Weingarten Surfaces Associated to Laguerre Minimal Surfaces

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Abstract

In the work [2], the author shows that every hypersurface in Euclidean space is locally associated to the unit sphere by a sphere congruence, whose radius function R is a geometric invariant of hypersurface. Here we define for any surface Σ its spherical mean curvature H_S which depends on principal curvatures of Σ and the radius function R. Then we consider two classes of surfaces: the ones with $H_S = 0$, called H_1-surfaces, and the surfaces with spherical mean curvature of harmonic type, named H_2-surfaces. We provide for each these classes a Weierstrass type representation depending on three holomorphic functions and we prove that the H_1-surfaces are associated to the minimal surfaces, whereas the H_2-surfaces are related to the Laguerre minimal surfaces. As application we provide a new Weierstrass type representation for the Laguerre minimal surfaces - and in particular for the minimal surfaces - in such a way that the same holomorphic data provide examples in H_1-surface/minimal surface classes or in H_2-surface/Laguerre minimal surface classes.

Introduction

An oriented surface S in the Euclidean space $\mathbb{R}^3$ is called a **Weingarten surface** if there is a differentiable relationship W between the Gaussian curvature K and the mean curvature H of S such that $W(H,K) \equiv 0$.

In the work [2] is established that for a hypersurface Σ in $\mathbb{R}^{n+1}$ satisfying $\langle p, N(p)\rangle \neq 1$, for all $p \in \Sigma$, there exists a sphere congruence for which Σ and the unit sphere $\mathbb{S}^n$ are envelopes. Such a surface Σ can be locally parameterized from a local parameterization of $\mathbb{S}^2$ as below.

Theorem 1: Let Σ be a Riemann surface and $X : \Sigma \to \mathbb{R}^3$ an immersion such that $\langle X(p), N(p)\rangle \neq 1$, for all $p \in \Sigma$, where N is the normal Gauss map of X. Consider also a parameterization $Y : U \subset \mathbb{R}^2 \to \mathbb{S}^2$ of the unit sphere given by $Y = \pi_-^{-1} \circ g$, where $g : \mathbb{C} \to \mathbb{C}_\infty$ is a holomorphic function such that $g' \neq 0$ and $\pi_-^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{-e_3\}$ is the inverse of stereographic projection. Then there exists a differentiable function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ associated to this parameterization, such that Σ can be locally parameterized by

$$X = \frac{1}{T}\Big(2g, 2-T\Big) - \frac{2(h+c)}{S}\eta, \tag{1}$$

where c is a nonzero real constant, $T = 1 + |g|^2$ and

$$\eta = \nabla_L h + hY, \quad S = \langle \eta, \eta\rangle = \left|\nabla_L h\right|^2 + h^2,$$

with

$$L_{ij} = \langle Y_{,i}, Y_{,j}\rangle = \frac{4|g'|^2}{T^2}\delta_{ij}, \quad T = 1+|g|^2, \ 1 \leq i,j \leq 2,$$

For such a hypersurface Σ, we define its **spherical radial curvatures s_i associated to $\mathbb{S}^n$** and **spherical mean curvature H_S associated to $\mathbb{S}^n$**, as follows:

$$s_i = \frac{1+k_i}{1-k_iR}, \quad H_S = \frac{1}{n}\sum_{i=1}^{n} s_i,$$

where k_i are the principal curvatures of Σ, $1 \leqslant i \leqslant n$, and R is a geometric invariant of Σ given by the radius function of the sphere congruence.

H_1-Surfaces and H_2-Surfaces

Let Σ be a surface and $X : \Sigma \to \mathbb{R}^3$ an immersion such that $\langle X(p), N(p)\rangle \neq 1$, for all $p \in \Sigma$, where N is the normal Gauss map of X. The surface Σ is called a *surface of null spherical mean curvature*, in short H_1**-surface**, if holds $H_S = 0$ and Σ is called a *surface with spherical mean curvature of harmonic type*, in short H_2**-surface**, if it satisfies

$$\Delta_\sigma\left[\frac{H_S}{\Psi - 1}\right] = 0,$$

where H_S is the spherical mean curvature of Σ and $\sigma = I + 2RII + R^2 III$, with I, II, III the fundamental forms of Σ.

Main Results

Next we have a characterization for the H_1 and H_2-surfaces.

1. Let Σ be a surface as in Theorem 1. Then Σ is a H_1-surface if and only if
$$h = \frac{\langle 1, A\rangle + \langle g, B\rangle}{1+|g|^2}, \tag{2}$$
where A is a holomorphic function and B is a holomorphic function such that $B(z) = \int (A'(z)g(z) - A(z)g'(z) + ic_1 g'(z))\,dz$, for c_1 a real constant.
2. Let Σ be a surface as in Theorem 1. Then Σ is a H_2-surface if and only if
$$h = \frac{\langle 1, A\rangle + \langle g, B\rangle}{1+|g|^2}, \tag{3}$$
where A, B are holomorphic functions.
3. In the conditions of Theorem 1, Σ is a H_1-surface if and only if η is a minimal surface.
4. In the conditions of Theorem 1, Σ is a H_2-surface if and only if η is a Laguerre minimal surface.
5. For h given as in (2), X is a Weierstrass type representation for the H_1-surfaces, whereas for h given as in (3), X is a Weierstrass type representation for the H_2-surfaces.

   In the conditions of Theorem (1), η can be rewrite as
$$\eta = \left(\frac{T}{2}\frac{\nabla h}{|g'|^2}g' - g\left\langle \nabla h, \frac{g}{g'}\right\rangle + \frac{2h}{T}g, \frac{(2-T)}{T}h - \left\langle \nabla h, \frac{g}{g'}\right\rangle\right) \tag{4}$$
6. From (3), we have that the expression (4) above is an alternative **Weierstrass representation for the minimal surfaces** when the function h is given as in (2).
7. From (4), we conclude that the expression (4) is a **Weierstrass representation for the Laguerre minimal surfaces** when h is given as in (3).

Conclusion

- The study of H_1-surfaces allows obtaining an alternative Weierstrass representation for the minimal surfaces depending on three holomorphic functions.
- The study of H_2-surfaces allows obtaining an alternative Weierstrass representation for the Laguerre minimal surfaces depending on three holomorphic functions.

References

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