About the shadowing property for group actions on manifolds.

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Abstract

We present an ongoing work on the relation between the C^0-stability and the shadowing property for actions of free groups by homeomorphisms on manifolds. Following [2] and [1], we aim to find a characterization of actions on the circle that have the shadowing property. In this direction, we present some partial results, mainly related to conditions on the minimal set.

Introduction

Shadowing or pseudo-orbit tracing is a well-developed topic in dynamical systems. We can find in [6] a comprehensive survey on the subject. This concept was generalized in [5] for finitely generated groups acting in a metric space. There are given conditions for the action of a finitely generated nilpotent group to have shadowing, so the following question is set: which groups admit an action satisfying the shadowing property? In [2] and [1], the shadowing property of action of free groups of rank $n \geq 2$ on a manifold M and its relation with C^0-stability is established when the dimension of M is greater than 2 (see Theorem 2 below). Remark that for the usual dynamical system (the action of $\mathbb{Z}$ by a homeomorphism) such relation was already known (see [7] and [8], for example). In [1], conditions about the minimal set are given for the action of the free group F_2 in $\mathbb{S}^1$ to have shadowing. Also in [1], an enlightening example is constructed. See Theorem 1 and Example 1 below. Based on these results, we present here some questions that, after our incipient studies, we believe can be answered - partially at least - affirmatively.

Discussions

Let G be a group and X be a topologial space. Let $\Phi : G \times X \to X$ be a map such that for each $g \in G$, $x \mapsto \Phi(g,x)$ is a homeomorphism (we denote this map by Φ_g) and $\Phi(g_1g_2,x) = \Phi(g_1(\Phi(g_2,x))$, i.e., $\Phi_{g_1g_2}(x) = (\Phi_{g_1} \circ \Phi_{g_2})(x)$ for all $g_1, g_2 \in G$ and $x \in X$. We shall call the map Φ an *action of the group G on X* and the triplet (G,X,Φ) a *dynamical system*. We define an *orbit of $x \in X$ under Φ* as the set $O(x) := \{\Phi_g(x) : g \in G\}$.

Given a group G, we call a *G-sequence* in X a function $g \mapsto x_g$, denoted by $\{x_g\}$.

Let (X,d) be a metric space. For a finitely generated group G, with a symmetric generator S, given $\delta > 0$, we say that a G-sequence $\{x_g\}$ is a *δ- pseudotrajectory* if

$$d(\Phi_s(x_g), x_{sg}) < \delta, \ \forall g \in G \text{ and } \forall s \in S.$$

Definition 1. *Let (X,d) be a metric space, G be a finitely generated group and (G,X,Φ) be a dynamical system. We say that Φ has the* shadowing property *if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ- pseudotrajectory $\{x_g\}$ there exists a point $y \in X$ such that*

$$d(x_g, \Phi_g(y)) < \varepsilon \ \forall g \in G.$$

A non-empty set $\Lambda \subset X$ is said to be *minimal for* Φ if $\overline{O(x)} = \Lambda$ for any $x \in \Lambda$.

Let S be a finite generator of a group G and (X,d) a compact metric space. We denote by $\mathsf{Act}(G,X)$ the set of actions of G in X and define a metric d_S on $\mathsf{Act}(G,X)$ by

$$d_S(\Phi,\Psi) := \sup_{\substack{x \in X \\ s \in S}}\{d(\Phi_s(x), \Psi_s(x))\}, \ \forall \Phi, \Psi \in \mathsf{Act}(G,X).$$

We say that an action $\Phi \in \mathsf{Act}(G,X)$ is C^0-stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\Psi \in \mathsf{Act}(G,X)$ satisfying $d_S(\Phi,\Psi) < \delta$, there exists a continuous surjective map $h : X \to X$ such that $d(h(x),x) < \varepsilon$ for all $x \in X$ and $h \circ \Psi_g = \Phi_g \circ h$ for every $g \in G$.

We denote by F_2 the free group of rank 2. By [3, Theorem 2.1.1], we know that a minimal set for a dynamical system $(G, \mathbb{S}^1, \Phi)$, where Φ_g is an orientation preserving homeomorphism in $\mathbb{S}^1$ for all $g \in G$, there could be only one of three possibilities: a finite orbit of Φ; $\mathbb{S}^1$; a Cantor set K. In the last case, K is the only minimal set for Φ and is contained in the set of accumulation points of every orbit.

The theorem and example below are from [1].

Theorem 1. *Let $(F_2, \mathbb{S}^1, \Phi)$ be a dynamical system. If there exists a minimal set Λ for Φ and Λ is not a Cantor set, then Φ does not have the shadowing property.*

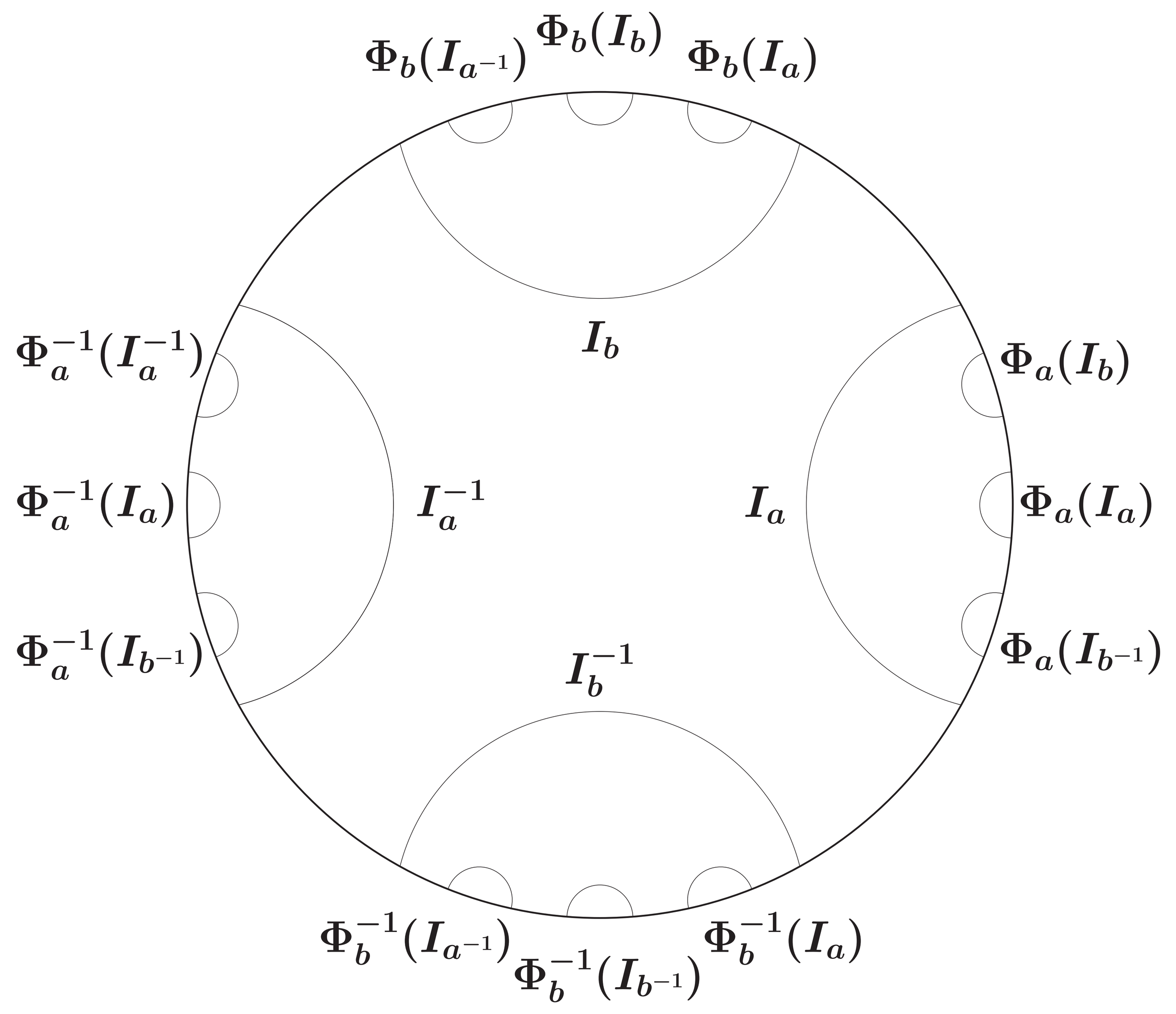
Example 1. *Let $S = \{a, b, a^{-1}, b^{-1}\}$ be a symmetric generator of F_2. We define continuous maps $\Phi_a, \Phi_b : S^1 \to S^1$ with the following properties:*

1. *Φ_a and Φ_b are north-south pole type homeomorphisms. We denote the two fixed points of Φ_a as N_a and S_a as well the two fixed points of Φ_b as N_b and S_b. Recall that $\Omega(\Phi_a) = \{N_a, S_a\}$ and $\Omega(\Phi_b) = \{N_b, S_b\}$.*
2. *There exist four open balls: $B_{N_a} = B(N_a, r_a)$, $B_{S_a} = B(S_a, l_a)$, $B_{N_b} = B(N_b, r_b)$ and $B_{S_b} = B(S_b, l_b)$, satisfying*
   - *$\overline{B_{N_i}} \cap \overline{B_{S_i}} = \emptyset$ for all $i \in \{a, b\}$.*
   - *$d(\Phi_i(x), \Phi_i(y)) > 2d(x,y)$ for all $x, y \in B_{N_i}$, $i \in \{a,b\}$.*
   - *$d(\Phi_i(x), \Phi_i(y)) < \frac{1}{2}d(x,y)$ for all $x, y \in B_{S_i}$, $i \in \{a,b\}$.*
   - *$\Phi_i(\partial B_{N_i}) \subset B_{S_i}$, for all $i \in \{a,b\}$.*
   - *$\overline{B_{N_b} \cup B_{S_b}} \subset (\overline{B_{N_a} \cup B_{S_a}})^C$.*

Let $I_a = \Phi_a(B_{N_a})$, $I_{a^{-1}} = \Phi_{a^{-1}}(B_{S_a})$, $I_b = \Phi_b((B_{N_b}))^c$ and $I_{b^{-1}} = \Phi_b^{-1}((B_{S_b}))^c$. Let $A_0 = I_a \cup I_{a^{-1}} \cup I_b \cup I_{b^{-1}}$ and

$$A_{n+1} = [\Phi_a(A_n) \cap \Phi_{a^{-1}}(A_n) \cap \Phi_b(A_n) \cap \Phi_{b^{-1}}(A_n)] \cap A_n.$$

The Cantor set $K = \bigcap_{n \geq 1} A_n$ is a minimal set for the system $(F_2, \mathbb{S}^1, \Phi)$ and the dynamical system has the shadowing property.



Inspired by Theorem 1 and Example 1, we ask:

Question 1. *If $(F_2, \mathbb{S}^1, \Phi)$ has a Cantor set K as its minimal set, and K is also the union of the sets of accumulation points of each and every orbit of Φ, is it true that the dynamical system has the shadowing property?*

About C^0-stability and shadowing, we have [2, Theorem A]:

Theorem 2. *Let M be a compact manifold of dimension greater than or equal to two and (F_2, M, Φ) be a dynamical system. If Φ is C^0-stable then Φ has the shadowing property.*

The proof of Theorem 2 uses the following lemma, [4, Lemma 13]:

Lemma 1. *Let M be a compact manifold of dimension greater or equal to two. Suppose a finite collection $\{(p_i, q_i) \in M \times M \ \forall i = 1, ..., r\}$ is specified together with a small $\lambda > 0$ such that $d(p_i, q_i) < \lambda$ for all $1 \leq i \leq r$, and if $i \neq j$ then $p_i \neq p_j$ and $q_i \neq q_j$. Then there exists a diffeomorphism $f : M \to M$ such that $d_{C^0}(f, id) < 2\pi\lambda$ and $f(p_i) = q_i \ \forall 1 \leq i \leq r$.*

Therefore, it is not possible to use the same approach to discuss whether the C^0-stability of a dynamical system $(F_2, \mathbb{S}^1, \Phi)$ implies that it has the shadowing property. This is our main question:

Question 2. *If a dynamical system $(F_2, \mathbb{S}^1, \Phi)$ is C^0-stable, does it have the shadowing property?*

In order to begin answering the Question 2, we may ask:

Question 3. *If a dynamical system $(F_2, \mathbb{S}^1, \Phi)$ is C^0-stable, is it true that for each $g \in G$, Φ_g is C^0-stable?*

Question 3 is interesting in itself. We have partial results that indicate that when for some $g \in G$, Φ_g has periodic points, then it is C^0-stable.

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