Hypersurfaces in product spaces belonging to the class $\mathcal{A}$

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Abstract

This poster deals mostly with hypersurfaces $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ in the class $\mathcal{A}$; here $\mathbb{Q}^n_\epsilon$ stands for a simply connected space form with constant sectional curvature $\epsilon \in \{-1, 0, 1\}$. Firstly, the definition of such hypersurfaces is presented. Secondly, two fundamental families of such hypersurfaces are introduced; hypersurfaces in the second family are built up from a smooth real-valued function with positive derivative and a parallel family of hypersurfaces. Thirdly, the local geometry of arbitrary hypersurfaces in the class $\mathcal{A}$ is explained. In fact, these hypersurfaces can be thought of as a collection of hypersurfaces of the two fundamental families that are smoothly attached. After this, we characterise hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, $\epsilon \in \{-1, 1\}$, belonging to the class $\mathcal{A}$ as codimension-2 submanifolds of $\mathbb{E}^{n+2}$ with nonzero T that have flat normal bundle; here $\mathbb{E}^{n+2}$ denotes either Lorentzian space $\mathbb{L}^{n+2}$ or Euclidean space $\mathbb{R}^{n+2}$ depending upon $\epsilon = -1$ or $\epsilon = 1$. Finally, as a corollary, a complete characterisation of constant angle hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ is provided.

Introduction

Let $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ be a hypersurface, N a (possibly local) unit normal to it, and S the shape operator in the direction of N. The *height function* $h \in C^\infty(M)$ of f is given by $h = \pi_2 \circ f$, where $\pi_2 : \mathbb{Q}^n_\epsilon \times \mathbb{R} \to \mathbb{R}$ is the standard projection. The gradient $\partial/\partial t := \operatorname{grad} \pi_2$ is characterised by $\partial/\partial t \equiv 1 \in \mathbb{R}$.

Denote by $\mathcal{A}$ the class of such hypersurfaces for which the gradient $T := \operatorname{grad} h$ is an eigenvector of S.

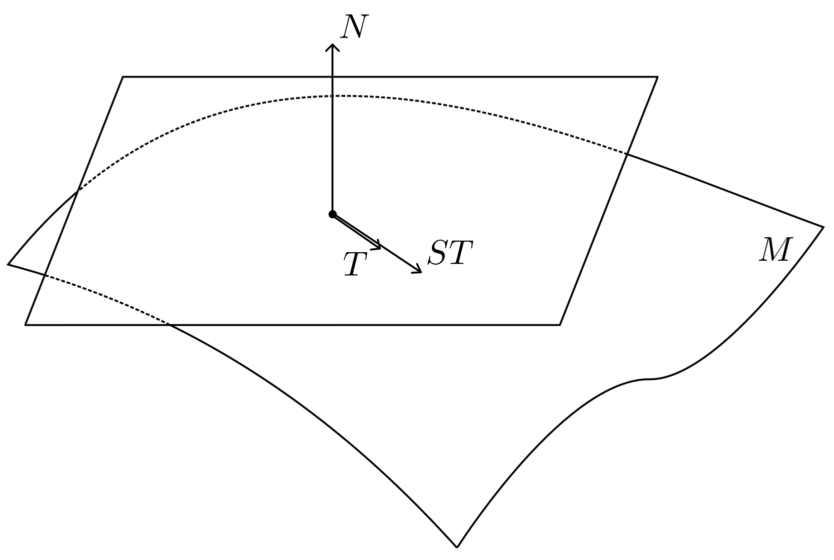

Figure 1: Hypersurface M in the class $\mathcal{A}$.

Fundamental Examples

Example 1. Products $M^{n-1} \times \mathbb{R}$, where $M^{n-1} \hookrightarrow \mathbb{Q}^n_\epsilon$ is a hypersurface.

Example 2. Given a hypersurface $g : M^{n-1} \to \mathbb{Q}^n_\epsilon$ admitting a unit normal N, and denoting by exp the exponential map of $\mathbb{Q}^n_\epsilon$, the parallel hypersurface $g_s : M \to \mathbb{Q}^n_\epsilon$, $s \in \mathbb{R}$, of g is determined by

$$g_s(x) = \exp_x(sN(x)).$$

Theorem 1. *If $a : I \to \mathbb{R}$ is a smooth function over the open interval $I \subset \mathbb{R}$ with $a' > 0$, the map $f : M^{n-1} \times I \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ determined by*

$$f(x, s) = g_s(x) + a(s)\frac{\partial}{\partial t},$$

and restricted to regular points, is a hypersurface in $\mathcal{A}$.

Local geometry

The *angle function* ν of f is characterised by

$$\nu = \left\langle N, \frac{\partial}{\partial t} \right\rangle$$

Whereas hypersurfaces in Example 1 correspond to the case $\nu \equiv 0$, hypersurfaces in Example 2 have $\nu \neq 0$ everywhere.

Theorem 2. *Any hypersurface $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$, $n \geq 2$, in the class $\mathcal{A}$ with $\nu \neq 0$ everywhere, coincides locally with a hypersurface in Example 2.*

Consequently, an arbitrary f in $\mathcal{A}$ coincides locally with either a hypersurface in Example 1 or a hypersurface in Example 2.

Alternative description

For $\epsilon \in \{1, -1\}$, $\mathbb{Q}^n_\epsilon$ is a hypersurface of either $\mathbb{R}^{n+1}$ or Lorentzian space $\mathbb{L}^{n+1}$, both are to be denoted by $\mathbb{E}^{n+1}$, depending upon $\epsilon = 1$ or $\epsilon = -1$. The product $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ is thus a hypersurface of $\mathbb{E}^{n+2}$. Consequently, a hypersurface $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ in $\mathcal{A}$ can be thought of as a submanifold with codimension 2 of $\mathbb{E}^{n+2}$.

Theorem 3. *At points of the hypersurface M where T is nonzero, T is an eigenvector of A as long as M has flat normal bundle in $\mathbb{E}^{n+2}$.*

Characterisation of constant angle hypersurfaces

We say that f is a *constant angle hypersurface* if ν is constant.

Example 3. Open subsets M of slices $\mathbb{Q}^n_\epsilon \times \{t\}$, $t \in \mathbb{R}$.

Example 4. Hypersurfaces in Example 1 and their open subsets.

Example 5. If we take the function a in Theorem 1 as being given by $a(s) = As + B$, where $A > 0$ and B are real numbers, the resulting f is a constant angle hypersurface.

Theorem 4. *A constant angle hypersurface $f : M^n \to \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is either a hypersurface in Example 3, one in Example 4, or a hypersurface in Example 5.*

References

[1] R. Tojeiro. On a class of hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. *Bull Braz Math Soc*, 41:199–209, 2010.

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