Infinite-Dimensional Genetic Algebras generated by Gibbs Measures

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Introduction and abstract

Genetic and evolution algebras are mathematical structures that model the genetic dynamics of populations. Gibbs measures are probability distributions that describe the equilibrium states of interacting systems in statistical mechanics. In this poster, we present a novel model of infinite-dimensional genetic algebras determined by configurations of finite spins on a countable set with their associated Gibbs measures. We show that this model preserves some properties of the finite-dimensional Gibbs algebras found in the literature and extends their results.

We also introduce the concept of infertility in the genetic dynamics when the configurations differ macroscopically. This leads to a decomposition of the algebra into a direct sum of fertile ideals with genetic realization. The proposed infinite-dimensional algebras are commutative, non-associative, with uncountable basis and zero divisors.

Basic Definitions

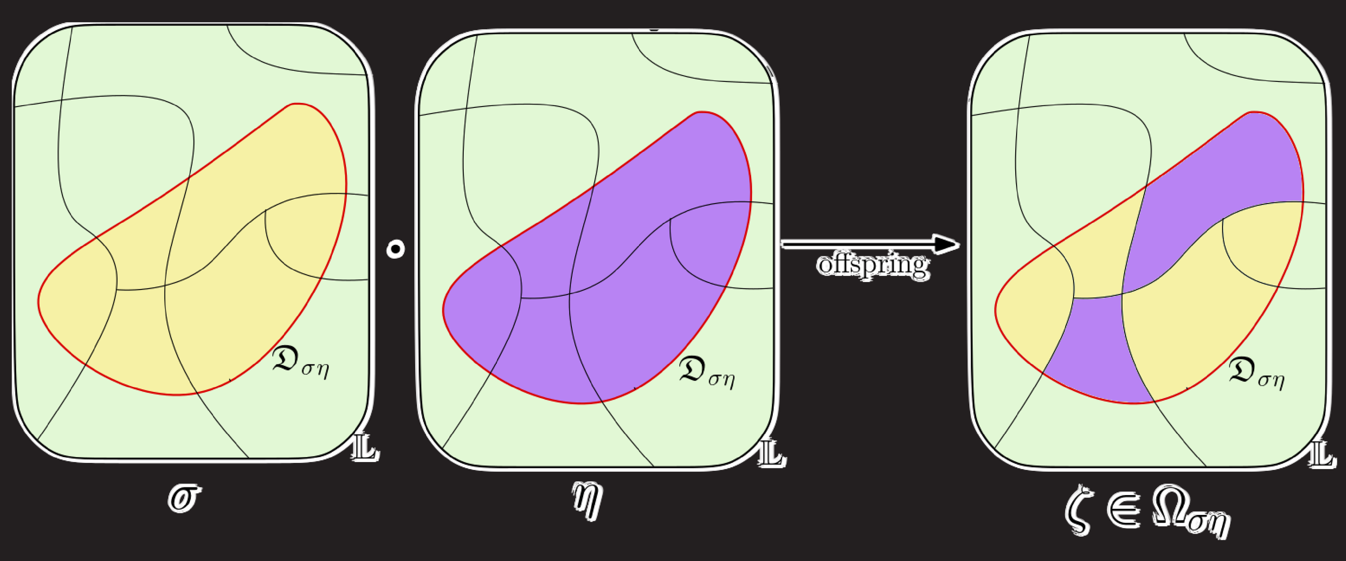
Let S be the finite set of spins of a countable set $\mathbb{L}$. Regard $\mathcal{L} \subseteq 2^{\mathbb{L}}$ as the set of finite subsets of $\mathbb{L}$. A configuration σ is an element of $\Omega = S^{\mathbb{L}}$. The interacting potential $\Phi = (\Phi_A)_{A\in\mathcal{L}}$ is a family of functions. By standard Gibbs measure theory, Φ determines the local Hamiltonians H_Λ^Φ for each $\Lambda \in \mathcal{L}$ and the local specification γ_Λ^Φ.

The set $\mathscr{G}(\Phi)$ is the set of all Gibbs measures μ determined by the local specification γ_Λ^Φ. A potential Φ exhibits phase transition when $|\mathscr{G}(\Phi)| > 1$.

Consider $\mathscr{C} \subseteq 2^{\mathbb{L}}$ to be a partition of $\mathbb{L}$. We call each $\Delta \in \mathscr{C}$ a cluster. Given $L \in 2^{\mathbb{L}}$ and $\sigma \in \Omega$, σ_L is the restriction of σ to L. The discrepancy set $\mathcal{D}_{\sigma\eta}$ consists of elements $x \in \mathbb{L}$ such that $\sigma(x) \neq \eta(x)$.

The offspring of σ and η is the set

$$\Omega_{\sigma\eta} = \{\xi \in \Omega\colon \forall \Delta \in \mathscr{C}(\xi_\Delta = \sigma_\Delta \text{ or } \xi_\Delta = \eta_\Delta) \text{ and } \mathcal{D}_{\sigma\eta} \in \mathcal{L}\}$$



Let $\mathfrak{B}_\Omega = \{e_\sigma\}_{\sigma\in\Omega}$ be a basis and let $\mathbb{K}$ be fixed with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

The $\mathscr{C}$-genetic Gibbs genetic algebra generated by $\mu \in \mathscr{G}(\Phi)$ on Ω is the $\mathbb{K}$-module $\mathcal{A}(\mathscr{C}, \mu, \Phi, \Omega)$ given by bilinear extension of

$$e_\sigma \cdot e_\eta = \sum_{\zeta\in\Omega_{\sigma\eta}} c_{\sigma\eta,\zeta} e_\zeta$$

where $c_{\sigma\eta,\zeta} = \dfrac{\mu(\zeta \mid \sigma_{\mathcal{D}_{\sigma\eta}^\complement})}{\mu(\Omega_{\sigma\eta} \mid \sigma_{\mathcal{D}_{\sigma\eta}^\complement})}$.

Consider $\mathtt{E}^\eta := \{\zeta \in \Omega\colon |\mathcal{D}_{\eta\zeta}| < +\infty\}$ and define the fertile ideal $\mathtt{F}^\eta = \langle \mathtt{E}^\eta \rangle$. It is related with infertility because $e_\sigma \cdot e_\eta = 0$ iff $\sigma \notin \mathtt{F}^\eta$. Let us fix $\widetilde{\Omega} \subseteq \Omega$ as the set that chooses a unique representative of each fertile ideal. Namely, for all $\eta \in \Omega$ there exists an unique $\sigma \in \widetilde{\Omega}$ s.t. $\mathtt{F}^\sigma = \mathtt{F}^\eta$. Therefore the results can be extended to all countable $\mathbb{L}$.

Main results

We list below some results obtained for the Gibbs algebra defined and studied in [1]. Before stating them, it is woth to point out that when $\mathbb{L}$ is finite $\mathcal{A} = \mathcal{A}(\mathscr{C}, \mu, \Phi, \Omega)$ is isomorphic to the finite dimensional Gibbs algebras found in the literature. Also, $\mathcal{A}$ is commutative and non-associative when $\dim \mathcal{A} \geq 3$ and $|\mathfrak{B}_\Omega| = |S|^{|\mathbb{L}|}$.

Theorems: Let $\mathcal{A} = \mathcal{A}(\mathscr{C}, \mu, \Phi, \Omega)$

- (*Decomposition into a sum fertile (Markov) ideals*)
  For all $\eta \in \Omega$
  $$e_\eta \mathcal{A} = \mathtt{F}^\eta$$
  and
  $$\mathcal{A} = \bigoplus_{\eta\in\widetilde{\Omega}} \mathtt{F}^\eta.$$
  Moreover, each $\mathtt{F}^\eta$ has a countable basis and $|\widetilde{\Omega}| = 2^{\aleph_0}$ when $|\mathbb{L}| = \aleph_0$.
- Let $\Phi \sim \Psi$ (or $\Phi \sim \tau^{-1}(\Psi)$ for τ a suitable transformation), then
  $$\mathcal{A}(\mathscr{C}, \mu, \Phi, \Omega) \simeq \mathcal{A}(\mathscr{C}, \mu', \Psi, \Omega).$$
- **Corollary** (*Stability under phase transition*)
  The Gibbs measures $\mu, \mu' \in \mathscr{G}(\Phi)$ generate the same $\mathscr{C}$-Gibbs algebra.
- Let $\mathtt{F}$ be a fertile ideal and $\pi_\mathtt{F} : \mathcal{A} \to \mathbb{K}$ a multiplicative functional given by linear expansion of $\pi_\mathtt{F}(e_\sigma) = 1$ when $e_\sigma \in \mathtt{F}$ and $\pi_\mathtt{F}(e_\sigma) = 0$ otherwise. Then
  $$\hom(\mathcal{A}, \mathbb{K}) = \{\pi_{\mathtt{F}^\sigma}\colon \sigma \in \widetilde{\Omega}\}.$$
  Therefore $(\mathcal{A}, \pi_\mathtt{F})$ if a weighted/baric algebra.
- (*Genetic algebras generated by products of Gibbs measures*)
  Let $\mathcal{A}_i = \mathcal{A}(\mathscr{C}_i, \mu_i, \Phi^i, \Omega_i)$ for $i \in \{1, \ldots, n\}$. Set
  $$\mathcal{A} = \mathcal{A}\left(\bigsqcup_{i=1}^n \mathscr{C}_i, \bigotimes_{i=1}^n \mu_i, \bigoplus_{i=1}^n \Phi^i, \Pi_{i=1}^n \Omega_i\right).$$
  Then $\mathcal{A}$ is isomorphic to the standard tensor algebra $\bigotimes_{i=1}^n \mathcal{A}_i$.
- Let $\mathscr{A}_f$ be a finite-dimensional subalgebra of $\mathcal{A}$ such that the interaction potential Φ has finite range such that
  $$\mathscr{A}_f = \bigoplus_{i=1}^n \mathtt{A}_i \quad \text{with } \mathtt{A}_i \text{ subalgebra of a fertile ideal } \mathtt{F}_i.$$
  Then there exists $\Lambda' \in \mathcal{L}$ such that every $\mathtt{A}_i$ is a subalgebra of the finite dimensional Gibbs algebra $\mathcal{A}(\mathscr{C}|_{\Lambda'}, \mu_{\Lambda'}, \Phi, S^{\Lambda'})$.

References

[1] C. F. Coletti, L. R. de Lima, and D. A. Luiz. Infinite-dimensional genetic and evolution algebras generated by Gibbs measures. *arXiv:2212.06450*, 2022.

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