On the cyclicity of monodromic tangential singularities: a look beyond the pseudo-Hopf bifurcation

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Abstract

The cyclicity problem consists in estimating the number of limit cycles bifurcating from a monodromic singularity of planar vector fields and is usually addressed by means of Lyapunov coefficients. For nonsmooth systems, besides the limit cycles bifurcating by varying the Lyapunov coefficients, monodromic singularities lying on the switching curve can always be split apart generating, under suitable conditions, a sliding region and an extra limit cycle surrounding it. This bifurcation phenomenon is called pseudo-Hopf bifurcation and has been used to increase the lower bounds for the cyclicity of monodromic singularities in Filippov vector fields. We aim to go beyond the pseudo-Hopf bifurcation by showing that the destruction of $(2k, 2k)$-monodromic tangential singularities give birth to at least k limit cycles surrounding sliding segments.

Introduction

The *cyclicity problem* is a classical and important topic in the qualitative theory of planar vector fields. It consists in estimating the number of limit cycles bifurcating from a monodromic singularity p (center or focus) when it is perturbed inside a given family $\mathcal{F}$ of vector fields. Such a number of limit cycles is called *cyclicity* of the monodromic singularity p in $\mathcal{F}$.

Limit cycles bifurcating from monodromic singularities are usually studied by means of the so-called Lyapunov coefficients, V_i's, which are determined by the coefficients in the power series expansion of the first-return map $\pi(x)$ around the singularity. More specifically, when π is analytic in a neighborhood of $x = 0$, $\pi(x) - x = \sum_{i\geq 1} V_i x^i$ for x close to 0.

In the nonsmooth context, besides the limit cycles bifurcating by varying the Lyapunov coefficients, monodromic singularities lying on the switching curve can always be split apart generating, under suitable conditions, a sliding region and an extra limit cycle surrounding it. This bifurcation phenomenon is called pseudo-Hopf bifurcation.

Consider the following piecewise smooth vector field:

$$Z(x,y) = \begin{cases} Z^+(x,y) = \begin{pmatrix} X^+(x,y) \\ Y^+(x,y) \end{pmatrix}, & y > 0, \\ Z^-(x,y) = \begin{pmatrix} X^-(x,y) \\ Y^-(x,y) \end{pmatrix}, & y < 0. \end{cases} \tag{1}$$

The Filippov's convention [1] will be assumed for trajectories of (1).

we say that (1) has a $(2k^+, 2k^-)$*-monodromic tangential singularity* at the origin, provided that:

C1. $X^{\pm}(0,0) \neq 0$, $Y^{\pm}(0,0) = 0$, $\dfrac{\partial^i Y^{\pm}}{\partial x^i}(0,0) = 0$ for $i \in \{1, \dots, 2k^{\pm} - 2\}$, and $\dfrac{\partial^{2k^{\pm}-1} Y^{\pm}}{\partial x^{2k^{\pm}-1}}(0,0) \neq 0$;

C2. $X^+(0,0)\dfrac{\partial^{2k^+-1} Y^+}{\partial x^{2k^+-1}}(0,0) < 0$ and $X^-(0,0)\dfrac{\partial^{2k^--1} Y^-}{\partial x^{2k^--1}}(0,0) > 0$;

C3. $X^+(0,0)X^-(0,0) < 0$.

Briefly speaking, **C1** and **C2** ensure that the origin is an invisible $2k^+$-multiplicity (resp. $2k^-$-multiplicity) contact between Σ and the vector field Z^+ (resp. Z^-) and **C3** ensures that a first-return map is well defined on Σ around the origin.

Objective

We aim to go beyond the pseudo-Hopf bifurcation for monodromic tangential singularities in Filippov vector fields by showing that the number of limit cycles generated by destroying the singularity increases with its degeneracy. More specifically, we will show that the destruction of $(2k, 2k)$-monodromic tangential singularities give birth to at least k limit cycles surrounding sliding segments (see Figure 1).

Results

Theorem 1. *Let k be a positive integer. Assume that the Filippov vector field Z, given by* (1)*, has a $(2k, 2k)$-monodromic tangential singularity at the origin with non-vanishing second Lyapunov coefficient V_2. Then, given $\lambda > 0$ and a neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$, there exist polynomials P^+ and P^-, with degree $2k-2$ and norm less than λ, and a neighborhood $I \subset \mathbb{R}$ of 0 such that, for every $b \in I$ satisfying $\operatorname{sign}(b) = -\operatorname{sign}(\delta V_2)$, the Filippov vector field*

$$\widetilde{Z}_b(x,y) = \begin{cases} \widetilde{Z}^+(x+b, y), & y > 0, \\ \widetilde{Z}^-(x,y), & y < 0, \end{cases} \tag{2}$$

with $\widetilde{Z}^{\pm}(x,y) = \begin{pmatrix} X^{\pm}(x,y) \\ Y^{\pm}(x,y) + X^{\pm}(x,y)P^{\pm}(x) \end{pmatrix},$

has k hyperbolic limit cycles inside U such that each one of these limit cycles surrounds a single sliding segment (see Figure 1). In addition, the hyperbolic limit cycles are stable (resp. unstable) provided that $V_2 < 0$ (resp. $V_2 > 0$).

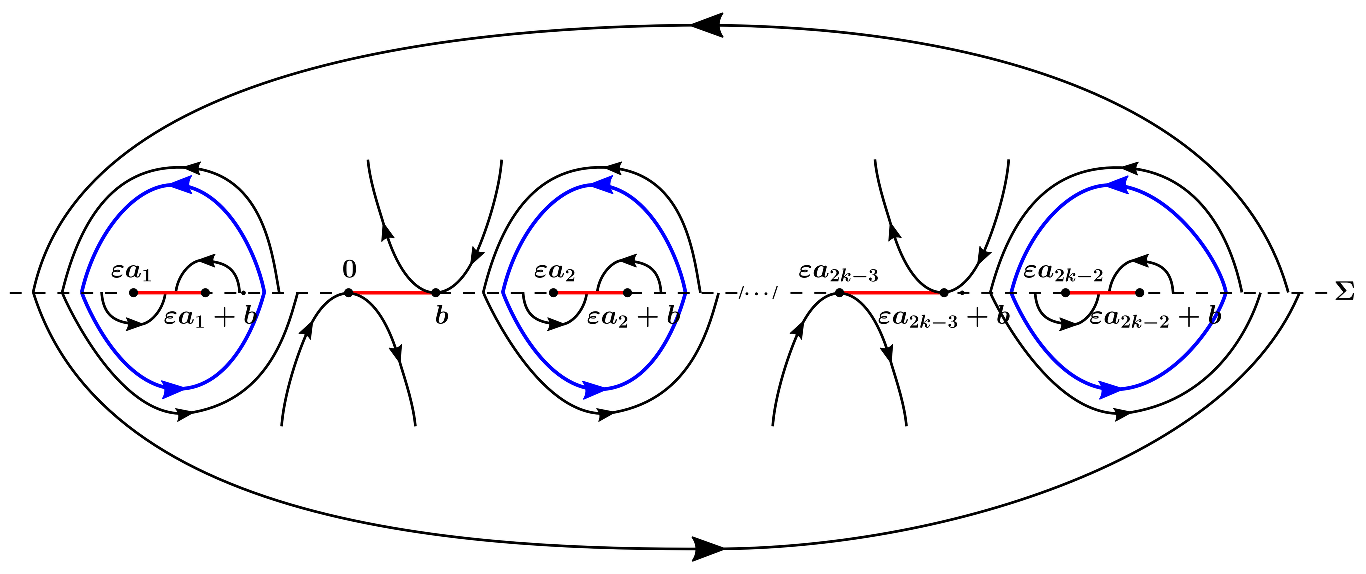

Figure 1: Illustration of k limit cycles surrounding k sliding segments predicted by Theorem 1. Continuous and dashed segments on Σ represent sliding and crossing regions, respectively.

References

[1] A. F. Filippov. *Differential equations with discontinuous righthand sides*, volume 18 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

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