Invariant tori and boundedness of solutions of non-smooth oscillators with Lebesgue integrable forcing term

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Abstract

We consider the Duffing-type equation

$$\ddot{x} + \mathrm{sign}(x) = p(t), \tag{1}$$

where sign stands for the standard sign function and p is Lebesgue integrable and T-periodic function.

We want to show that all solutions of (1) are bounded, provided that $p(t)$ has a vanishing average.

We achieve our aim by showing the existence of a infinite collection of nested invariant tori, which in turn are foliated by periodic orbits.

Statements and Main Result

The differential equation (1) can be seen as the vector field

$$\begin{cases} \phi' = 1, \\ x' = y, \\ y' = -\mathrm{sign}(x) + p(\phi). \end{cases} \tag{2}$$

- The phase space is $M = \mathbb{S}^1 \times \mathbb{R}^2$, with $\mathbb{S}^1 = \mathbb{R}/T\mathbb{Z}$.
- We define the integrals

$$P_1(t) := \int_0^t p(s)\mathrm{d}s \quad \text{and} \quad P_2(t) := \int_0^t P_1(s)\mathrm{d}s,$$

and, as usual, let $\overline{p}$ denote the average of $p(t)$, i.e.

$$\overline{p} := \frac{1}{T}\int_0^T p(s)\mathrm{d}s = \frac{P_1(T)}{T}.$$

Notice that the function $P_1(t)$ is continuous and the function $P_2(t)$ is continuously differentiable.

- The plane $\Sigma := \{(\phi, x, y) \in M : x = 0\}$ is a region of discontinuity of the vector field (2).
- **Solutions:** Equation (1) matches all the necessary conditions to the existence and uniqueness of its solutions, which in turn are only continuous in t.

Theorem A. *Suppose that $p(t)$ is a Lebesgue integrable T-periodic function satisfying $\overline{p} = 0$. Then, there exists a sequence $\mathbb{T}_n \subset \mathbb{S}^1 \times \mathbb{R}^2$ of nested invariant tori of the vector field (2) satisfying:*

$$M = \bigcup_{n\in\mathbb{N}} \mathrm{int}(\mathbb{T}_n),$$

where $\mathrm{int}(\mathbb{T}_n)$ denotes the region enclosed by $\mathbb{T}_n$. In addition, for each $n \in \mathbb{N}$, the torus $\mathbb{T}_n$ is foliated by periodic solutions.

Corollary 1. *Suppose that $p(t)$ is a Lebesgue integrable T-periodic function satisfying $\overline{p} = 0$. Then, for each $(t_0, x_0, \dot{x}_0) \in \mathbb{R} \times \mathbb{R}^2$,*

$$\sup_{t\in\mathbb{R}}\{|x(t; t_0, x_0, \dot{x}_0)| + |\dot{x}(t; t_0, x_0, \dot{x}_0)|\} < \infty,$$

where $x(t; t_0, x_0, \dot{x}_0)$ denotes the solution of (1) with initial condition $(t_0, x_0, \dot{x}_0)$.

Preliminary results

For each $n \in \mathbb{N}$, define the functions $y_n^+ : [0, T] \to \mathbb{R}$ and $y_n^- : [0, T] \to \mathbb{R}$ by

$$y_n^{\pm}(\phi_0) = \pm\frac{nT}{2} + P_1(\phi_0) - \frac{P_2(T)}{T}$$

and, for each $n \in \mathbb{N}$, such that $y_n^-(\phi_0) < y_n^+(\phi_0)$ for every $\phi_0 \in [0, T]$, define the surface

$$\mathcal{T}_n := \mathcal{T}_n^+ \cup \mathcal{T}_n^-,$$

where

$$\mathcal{T}_n^{\pm} := \{(\phi_0, \Psi_n^{\pm}(\phi_0, y_0), y_0) : \phi_0 \in \mathbb{R},\ y_0 \in [y_n^-(\phi_0), y_n^+(\phi_0)]\},$$

and

$$\Psi_n^{\pm}(\phi_0, y_0) := \frac{1}{8}\left(\pm n^2T^2 \mp 4y_0^2 - 8P_2\left(\frac{nT}{2} \pm y_0 \mp P_1(\phi_0) \pm \frac{P_2(T)}{T} + \phi_0\right) + 4P_2(T)\left(n \pm \frac{P_2(T)}{T^2}\right) - 4P_1(\phi_0)(\pm P_1(\phi_0) \mp 2y_0) + 8P_2(\phi_0)\right).$$

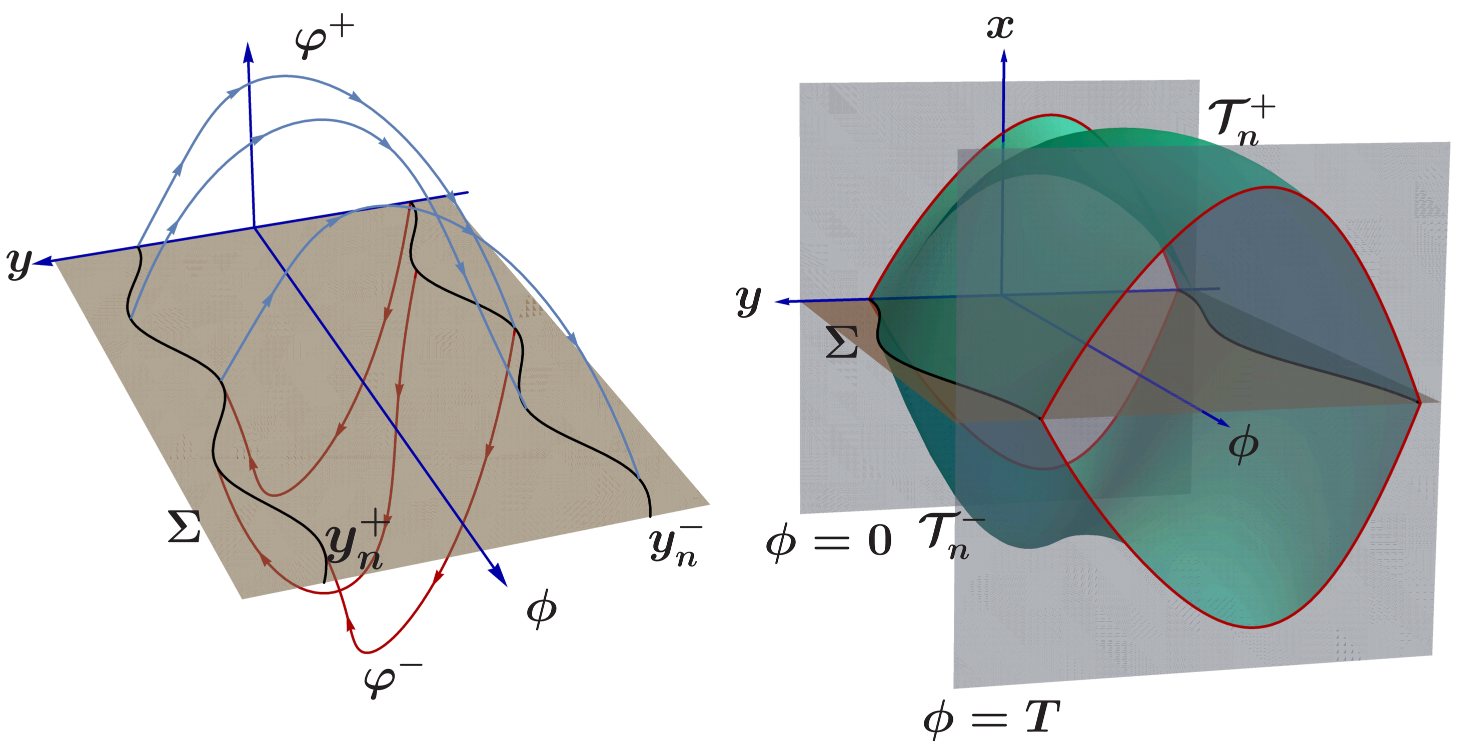
Lemma 1 (Fundamental Lemma). *Let $n \in \mathbb{N}$ be fixed. Assume that, for every $\phi_0 \in [0, T]$,*

$$\left|TP_1(\phi_0) - P_2(T)\right| < \frac{nT^2}{2}$$

and

$$\left|tP_2(T) + TP_2(\phi_0) - TP_2(t + \phi_0)\right| < \frac{T}{2}t(nT - t),\ \ t \in (0, nT).$$

Then, $\mathcal{T}_n$ is an invariant torus of the vector field (2). Moreover, $\mathcal{T}_n$ is foliated by $2nT$-periodic orbits.



Proof of the Main Result

The proof of Theorem A will follow as an immediate consequence of the next result, which will provide the existence of $n^* \in \mathbb{N}$ such that the conditions of the Fundamental Lemma are satisfied for every $n \geq n^*$. Accordingly, the sequence of invariant tori stated by Theorem A will be given by $\mathbb{T}_n := \mathcal{T}_{n+n^*}$, $n \in \mathbb{N}$.

Proposition 1. *Let $p(t)$ be a Lebesgue integrable T-periodic function such that $\overline{p} = 0$. Then, there exists $n^* \in \mathbb{N}$ such that $\mathcal{T}_n$ is an invariant torus of (2) for every $n \geq n^*$.*

By assuming $p(t)$ to be an L^∞-function on $[0, T]$, instead of just Lebesgue integrable, we show that the surface $\mathcal{T}_n$ is an invariant torus of (2) for every $n \in \mathbb{N}$ bigger than $||p||_{L^\infty}$.

Proposition 2. *Let p be a T-periodic function with vanishing average and suppose that there exists $M > 0$ such that $||p||_{L^\infty} < M$. Then, the surface $\mathcal{T}_n$ is an invariant torus of (2) for all $n \in \mathbb{N}$ satisfying $n \geq M$.*

References

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