Spectral study about Bopp-Podolsky equationns with Landesman-Lazer condition

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Abstact

In this paper we are concerned in the problem

$$\begin{cases}\alpha\Delta^2 u+\beta\Delta u=\mu u+\tau h(x,u) & \text{in } \Omega,\\ \mathcal{B}u=0 & \text{on } \partial\Omega,\end{cases} \tag{BL}$$

$$\mathcal{B}u=\begin{cases}u, & \text{if } \alpha=0\\ u=\Delta u, & \text{if } \alpha>0\end{cases}$$

where Ω is bounded smooth domain of $\mathbb{R}^N$, $\alpha\ge 0$ and $-\infty<\beta<\alpha\lambda_1$ where λ_1 is the first eigenvalue of the classical Dirichlet problem.

We study the existence of solutions in the intervals where (BL) presents resonance, where the solutions are near the first eigenvalue, of the Bi-harmonic problem $\alpha\Delta^2 u+\beta\Delta u$ in $\mathbb{H}$, and the parameters μ and τ are near the zero.

A minimization Problem

Let $h:\bar{\Omega}\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function that satisfies:

- **Boundedness.** There exists $f\in L^\sigma(\Omega)$, such that $|h(x,s)|\le f(x)$, for every $s\in\mathbb{R}$, a. e. in Ω.
- **Lipschitz continuity.** There exists $\zeta\in L^\sigma(\Omega)$, such that $|h(x,s_1)-h(x,s_2)|\le\zeta(x)\,|s_1-s_2|$, for every $s_1,\ s_2\in\mathbb{R}$, a. e. in Ω,

with $\sigma>\left\{\frac{N}{2},1\right\}$.

Landesman-Lazer conditions.

There exist $k\in\mathbb{N}$ and $t_i\in\mathbb{R}$, $t_i<t_{i+1}$, $i=1,\dots,k$, such that

$$\left[\int_\Omega h(x,t_i\phi_1)\phi_1 dx\right]\left[\int_\Omega h(x,t_{i+1}\phi_1)\phi_1 dx\right]<0, \tag{LL}$$

where ϕ is a positive eigenfunction associated to μ_1. There exist real numbers t_1 and t_2, with $t_1<t_2$. such that

$$\int_\Omega h(x,t_1\phi_1)\phi_1 dx>0>\int_\Omega h(x,t_2\phi_1)\phi_1 dx. \tag{LL-1}$$

$$\int_\Omega h(x,t_1\phi_1)\phi_1 dx<0<\int_\Omega h(x,t_2\phi_1)\phi_1 dx. \tag{LL-2}$$

Weak solutions of (BL) are critical points of the functional

$$I_{\mu,\tau}(u)=\frac{1}{2}\|u\|^2_{\alpha,\beta}-\frac{\mu}{2}|u|_2^2-\tau\int_\Omega H(x,u)dx$$

where $H(x,t)=\int_0^t h(x,s)ds$ and $0<\mu<\mu_1$. Due to **boundedness** the functional $I_{\mu,\tau}$ is of class $C^1(\mathbb{H})$ and weakly lower semi-continuous.

The space $\mathbb{H}:=H_0^1(\Omega)\cap H^2(\Omega)$ endowed with the scalar product

$$(u,v)_{\alpha,\beta}=\alpha\int_\Omega\Delta u\Delta v dx-\beta\int_\Omega\nabla u\nabla v dx$$

is a Hilbert, reflexive and separable space. Throughout this work, we denote by

$$|u|_p:=\left(\int_\Omega|u|^p dx\right)^{\frac{1}{p}} \text{ and } \|\phi\|:=\left(\int_\Omega|\nabla\phi|^2 dx\right)^{\frac{1}{2}},$$

the norms of the spaces $L^p(\Omega)$ and $H_0^1(\Omega)$ respectively.

We are going to prove the following result.

Theorem 1. *Assume that* ***boundedness*** *and LL-1 hold. Then there exists positive constants τ^* and θ^* such that, for every $\tau\in(0,\tau^*)$ and $|\mu-\mu_1|<\tau\theta^*$, Problem (BL) has a weak solution $u_\tau=\kappa\phi_1+v$, with $\kappa\in(t_1,t_2)$ and $v\in\langle\phi\rangle^\perp$.*

Existence of a minimum solution

Lemma 1. *Assume the* ***boundedness*** *condition. Then for every $\tau>0$, the functional $I_{\mu,\tau}$ is bounded from below and coercive on*

$$C:=\left\{u=\kappa\phi_1+v;\kappa\in[t_1,t_2],v\in\langle\phi\rangle^\perp\right\}.$$

Corollary 1. *For every $\tau>0$, there exists $u_\tau\in C$ such that $I_{\mu,\tau}(u_\tau)=m_C$. Moreover, for every $\tau>0$ and $\kappa\in[t_1,t_2]$, there exist $v_\tau\in\langle\phi_1\rangle^\perp$ such that $I_{\mu,\tau}(\kappa\phi+v_\tau)=m_\kappa$.*

Lemma 2. *Assume that* ***boundedness***. *Then, given $\delta>0$, there exists $\tau_1>0$ such that $\|v\|<\delta$, for every $v\in S_{t_1}\cup S_{t_2}$.*

Following the last result we can assume that the parameter μ is around of μ_1. We will look for estimates to calculate the interval we will consider the interval $|\mu-\mu_1|$.

End of the proof of Theorem 1.

By Corollary 1, for every $\tau>0$ there exists a weak solution of BL, $u_\kappa=\kappa\phi_1+v_\tau\in C$, then $m_C=I_{\mu,\tau}(u_\kappa)$. Because C is a convex set and I is coercive and weak lower semicontinuous m_C is unique, then $m_C=m_\kappa$. It remains to prove that there exists $\tau^*>0$ and $\theta^*>0$ such that u_τ is inside of C and $|\mu-\mu_1|<\tau\theta^*$.

The condition implies that the functional

$$Q(u)=\int_\Omega h(x,u)\phi_1 dx, \text{ with } u\in\mathbb{H}$$

is continuous. By the hypothesis LL-1, implies that for some $a>0$ and $\delta>0$ such that for every $v\in\langle v\rangle^\perp$ with $\|v\|<\delta$,

$$\begin{aligned}&\int_\Omega h(x,t_1\phi_1+v)\phi_1 dx>a>0\\ &>-a>\int_\Omega h(x,t_2\phi_1+v)\phi_1 dx.\end{aligned} \tag{1}$$

Given $\delta>0$ above, by Lemma 2 there exists $\tau^*=\tau_1$ by Considering $\tau\in(0,\tau^*)$, such that $v\in S_{t_1}\cup S_{t_2}$ and $\|v\|_{\alpha,\beta}<\delta$. Since $t_1<t_2$, being ϕ_1 the first eigenfunction of the eigenvalue problem BL, taking $0<\mu<\mu_1$ and using the inequality (1) we obtain that

$$\begin{aligned}&\left\langle I'_{\mu,\tau}(t_1\phi_1+v)-I'_{\mu,\tau}(t_2\phi_1+v),\phi_1\right\rangle\\ &=\langle t_1\phi_1,\phi_1\rangle-\mu\int_\Omega(t_1\phi_1+v)\phi_1-\tau\int_\Omega h(x,t_1\phi_1+v)\phi_1\\ &-\langle t_2\phi_1,\phi_1\rangle+\mu\int_\Omega(t_2\phi_1+v)\phi_1+\tau\int_\Omega h(x,t_2\phi_1+v)\phi_1\\ &<(t_1-t_2)\|\phi_1\|^2_{\alpha,\beta}-\frac{\mu}{\mu_1}(t_1-t_2)\|\phi_1\|^2_{\alpha,\beta}-2a\tau\\ &=(t_1-t_2)\left(1-\frac{\mu}{\mu_1}\right)\|\phi_1\|^2_{\alpha,\beta}-2a\tau<0,\end{aligned} \tag{2}$$

then $|\mu_1-\mu|<\tau\frac{2a\mu_1}{t_2-t_1}$.

For the existence of solution under hypothesis (LL-1) and h $L^\sigma(\Omega)$-locally bounded is used an approximation technique, the existence of a solution for the original problem is obtained by finding a local minimum for the functional associated with an appropriated truncation for h. Under the hypothesis (LL-2), the solution is obtained via Lyapunov-Schmidt reduction method.

Referências

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