Dynamics and geometry on surfaces

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Introduction

In [1], a continuous family of sphere homeomorphisms was constructed which is a mild quotient of the inverse limit of the tent family of interval endomorphisms. This family has the *unimodal generalized pseudo-Anosov (ugpA)* maps as a countable, dense subfamily, whose spheres of definition have well-defined (and much studied) geometric structures. In this work, we study the family of unimodal generalized pseudo-Anosov maps and show that the associated surface has interesting properties, such as being Ahlfors regular and linearly locally contractible.

Generalized pseudo-Anosov maps

The article [3] introduced the notion of *generalized pseudo-Anosov (gpA)* maps, which generalize the pseudo-Anosov (pA) transformations introduced by Thurston by allowing the presence of an infinite number of singularities, but with only a finite number of accumulation points. As in the pseudo-Anosov case, the conic-flat structure continues to exist away from the accumulation points of singularities and it is possible to prove [5] that the induced complex structure extends uniquely over the latter. The notion of gpA maps appeared naturally in the dynamical study of the natural extension of multimodal endomorphisms of the interval with finite critical orbits.

More formally, given a homeomorphism $g : S \to S$ defined on a differentiable surface S, we say that g is a *generalized pseudo-Anosov homeomorphism* if it satisfies the following conditions: there exist a pair $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ of transverse measured foliations with singularities – either modeled on pronged singularities or accumulations of such, of which there are only finitely many – and a real number $\lambda > 1$ such that

$$g(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \frac{1}{\lambda}\mu^s)$$
$$g(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u).$$

In [4] it was shown that to every tent map of the interval with finite critical orbit it was possible to associate a gpA map of the sphere. This family of gpA homeomorphisms associated with post-critically finite unimodal endomorphisms was called the *unimodal* gpA family.

Definitions of regularity

Recall that a metric space X is *Ahlfors Q-regular* if there is a constant $C > 0$ such that the Q-dimensional Hausdorff measure $\mathcal{H}^Q$ of every closed r-ball $\bar{B}(a,r)$ satisfies

$$C^{-1}r^Q \le \mathcal{H}^Q(\bar{B}(a,r)) \le Cr^Q$$

when $0 < r \le \operatorname{diam}(X)$.

A metric space (Z,d) is called *λ-linearly locally contractible* where $\lambda \ge 1$, if every ball $B(a,r)$ in Z with $0 < r \le \operatorname{diam} Z/\lambda$ is contractible inside $B(a,\lambda r)$, i.e., there exists a continuous map $H : B(a,r)\times[0,1] \to B(A,\lambda r)$ such that $H(\cdot,0)$ is the identity on $B(a,r)$ and $H(\cdot,1)$ is a constant map. Furthermore, a metric space (Z,d) is called *λ-LLC* for $\lambda \ge 1$ (LLC stands for linearly locally connected) if the following two conditions are satisfied:

(λ-LLC$_1$) If $B(a,r)$ is a ball in Z and $x, y \in B(a,r)$, then there is a continuum $E \subset B(a,\lambda r)$ containing x and y.

(λ-LLC$_2$) If $B(a,r)$ is a ball in Z and distinct points $x,y \in Z \setminus B(a,r)$, then there is a continuum $E \subset Z \setminus B(a,r/\lambda)$ containing x and y.

In [2] it was proven that linearly local contractibility implies the LLC condition for compact connected topological n-manifolds, and is equivalent to it when $n = 2$.

Main theorems

The identification pattern in the theorem below was inspired by the construction of generalized pseudo-Anosov maps.

Theorem 1. *Consider the unit square and a pairing that glues its horizontal sides, which folds the right side in half and at the left side there is the pattern given by Figure 1. The surface quotient is Ahlfors regular and LLC.*

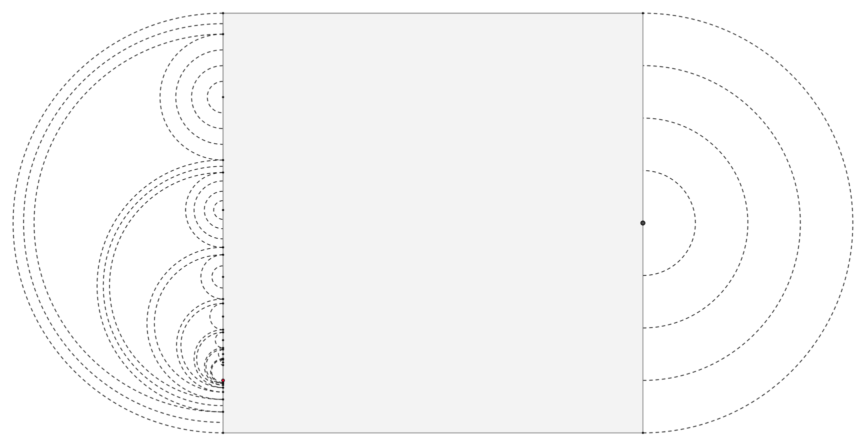

Figure 1: A example of a surface S.

Consider the unit square and a pairing that glues its horizontal sides and at the left side, there is the pattern given by Figure 2. A strip is placed between two conic points, as indicated in Figure 2. By placing this strip, the length of the upper part of the right side of the square becomes smaller than the lower one, so we need to make a modification in the identification of the right side of the square.

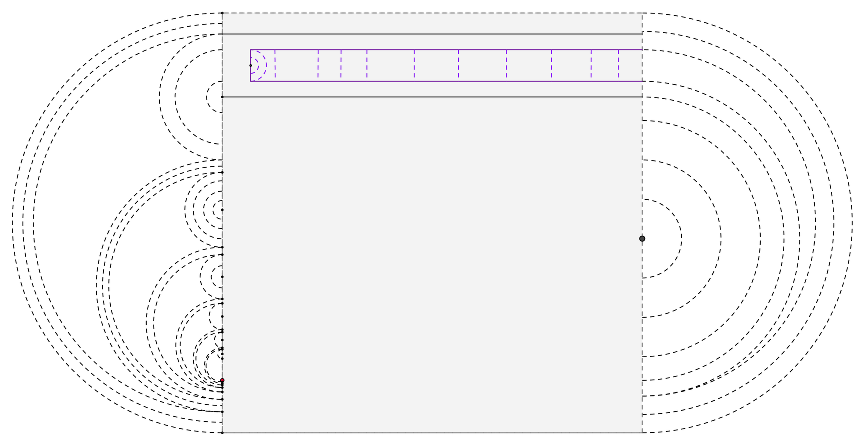

Figure 2: A example of a surface S.

Theorem 2. *The surface quotient as described is Ahlfors regular and LLC.*

The general idea is to place a countable number of strips like this in the region between two conic points without specifying the ends of the strips

What next?

We are working towards proving that the Ahlfors regularity and LLC constants are uniform (within 'height intervals') in the ugpA family. This will enable us to prove equicontinuity and, thus, to extract limits in the family. The final goal is to provide a completion of the ugpA family.

References

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