a complete list of codimension–one impasse singularities for constrained systems defined on $\mathbb{R}^3$. Moreover, we presented all normal forms of these systems.

### References


[1] Jaume Llibre, Jorge Sotomayor, and Michail Zhitomirskii. Impasse bifurcations of constrained systems. volume 31 of *Fields Inst. Commun.*, pages 235–255. Amer. Math. Soc., Providence, RI, 2002.

[2] Jorge Sotomayor and Michail Zhitomirskii. Impasse singularities of differential systems of the form $A(x)x' = F(x)$. volume 169, pages 567–587. 2001.

[3] M. Zhitomirskiĭ. Local normal forms for constrained systems on $2$-manifolds. *Bol. Soc. Brasil. Mat. (N.S.)*, 24(2):211–232, 1993.